Book: Dynamical Oceanography
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Chapter: 11, Exercise: 11.3: Properties of equatorial waves
Version: 2

a. Determine the dimensional vertical velocity field associated with the equatorial Kelvin wave and the $j = 1$ Rossby wave.

We consider the reduced gravity model of section 11.2.2. The pressure in the second layer is given by

$$p_{2*} = p_*|_{-H+\zeta_*} + (-H + \zeta_* - z)\,(\rho + \Delta\rho)\,g = h_*\rho g + (-H + \zeta_* - z)\,(\rho + \Delta\rho)\,g.$$

Since the second layer is motionless, horizontal derivatives vanish and hence

$$0 = \rho\frac{\partial(\eta_* - \zeta_*)}{\partial x_*} + (\rho + \Delta\rho)\,\frac{\partial \zeta_*}{\partial x_*} \Rightarrow \frac{\partial \zeta_*}{\partial x_*} = -\frac{\rho}{\Delta\rho}\frac{\partial \eta_*}{\partial x_*}$$

and similarly for the y-derivative. So, $\zeta_* = -\frac{\rho}{\Delta\rho}\eta_*$, and $h_* = H + \eta_*\left(1 + \frac{\rho}{\Delta\rho}\right) = H - \zeta_*\left(1 + \frac{\Delta\rho}{\rho}\right)$.

The vertical velocities at the top and at the thermocline are given by (11.2d) and (11.4b):

$$w_*|_{\eta_*} = \frac{D\eta_*}{dt_*}, \qquad w_*|_{-H+\zeta_*} = \frac{D\zeta_*}{dt_*},$$

and are hence both proportional to $\frac{Dh_*}{dt_*}$, which in the linearized model of section 11.4 simplifies to

$$\frac{\partial h_*}{\partial t_*} = \frac{Hc_0}{L}\frac{\partial h}{\partial t}.$$

Because horizontal velocities in the upper layer are independent of z, the vertical velocity is linear in z (continuity equation). So, the vertical velocity field is completely determined by h. (N.B.: unlike MdT's claim made during the tutorial session, the horizontal velocities in section 11.2 cannot a priori be assumed independent of z. However, for the linearized, unforced, nondissipative equations in section 11.4 this is indeed the case; *make sure you understand this!*)

For the Kelvin wave, we have

$$h_K(x, y, t) = \hat{h}_K(y)e^{i(kx-\sigma t)}\ ;\ \hat{h}_K(y) = H_K e^{-y^2/2}\ ;\ \sigma = k$$

where H_K is an arbitrary constant.

For the $j = 1$ Rossby wave, we consider the low-frequency / long wave limit, for which the dispersion relation is given by (11.39):

$$\sigma = -\frac{k}{3}.$$

In order to find the meridional structure of h, we combine equations (11.19a) and (11.19c), and subsequently substitute the travelling wave solutions given by (11.20) to find

$$\frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial t^2} = y\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y \partial t},$$

$$\hat{h}\left(\sigma^2 - k^2\right) = i\left(yk\hat{v} - \sigma\hat{v}'\right) \Rightarrow \hat{h} = \frac{-1}{8i\sigma}\left(3y\hat{v} + \hat{v}'\right).$$

Using expressions (11.27), (11.35a) and (11.35b) then gives

$$\hat{h}_R = \frac{-1}{8i\sigma}\left(3\left(\sqrt{\frac{1}{2}}\psi_0 - \psi_2\right) + \left(\sqrt{\frac{1}{2}}\psi_0 + \psi_2\right)\right) = \frac{1}{2\sqrt{2}i\sigma}\left(\frac{\psi_2(y)}{\sqrt{2}} - \psi_0(y)\right),$$

where the $\psi_i(y)$ are the Hermite functions in (11.27). Then $h_R(x, y, t) = \hat{h}_R(y)e^{i(kx-\sigma t)}$.

b. Determine the ratio of the sea surface height amplitude and the thermocline depth amplitude for both waves as in a.

When the sea surface height amplitude is given by ϵ (positive upward) and the thermocline amplitude by δ (positive downward), then we find

$$\delta = \epsilon \frac{\rho}{\Delta\rho}$$

where $\Delta\rho$ is the density difference between the upper and lower ocean lower. With $c_0 = \sqrt{g'H} = 2\ \mathrm{ms}^{-1}$ and $H = 100$ m, it follows that $\Delta\rho/\rho = c_0^2/(gH) = 4 \times 10^{-3}$.

c. During an El Niño the eastern Pacific thermocline can deepen by 50 m. Compute the amplitude of the sea surface height anomaly during such an event.

With $\delta = 50$ m, we find

$$\epsilon = \delta \frac{\Delta\rho}{\rho} = 20 \text{ cm}.$$