**Book: Dynamical Oceanography**
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**Chapter: 3, Exercise: 3.4: Equations of motion**
**Version: 2**

*It is a useful exercise to derive the equations of motion (3.32a-c) from the specification of the Navier-Stokes equations in spherical coordinates $(r, \vartheta, \varphi)$ such as provided, for example in appendix 2 of Batchelor (2000).*

*First consider the isotropic case for which $A_H = A_V$.*

*a. Give an explicit expression for the inertial terms in (3.32a-c).*

The expression for the inertial term in (3.32a) is

$$\frac{Du}{dt} - \frac{uv}{r}\tan\theta + \frac{uw}{r} = \frac{\partial u}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial u}{\partial \phi} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{uv}{r}\tan\theta + w\frac{\partial u}{\partial z} + \frac{uw}{r}$$

and the other terms follow similarly.

*b. Give an explicit expression for the term $\mathcal{F}_I^\phi$ in (3.32a-c).*

See Batchelor (2000), appendix 2.

*Next consider the nonisotropic case, but now in Cartesian coordinates.*

*c. Give an explicit expression for the term $\mathcal{F}_I^\phi$ in (3.32a-c) using (3.18).*

As we need to compute the divergence of the stress tensor $\mathcal{T}$, we use the identity for any vector $\mathbf{q}$,

$$(\nabla \cdot \mathcal{T})\mathbf{q} = \nabla \cdot (\mathcal{T}\mathbf{q})$$

For the component $\mathcal{F}_I^\phi$ or $\mathcal{F}_I^x$ in Cartesian coordinates, we can take $\mathbf{q} = \mathbf{e}_x$, the unit vector in $x-$direction. To compute $\mathcal{T}(1,0,0)^T$ we have to compute

$$\nabla_H \otimes \mathbf{v} + (\nabla_H \otimes \mathbf{v})^T = \begin{pmatrix} 2u_x & v_x + u_y & w_x \\ u_y + v_x & 2v_y & w_y \\ w_x & w_y & 0 \end{pmatrix}.$$

where $u_x = \partial u/\partial x$, etc., and

$$\nabla_z \otimes \mathbf{v} + (\nabla_z \otimes \mathbf{v})^T = \begin{pmatrix} 0 & 0 & u_z \\ 0 & 0 & v_z \\ u_z & v_z & 2w_z \end{pmatrix}.$$

It then follows that

$$\mathcal{T}\begin{pmatrix}1\\0\\0\end{pmatrix} = A_H\begin{pmatrix}2u_x\\u_y+v_x\\w_x\end{pmatrix} + A_V\begin{pmatrix}0\\0\\u_z\end{pmatrix}$$

and hence

$$(\nabla \cdot \mathcal{T})\begin{pmatrix}1\\0\\0\end{pmatrix} = A_H(2u_{xx} + v_{xy} + u_{yy} + w_{xz}) + A_V u_{zz}$$

When the flow is incompressible, we can use the continuity equation $u_x + v_y + w_z = 0$ to get the familiar form

$$(\nabla \cdot \mathcal{T})\begin{pmatrix}1\\0\\0\end{pmatrix} = A_H(u_{xx} + u_{yy}) + A_V u_{zz}$$