Book: Dynamical Oceanography
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Chapter: 5, Exercise: 5.2: An alternative approach to the bottom Ekman layer
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Consider an ocean with water of constant density ρ above a flat bottom (at $z_* = 0$). The horizontal velocities outside the Ekman layer (where the flow is hydrostatic and geostrophic) are indicated by (u_{g*}, v_{g*}). The total velocity can be written as

$$u_* = u_{g*} + u_{E*}; \qquad v_* = v_{g*} + v_{E*}$$

where u_{E*} en v_{E*} are the horizontal Ekman velocities.

a. Show that the Ekman velocities can be determined from

$$-f_0\, v_{E*} = A_V \frac{\partial^2 u_{E*}}{\partial z^2}$$

$$f_0\, u_{E*} = A_V \frac{\partial^2 v_{E*}}{\partial z^2}$$

If we neglect inertia and horizontal friction, the momentum equations become

$$-fv = -\frac{1}{\rho} p_x + A_V u_{zz}$$

$$fu = -\frac{1}{\rho} p_y + A_V v_{zz}$$

where $p_x = \partial p/\partial x$, $u_{zz} = \partial^2 u/\partial z^2$, etc. In the interior we have the geostrophic balance, given by:

$$-fv = -fv_g = -\frac{1}{\rho} p_x$$

$$fu = fu_g = -\frac{1}{\rho} p_y$$

The geostrophic flow u_g, v_g and the horizontal gradients are independent of z, because:

$$-f(v_g)_z = -\frac{1}{\rho} p_{xz} = \frac{g}{\rho} (\rho)_x = 0$$

$$f(u_g)_z = -\frac{1}{\rho} p_{yz} = \frac{g}{\rho} (\rho)_y = 0$$

due to the assumption of homogeneous density. Because the momentum equations are linear in the velocity we can superpose the velocity field in a geostrophic and an Ekman part:

$$u = u_g + u_E, v = v_g + v_E$$

If we subtract the geostrophic balance from the momentum balance, we end up with:

$$-fv_E = A_V (u_E + u_g)_{zz} = A_V (u_E)_{zz}$$

$$fu_E = A_V (v_E + v_g)_{zz} = A_V (v_E)_{zz}$$

So, the Ekman velocity arises from a balance between the Coriolis force and the frictional force.

b. Determine the Ekman transport $\vec{M}_{E*} = (M^x_{E*}, M^y_{E*})$ through vertical integration and show that the direction of this transport is perpendicular to the bottom shear stress.

Cross differentiation the result under a. gives one equation for u_E:

$$f^2 u_E + A_V^2 (u_E)_{zzzz} = 0$$

Solutions of the form

$$u_E = u_0 e^{\lambda z}$$

lead to

$$f^2 + A_V^2 \lambda^4 = 0$$

so that we find for λ:

$$\lambda = \pm(1 \pm i)\frac{1}{\delta_E}$$

in which the Ekman depth δ_E is given by:

$$\delta_E = \sqrt{\frac{2A_v}{|f|}}$$

Because we do not allow exponentially growing solutions with z, (the influence of friction vanishes in the interior) two solutions are unphysical. The other two solutions are fixed by the condition that the velocity should vanish at the boundary, so $u_E = -u_g$ at the boundary. If we assign a coordinate system to the flow that originates at the bottom the following solution is found:

$$u = u_g - e^{-z/\delta_E}(u_g \cos\frac{z}{\delta_E} + \frac{f}{|f|} v_g \sin\frac{z}{\delta_E})$$

$$v = v_g - e^{-z/\delta_E}(v_g \cos\frac{z}{\delta_E} - \frac{f}{|f|} u_g \sin\frac{z}{\delta_E})$$

Close to the boundary the solution is:

$$u = (u_g - \frac{f}{|f|} v_g)\frac{z}{\delta_E}$$

$$v = (\frac{f}{|f|} u_g + v_g)\frac{z}{\delta_E}$$

so the solution is at 45^0 of the interior geostrophic flow, rotated to the left. (That is, for the northern hemisphere. For the souther hemisphere, the rotation is to the right). Going further up in the water column both velocity components increase, to reach a maximum at $z = 3\pi\delta_E/4$. This maximum is in fact larger than the geostrophic interior flow. From then on the velocity decreases again to the interior geostrophic velocity.
The Ekman transport in the Ekman layer can be found by integrating the Ekman velocity over this layer:

$$M_E^x = \int_0^\infty (u - u_g)dz = -\frac{\delta_E}{2}(u_g + \frac{f}{|f|} v_g)$$

$$M_E^y = \int_0^\infty (v - v_g)dz = \frac{\delta_E}{2}(\frac{f}{|f|} u_g - v_g)$$

So, the Ekman transport is proportional to the layer depth and the geostrophic interior velocity.

c. Show that for $v_{g*} = 0$ and constant u_{g*}, the solution of the Ekman layer velocities is given by

$$u_{E*} = -u_{g*} e^{-z_*/\delta_E} \cos\frac{z_*}{\delta_E}$$

$$v_{E*} = u_{g*} e^{-z_*/\delta_E} \sin\frac{z_*}{\delta_E}$$

where $\delta_E = \sqrt{\frac{2A_V}{f_0}}$ *is the bottom Ekman layer thickness.*
For $v_g = 0$, the solution of the Ekman layer velocities is given by

$$u_E = -u_g e^{-z/\delta_E} \cos \frac{z}{\delta_E}$$

$$v_E = u_g e^{-z/\delta_E} \sin \frac{z}{\delta_E}$$

d. Check that the total velocity field (u, v) *satisfies the boundary conditions*

$$(z = 0) : u_* = v_* = 0$$

$$(z \gg 1) : u_* = u_{g*}; \quad v_* = 0$$

and make a sketch of the total velocity field in the Ekman layer.

These can be easily checked by using $u = u_E + u_g$ and $v = v_E + v_g$ and using the expressions for u_E and v_E as obtained under c.