Book: Dynamical Oceanography
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Chapter: 3, Exercise: 3.5: Damped inertial motion
Version: 2

We consider the situation where a particular wind stress has driven a flow in an ocean basin (containing water of constant density) for a while and then suddenly ceases. At this point there are no external forces acting on the water. There is a linear friction damping the motion with a friction coefficient r. The horizontal momentum equations are

$$\frac{Du}{dt} = 2\Omega v \sin\theta - ru$$

$$\frac{Dv}{dt} = -2\Omega u \sin\theta - rv$$

where D/dt is the material derivative. Assume that a water parcel has a horizontal velocity $(u, v) = (0, v_0)$ at $t = 0$.

a. Show that

$$\frac{D}{dt}(u^2 + v^2) = -2r(u^2 + v^2)$$

After multiplying the first equation with u and the second one with v we obtain:

$$\frac{1}{2}\frac{Du^2}{Dt} = 2\Omega uv \sin\theta - ru^2$$

$$\frac{1}{2}\frac{Dv^2}{Dt} = -2\Omega uv \sin\theta - rv^2$$

Adding both equations it follows:

$$\frac{D(u^2 + v^2)}{Dt} = -2r(u^2 + v^2) \tag{1}$$

We now search for solutions of the form

$$(u(t), v(t)) = e^{-rt}(c_1 \sin(\alpha t + \Psi_1), c_2 \cos(\alpha t + \Psi_2)).$$

with constants α, c_1, c_2 and Ψ_2.

b. Determine $(u(t), v(t))$ and explain what kind of motion of the water parcel results.

In the following, let $f = 2\Omega \sin\theta$. We substitute the given Ansatz into the momentum equations to find

$$-ru + \alpha c_1 e^{-rt} \cos(\alpha t + \psi_1) = f c_2 e^{-rt} \cos(\alpha t + \psi_2) - ru$$

$$-rv - \alpha c_2 e^{-rt} \sin(\alpha t + \psi_2) = -f c_1 e^{-rt} \cos(\alpha t + \psi_1) - rv$$

for any t. Hence

$$\alpha c_1 \cos(\alpha t + \psi_1) = f c_2 \cos(\alpha t + \psi_2)$$

$$\alpha c_2 \sin(\alpha t + \psi_2) = f c_1 \cos(\alpha t + \psi_1)$$

This implies that $\psi_1 = \psi_2 = \psi$, $c_1 = c_2 = c$ and $\alpha = f$. Using the assumption that for $t = 0$ $u = 0, v = v_0$ we find from the expression of the solutions at $t = 0$

$$0 = c \sin\psi$$

$$v_0 = c \cos\psi$$

So $c = v_0$ and $\psi = 0$. Finally, the solutions are:

$$u(t) = v_0 e^{-rt} \sin ft$$

$$v(t) = v_0 e^{-rt} \cos ft$$

For convenience define $\zeta(t) = v(t) + \imath u(t) = v_0 e^{(\imath f - r)t}$. Integrate $\zeta(t)$ to find

$$\int_0^t \zeta(t')dt' = \frac{-v_0(\imath f + r)}{r^2 + f^2} e^{(\imath f - r)t'} \Big|_0^t = \frac{v_0(\imath f + r)}{r^2 + f^2} \left[1 - e^{-rt} \left(\cos ft + \imath \sin ft\right)\right]$$

The paths of the water parcels in the x, y-plane are determined by

$$x(t) = \mathrm{Im} \int_0^t \zeta(t')dt' = \frac{v_0}{r^2 + f^2} \left[f - e^{-rt} \left(f \cos ft + r \sin ft\right)\right]$$

$$y(t) = \mathrm{Re} \int_0^t \zeta(t')dt' = \frac{v_0}{r^2 + f^2} \left[r - e^{-rt} \left(r \cos ft - f \sin ft\right)\right]$$

For $r = 0$, the water parcels move on a circle with a radius v_0/f. When $r > 0$ the water parcels move in spirals towards a fixed point.