Book: Dynamical Oceanography
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Chapter: 1, Exercise: 1.3: Buoyancy frequency
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Consider a stratified water column with a density profile $\rho(z)$.

a. Derive the equation of motion for a water parcel that, without any exchange of heat and salt with its environment, is subjected to a small initial vertical displacement.

The vertical acceleration a_z is given by the following expression

$$a_z = \frac{d^2 Z}{dt^2} = \frac{g}{\rho_1}[\frac{\partial \rho}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial T}(\frac{\partial T}{\partial z} + \Gamma)]\Delta z,$$

where Γ is the adiabatic temperature gradient and Z is the vertical position and can be written as a reference depth plus a deviation of it, i.e.,

$$Z(t) = z_0 + \Delta z(t)$$

Here, Δz is a small vertical displacement of the water parcel with respect to the reference level z_0.

At $t = 0$, the position of the parcel is $z = z_0$ and the velocity of the parcel is zero.

b. Show that the buoyancy frequency N can be seen as the characteristic oscillation frequency of the water parcel in a stably stratified water column.

We can rewrite the equation for a_z above as

$$\frac{d^2 \Delta z}{dt^2} = -N^2 \Delta z,$$

where

$$N^2 = -\frac{g}{\rho}[\frac{\partial \rho}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial T}(\frac{\partial T}{\partial z} + \Gamma)].$$

If $N^2 > 0$, the ocean water is stable layered and the equation above has a solution of the form

$$\Delta z(t) = A \cos Nt + B \sin Nt,$$

where A and B are constants and N is the so-called Brunt-Väisälä frequency. This solution represents a so-called buoyancy oscillation with a frequency equal to N.

If $N^2 < 0$, the ocean water is unstably stratified and the water parcel does not return to its original equilibrium position. The solution for Δz is then given by

$$\Delta z(t) = Ae^{-Nt} + Be^{Nt}$$

where A and B are constants.

Consider now the same situation but in the presence of friction that is linearly related to the velocity of the water parcel.

c. Derive in this case also the equation of motion for the water parcel. When is the water column unstably stratified?

Let the coefficient of linear friction be indicated by γ, then because the velocity is given by dz/dt, the equation for Δz becomes

$$\frac{d^2 \Delta z}{dt^2} = -N^2 \Delta z - \gamma \frac{dz}{dt}.$$

This is an equation for a so-called damped linear oscillator. The characteristic polynomial of this equation is given by

$$\lambda^2 + \gamma\lambda + N^2 = 0$$

which has roots

$$\lambda_{1,2} = -\frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4N^2})$$

Let $D = \gamma^2 - 4N^2$. When $0 < N^2 < \gamma^2/4$, then $D > 0$ and there are two real roots with negative real parts (the overdamped solutions) and hence the water column is statically stable. When $N^2 < 0$ then $D > 0$, there are two real solutions, but one with positive real part and hence the water column is statically unstable. When $N^2 > \gamma^2/4$, then the solutions are complex conjugate with negative real parts and hence the water column is also stable. Formally one should also the cases where $D = 0$ where critically damped solutions exist, but this is not important for the exercise here.