Book: Dynamical Oceanography
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Chapter: 5, Exercise: 5.5: Steady geostrophic flow
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Consider a thin layer of water that rotates with a constant angular velocity Ω; the layer is bounded from below by a flat bottom. Initially, the water is motionless and the deviation h of the dimensionless sea surface from its equilibrium value is zero. At $t = 0$, the sea surface is deformed according to

$$h(x, y) = (-\mathcal{H}(x) + \mathcal{H}(-x))h_0$$

where h_0 is a constant and $\mathcal{H}$ is the Heaviside function. During the development of the flow, the Rossby number ϵ remains small.

a. Use quasi-geostrophic theory and formulate the equations which determine the geostrophic velocities u^0, v^0 and surface deformation h^0.

Assuming that u_* and v_* are depth-independent, the dimensional equations are:

$$\frac{Du_*}{Dt_*} - fv_* = -g\frac{\partial h_*}{\partial x_*}$$

$$\frac{Dv_*}{Dt_*} + fu_* = -g\frac{\partial h_*}{\partial y_*}$$

$$\frac{\partial H_*}{\partial t_*} + \nabla_* \cdot (H_* u_*) = 0$$

where the depth-integrated equation for continuity is used. The total water depth is $H_* = h_* + D$. We scale the following variables as:

$$u_* = Uu \ ; \ v_* = Uv \ ; \ x_* = Lx \ ; \ y_* = Ly \ ; \ h_* = \varepsilon FDh$$

After substituion of the scaled variables into the equations above we find

$$\frac{U^2}{L}\frac{du}{dt} - fUv = -\frac{\varepsilon FDg}{L}\frac{\partial h}{\partial x}$$

$$\frac{U^2}{L}\frac{dv}{dt} + fUu = -\frac{\varepsilon FDg}{L}\frac{\partial h}{\partial y}$$

$$\varepsilon FD\frac{U}{L}\frac{\partial h}{\partial t} + (D + \varepsilon FDh)\frac{U}{L}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{U}{L}\varepsilon FDu\frac{\partial h}{\partial x} + \frac{U}{L}\varepsilon FDv\frac{\partial h}{\partial y}$$

Now we use that

$$\varepsilon F = \frac{U}{fL}\frac{f^2L^2}{gD} = \frac{UfL}{gD}$$

and we find the following dimensionless equations:

$$\varepsilon\frac{du}{dt} - v = -\frac{\partial h}{\partial x}$$

$$\varepsilon\frac{dv}{dt} + u = -\frac{\partial h}{\partial y}$$

$$\varepsilon F\frac{\partial h}{\partial t} + (1 + \varepsilon Fh)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \varepsilon Fu\frac{\partial h}{\partial x} + \varepsilon Fv\frac{\partial h}{\partial y} = 0$$

Next we expand u, v and h in parameter ε:

$$u = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + ...$$

$$v = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + ...$$
$$h = h^0 + \varepsilon h^1 + \varepsilon^2 h^2 + ...$$

This leads to the first order balance:

$$v^0 = \frac{\partial h^0}{\partial x}$$
$$-u^0 = \frac{\partial h^0}{\partial y}$$
$$\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} = 0$$

Assume now that the amplitudes of velocities are so small that products of these quantities can be neglected.

b. Determine the linear equation describing the evolution of h^0.

We don't need the assumption that the products are small. The order ε balance in the continuity equation simply becomes:

$$F\frac{\partial h^0}{\partial t} + \frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y} = 0$$

Cross differentiating both order ε momentum equations we obtain the following equation for relative vorticity:

$$\frac{d\zeta^0}{dt} = -(\frac{\partial u^1}{\partial x} + \frac{\partial v^1}{\partial y})$$

where $\zeta^0 = \partial v^0/\partial x - \partial u^0/\partial y$ is the relative vorticity. Combining this voriticity equation and the order epsilon continuity equation above we obtain an equation for h^0 as

$$F\frac{\partial h^0}{\partial t} - \frac{d\zeta^0}{dt} = 0$$

or equivalently as

$$\frac{d}{dt}[Fh^0 - \zeta^0] = 0$$

Consider now the special case that the steady state field h^0 does not depend on y.

c. Determine the steady state field h^0.

In this case, h^0 is only a function of x therefore, $\partial v^0/\partial x = \partial^2 h^0/\partial x^2$. From now on, use the assumption of vanishing products of small quantities. We finally get the following equation for h^0:

$$\frac{\partial}{dt}[Fh^0 - \frac{\partial^2 h^0}{\partial x^2}] = 0$$

which is just an expression of potential vorticity conservation $\Pi = \partial^2 h^0/\partial x^2 - Fh^0$.

At $t = 0$ we have

$$x > 0: \; h^0 = -h_0 \text{ and } \Pi = Fh_0$$
$$x < 0: \; h^0 = h_0 \text{ and } \Pi = -Fh_0$$

For $t \to \infty$, it follows from conservation of potential vorticity that

$$x > 0: \; Fh^0 - h^0_{xx} = -Fh_0$$
$$x < 0: \; Fh^0 - h^0_{xx} = Fh_0$$

The general solution solution of these equations is

$$h^0(x) = Ae^{\sqrt{F}x} + Be^{-\sqrt{F}x} + Ch_0$$

The bounded solution for $x > 0$ is:

$$h^0(x) = B_1 e^{-\sqrt{F}x} - h_0$$

and for $x < 0$,

$$h^0(x) = A_1 e^{\sqrt{F}x} + h_0$$

From the continuity and differentiability conditions of h^0 at $x = 0$ we get:

$$A_1 + h_0 = B_1 - h_0$$

$$A_1\sqrt{F} = -B_1\sqrt{F}$$

The solution for h^0 is then

$$x > 0: \ h^0(x) = h_0[e^{-\sqrt{F}x} - 1]$$

$$x < 0: \ h^0(x) = h_0[1 - e^{\sqrt{F}x}]$$

d. Determine and sketch the geostrophic velocities u^0 and v^0.

The solution for v^0 ($u^0 = 0$ because h is not depending on y) can be found from $v^0 = \partial h/\partial x$, i.e.,

$$x > 0: \ v^0(x) = -\sqrt{F}h_0 e^{-\sqrt{F}x}$$

$$x < 0: \ v^0(x) = -\sqrt{F}h_0 e^{\sqrt{F}x}$$