**Book: Dynamical Oceanography**
**Author: Dijkstra, H. A.; email: H.A.Dijkstra@uu.nl**
**Chapter: 4, Exercise: 4.1: Helmholtz theorem**
**Version: 2**

*Let $C_1$ and $C_2$ be two curves on a closed vortex tube, $S_i$ be the surfaces enclosed by the $C_i$, $S$ be the total surface of the tube and $V$ be the total volume enclosed by $S$ (see Fig. 4.1). From the identity $\nabla \cdot \omega = 0$, show that*

$$\Gamma_1 = \int_{C_1} \mathbf{v} \cdot \mathbf{ds} = \int_{C_2} \mathbf{v} \cdot \mathbf{ds} = \Gamma_2$$

The quantities $\Gamma_1$ and $\Gamma_2$ are defined as

$$\Gamma_1 = \int_{C_1} \vec{v} \cdot \vec{ds} = \int_{S_1} (\vec{\nabla} \times \vec{v}) \cdot \vec{n} dA = \int_{S_1} \vec{\omega} \cdot \vec{n} dA$$

$$\Gamma_2 = \int_{C_2} \vec{v} \cdot \vec{ds} = \int_{S_1} (\vec{\nabla} \times \vec{v}) \cdot \vec{n} dA = \int_{S_2} \vec{\omega} \cdot \vec{n} dA$$

where $C_1$ and $C_2$ are two closed curves on a vortex tube, $\Gamma$ is the circulation of the velocity filed with respect to a closed curve $C$, $S_i$ are the surfaces enclosed by $C_i$ and $\vec{n}$ is the normal vector on the surface, whose direction is defined by the orientation of the curves $C_i$ (right-hand rule).
Because the vorticity field is divergence free we have:

$$0 = \int_V \vec{\nabla} \cdot \vec{\omega} \; dV = \int_S \vec{\omega} \cdot \vec{n} \; dS$$

in which $S$ is the total surface of the tube and $V$ is the total volume enclosed by $S$. Now $\vec{n}$ is the *outward directed* normal on the surface. We can write the surface $S$ as $S_1 + S_2 + S_{lat}$, where $S_{lat}$ is the tube surface. Because the tube surface is everywhere perpendicular to the vortex lines, we have $\vec{\omega} \cdot \vec{n} = 0$. Hence,

$$0 = \int_{S_1} \vec{\omega} \cdot \vec{n} dA + \int_{S_2} \vec{\omega} \cdot \vec{n} dA = \Gamma_1 - \Gamma_2$$

where the minus sign results from the different orientation of $\vec{n_1}$ and $\vec{n_2}$ to $S_1$ and $S_2$.