Book: Dynamical Oceanography
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Consider a flow with velocity field $\mathbf{v} = (u, v, w)$ *in a horizontally unbounded layer of water. The water has a constant density* ρ *and rotates with an angular velocity* Ω *around the z-axis.*

a. Give the vorticity equation of this flow.

Since the water has a constant density ρ, the baroclinic effects are zero and the velocity field is divergence free. In addition, random mixing (diffusion) of vorticity is neglected. The vorticity equation of this flow becomes:

$$\frac{\partial \vec{\omega_a}}{\partial t} = -\vec{v} \cdot \vec{\nabla}\vec{\omega_a} + \vec{\omega_a} \cdot \vec{\nabla}\vec{v}$$

where the absolute vorticity of the fluid is the sum between the relative vorticity and the planetary vorticity

$$\vec{\omega_a} = \vec{\omega}_{rel} + \vec{\omega}_{plan} = \vec{\omega} + 2\vec{\Omega}.$$

We will focus now on variations in the flow on a time scale τ*, a horizontal velocity scale* U *and a length scale* L*.*

b. Estimate the order of magnitude of the terms in the vorticity equation.

Using primes ($'$) to denote dimensionless quantities, the vorticity equation becomes:

$$\frac{1}{\tau}\frac{\partial}{\partial t'}(2\Omega\vec{\Omega}' + \frac{U}{L}\vec{\omega}') = -\frac{U}{L}(\vec{v}' \cdot \vec{\nabla}'(2\Omega\vec{\Omega}' + \frac{U}{L}\vec{\omega}')) + \frac{U}{L}(2\Omega\vec{\Omega}' + \frac{U}{L}\vec{\omega}') \cdot \vec{\nabla}'\vec{v}'$$

Let $f = 2\Omega$*. There is a special type of flow in case* $\tau \gg 1/f$ *and* $\epsilon = U/(fL) \ll 1$*.*

c. Why is this case so special when considering the vorticity equation?

Divide the result above by f^2 to find

$$\frac{1}{\tau f}\frac{\partial}{\partial t'}(\vec{\Omega}' + \epsilon\vec{\omega}') = -\epsilon(\vec{v}' \cdot \vec{\nabla}'(\vec{\Omega}' + \epsilon\vec{\omega}')) + \epsilon(\vec{\Omega}' + \epsilon\vec{\omega}') \cdot \vec{\nabla}'\vec{v}'$$

Note that the dimensionless absolute vorticity $\vec{\omega_a}'$ becomes

$$\vec{\omega_a}' = \vec{\Omega}' + \varepsilon\vec{\omega}'.$$

Of special interest are those cases in which the relative vorticity is much smaller than the planetary vorticity, i.e. $\epsilon \ll 1$; and where the local time derivative is much smaller than the effects of planetary vorticity on the flow, i.e. $\tau f \gg 1$. Assuming that $\tau f = O(\epsilon^{-1})$ or larger, the lowest order vorticity equation is:

$$\vec{\Omega}' \cdot \vec{\nabla}'\vec{v}' = 0 \qquad \text{or} \qquad (2\vec{\Omega} \cdot \vec{\nabla})\vec{v} = 0,$$

which is the Taylor-Proudman theorem. In dimensional quantities, the component equations can be written as

$$\frac{\partial u}{\partial z} = 0 \ ; \ \frac{\partial v}{\partial z} = 0 \ ; \frac{\partial w}{\partial z} = 0$$

and from the last equation it follows, since the flow is incompressible, that $\partial u/\partial x + \partial v/\partial y = 0$. The interpretation of the Taylor-Proudman equation is that the derivatives of all velocity components in the direction of the planetary vorticity are zero. The flow does not vary in the direction of the background velocity field, and it is divergence free perpendicular to that field. The motion is purely 2-dimensional, and can be visualized a moving columns, in each column is directed parallel to the $\vec{\Omega}$ field.

d. In an initially motionless layer, a sphere of radius R, with $R < L$, is moved over the bottom with a velocity U. After a while a steady flow results with $\epsilon \ll 1$. Make a sketch of this flow.

A wonderful flow patterns emerges in the following situation, as a direct consequence of the theorem. As $w = 0$ at the wall, it has to be zero everywhere. In a barotropic fluid a solid sphere is pulled slowly in a direction perpendicular to the rotation axis. The fluid has to move around the solid sphere, but because the motion has to be purely 2-dimensional, fluid cannot over or under the sphere (seen along the rotation axis). The only solution that fulfills these conditions it such that the fluid above and below the sphere has to move with the sphere, parallel to the rotation axis. So, for the fluid the sphere is in fact a cylinder, dragging fluid along the rotation axis.