Book: Dynamical Oceanography
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Chapter: 10, Exercise: 10.5: Barotropic and baroclinic instability
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The relation between the condition of the motion of fluid particles with respect to isopycnals and the instability condition in the Eady model ($\mu < \mu_c$) is not immediately transparent.

a. Show that

$$\frac{\frac{\partial \rho_*}{\partial y_*}}{\frac{\partial \rho_*}{\partial z_*}} = -\frac{D\epsilon}{SL}\frac{\partial \bar{\rho}}{\partial y} + \mathcal{O}(\frac{D}{L}\epsilon F)$$

We start from (8.19),

$$\rho_* = \bar{\rho}_* \left(1 + \epsilon F \rho\right)$$

and derive

$$\frac{\partial \rho_*}{\partial z_*} = \frac{1}{D}\frac{d\bar{\rho}_*}{dz}\left(1 + \epsilon F \rho\right) + \frac{\bar{\rho}_*}{D}\epsilon F \frac{\partial \rho}{\partial z}$$

and

$$\frac{\partial \rho_*}{\partial y_*} = \frac{\bar{\rho}_*}{L}\epsilon F \frac{\partial \rho}{\partial y}.$$

In additon, we use the definition of the Burger number S and (8.22),

$$S = \frac{N^2 D}{gF} = -\frac{1}{\bar{\rho}_* F}\frac{d\bar{\rho}_*}{dz}$$

to find

$$\frac{\frac{\partial \rho_*}{\partial y_*}}{\frac{\partial \rho_*}{\partial z_*}} = \frac{\bar{\rho}_* \epsilon F D}{L \frac{d\bar{\rho}_*}{dz}} \frac{\frac{\partial \rho}{\partial y}}{1 + \epsilon F\left(\rho + \left(\frac{1}{\bar{\rho}_*}\frac{d\bar{\rho}_*}{dz}\right)^{-1}\frac{\partial \rho}{\partial z}\right)} = -\frac{D}{L}\frac{\epsilon}{S}\frac{\partial \rho}{\partial y}\left(1 + \epsilon F\left(\rho + \left(\frac{1}{\bar{\rho}_*}\frac{d\bar{\rho}_*}{dz}\right)^{-1}\frac{\partial \rho}{\partial z}\right)\right)^{-1}$$

For the Eady basic zonal flow, $\rho = \bar{\rho}(y, z) = y$, and hence

$$\frac{\frac{\partial \rho_*}{\partial y_*}}{\frac{\partial \rho_*}{\partial z_*}} = -\frac{D}{L}\frac{\epsilon}{S}\frac{d\bar{\rho}}{dy} + \mathcal{O}\left(\frac{D}{L}\frac{\epsilon}{S}\epsilon F\right)$$

where the latter order estimate arises by developing $1/(1 + \epsilon F \bar{\rho})$ into a Taylor series.

b. Derive for this case that the acceleration a_* is

$$a_* = -\mathbf{g}.\mathbf{P}\frac{\zeta_*}{\rho_*}\frac{\partial \rho_*}{\partial z_*}(1 - \frac{D\epsilon}{SL}\frac{\partial \bar{\rho}}{\partial y}\frac{\eta_*}{\zeta_*})$$

We directly substitute the result under a. into (10.44) to get the expression above.

c. Subsequently show that η_* and ζ_* are related to the perturbation velocities through

$$\frac{\eta_*}{\zeta_*} = \frac{\tilde{v}_*}{\tilde{w}_*} = \frac{L}{D\epsilon}\frac{\tilde{v}^0}{\tilde{w}^1}$$

where $\tilde{v}^0$ and $\tilde{w}^1$ are the dimensionless velocities of the perturbations; these are determined from the eigensolutions with positive growth factor.

For small perturbations (that arise during a small time step Δt), perturbations are related to perturbation velocities as $\tilde{\eta}_* = \tilde{v}_* \Delta t$ and $\tilde{\zeta}_* = \tilde{w}_* \Delta t$. Hence,

$$\frac{\eta_*}{\zeta_*} = \frac{\tilde{v}_*}{\tilde{w}_*}$$

As $\tilde{v}_* = \tilde{v}_*^0 + ...$ and $\tilde{w}_* = \epsilon \frac{D}{L} \tilde{w}_*^1 + ...$ because of the scaling of the quasi-geostrophic equations we find

$$\frac{\eta_*}{\zeta_*} = \frac{\tilde{v}_*^0}{\tilde{w}_*^1} \frac{L}{\epsilon D}$$

d. Show that the background flow is unstable when

$$\frac{\partial \bar{\rho}}{\partial y} > S \tan \phi$$

where S is the Burger number.

The result under c. can be written as

$$\frac{\eta_*}{\zeta_*} = \frac{1}{\tan \phi} \frac{L}{\epsilon D}$$

where $\tan \phi$ is as in section 10.4 the angle with the horizontal under which the fluid particles move. From the expression for the acceleration under b. the condition for a positive acceleration into the direction of the motion of the perturbations is

$$\frac{D\epsilon}{SL} \frac{\partial \bar{\rho}}{\partial y} \frac{\eta_*}{\zeta_*} = \frac{D\epsilon}{SL} \frac{\partial \bar{\rho}}{\partial y} \frac{1}{\tan \phi} \frac{L}{\epsilon D} > 1$$

and hence

$$S < \frac{1}{\tan \phi} \frac{\partial \bar{\rho}}{\partial y}$$

e. Explain now why there is instability only when $\mu < \mu_c$, where $\mu^2 = (k^2 + [(n+1/2)\pi]^2) S$ (section 10.3).

For fixed k and n, we find that S is proportional to μ^2 as

$$S = \frac{\mu^2}{k^2 + [(n+1/2)\pi]^2}$$

The condition derived under d. indicates that S should be smaller than a certain number which depends on the motion of the fluid particles with respect to the background density gradient. The equation above then shows that also μ^2 should be smaller than a specific value to get instability.