**Book: Dynamical Oceanography**
**Author: Dijkstra, H. A.; email: H.A.Dijkstra@uu.nl**
**Chapter: 6, Exercise: 6.4: Fofonoff inertial flow**
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*Consider a general pure inertial quasi-geostrophic flow in a homogeneous ocean. This flow satisfies the barotropic vorticity equation with zero wind stress ($\mathbf{T} = \mathbf{0}$) and without frictional effects ($\delta_S = \delta_M = 0$).*

*a. Show that the resulting equation is given by*

$$\mathbf{u} \cdot \nabla \left( \left( \frac{\delta_I}{L} \right)^2 \nabla^2 \psi + y \right) = 0$$

*where $\mathbf{u}$ is the horizontal velocity vector. Give a physical interpretation of this vorticity balance.*

Equation (6.22) can be written as:

$$\left( \frac{\delta_I}{L} \right)^2 \left[ \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \right] \nabla^2 \psi + \frac{\partial \psi}{\partial x} = 0$$

Here $\psi = p$ is the geostrophic streamfunction, $\nabla^2 \psi$ is the relative vorticity and $\partial \psi / \partial x$ is the planetary vorticity advection. It is easily verified that the above can also be written as

$$\left[ \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \right] \left( \left( \frac{\delta_I}{L} \right)^2 \nabla^2 \psi + y \right) = 0$$

Replacing $\frac{\partial \psi}{\partial x} = v$ and $-\frac{\partial \psi}{\partial y} = u$ the answer follows. This means that $(\frac{\delta_I}{L})^2 \nabla^2 \psi + y$, the potential vorticity, is a conserved quantity along streamlines.

*Consider now an inertial current and the situation that the ocean is bounded at $x = 0$ by a coast. The flow is purely zonal for $x \to \infty$ and it has a dimensionless horizontal velocity field given by $u = -1$ and $v = 0$.*

*b. Demonstrate that in this case, the absolute vorticity is a linear function of the streamfunction.*

To show that the absolute vorticity for these types of flows is a function of the streamfunction, it is convenient to use the Jacobian $J$. This quantity is defined as follows:

$$J_{x,y}(f, g) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}$$

where $f$ and $g$ are two arbitrary functions which depend on $x$ and $y$. The Jacobian $J$ has the following properties:

$$J_{x,y}(f, f) = 0$$
$$J_{x,y}(\psi, G(\psi)) = 0$$

where $G$ is an arbitrary operator. With these properties the vorticity equation can be written as

$$J_{x,y}(\psi, (\frac{\delta_I}{L})^2 \nabla^2 \psi + y) = 0$$

Therefore, the absolute vorticity is an arbitrary function but unique function of the streamfunction. The flow is purely zonal for $x \to \infty$ and it has a dimensionless horizontal velocity filed given by $u = -1$ and $v = 0$. The latter means that $\partial \psi / \partial y = 1$ therefore, $\psi = y + c$. For $x \to \infty$ we have $\nabla^2 \psi = 0$ and using $J_{x,y}(\psi, G(\psi)) = 0$ we determine that the absolute vorticity $G(\psi) = y$. Hence

$$y = G(\psi) = \psi - c$$

and the absolute vorticity is a linear function of the streamfunction.

*c. Consider the western boundary layer (with $\delta_I \ll L$) and show that the total solution is given by*

$$\psi_W(x,y) = (y - y_0)(1 - e^{-xL/\delta_I})$$

*where $y_0$ is an arbitrary north-south location for which $\psi = 0$ for all $x$. From now, take $y_0 = 0$.*

Perpendicular to the coast we have that velocity is zero, so $u = 0$ for $x = 0$. To achieve this the flow will produce relative vorticity close to $x = 0$. To let inertial effects join in, the equations need to be rescaled. We introduce the following scale:

$$\lambda = \frac{x}{l}$$

where $l = \frac{\delta_I}{L}$. The equations now become:

$$\frac{\partial^2 \psi_W}{\partial \lambda^2} + \left(\frac{\delta_I}{L}\right)^2 \frac{\partial^2 \psi_W}{\partial y^2} + y = \psi_W - c$$

We write now $\psi_W = \psi_I + \widehat{\psi}$, where $\psi_I = y + c$ is the streamfunction far from the coast and $\widehat{\psi}$ is the boundary layer solution. We also consider that $l^2$ is a small parameter and we can expand $\widehat{\psi}$ in this parameter. We then arrive at the following lowest order equation for the boundary layer:

$$\frac{\partial^2 \widehat{\psi_0}}{\partial \lambda^2} = \widehat{\psi_0}$$

The general solution is:

$$\widehat{\psi_0} = A(y)e^{\lambda} + B(y)e^{-\lambda}$$

Now we apply the boundary conditions and we consider the solution to be limited for $x \to \infty$. The latter brings $A(y) = 0$. Using that $u = 0$ at $x = 0$ we have:

$$-\frac{\partial \psi_W}{\partial y} = 0$$

$$-\frac{\partial}{\partial y}\left[\psi_I + \widehat{\psi_0}\right] = -1 - \frac{\partial B(y)}{\partial y} = 0$$

From this follows that $B(y) = -y + d$. If we further assume that $\psi_W = 0$ at $y = y_0$ the total solution is given by:

$$\psi_W = (y - y_0)\left(1 - e^{-\frac{xL}{\delta_I}}\right)$$

where $y_0$ is an arbitrary north-south location for which $\psi = 0$ for all $x$.

*Consider now the new situation in which there is also a continental boundary at $x = 1$. Between the continents, the flow is still purely zonal.*

*d. Show that the total solution (with western and eastern boundary layers) can be written as*

$$\psi_{WE}(x,y) = y(1 - e^{-xL/\delta_I} - e^{-(1-x)L/\delta_I})$$

The analysis at the eastern boundary $x = 1$ is similar to that at $x = 0$; we introduce a coordinate $\mu = (1 - x)/l$, scale accordingly and match up with the interior solution. With $y_0 = 0$, we then find as total solution

$$\psi_{WE}(x,y) = y(1 - e^{-xL/\delta_I} - e^{-(1-x)L/\delta_I})$$

*Finally, we consider a closed basin with a southern boundary at $y = -1$ and a northern boundary at $y = 1$ (where $\psi = 0$).*

*e. Why are there boundary layers at these northern and southern boundary? Show that the solution is given by*

$$\psi_B(x, y) = \psi_{WE}(x, y) + c_1 e^{-(1-y)L/\delta_I} + c_2 e^{-(y+1)L/\delta_I}$$

*and determine the constants $c_1$ and $c_2$.*

The flow at the western and eastern boundaries is purely meridional and because of mass conservation, it has to turn near the northern and southern boundaries. With $\delta = \delta_I/L$, the governing equation can be written as

$$\delta^2 \nabla^2 \psi + y = \psi$$

Near the northern boundary, where we use the notation $\psi_N$ for the streamfunction, we rescale $\eta = (1 - y)/\delta$ and this equation becomes

$$\delta^2 \left( \frac{\partial^2 \psi_N}{\partial x^2} + \frac{1}{\delta^2} \frac{\partial^2 \psi_N}{\partial \eta^2} \right) + 1 - \delta\eta = \psi_N$$

When we write $\psi_N = \psi_{EW} + \phi_N$ the resulting first order balance is

$$\frac{\partial^2 \phi_N}{\partial \eta^2} = \phi_N$$

having the solution

$$\phi_N(x, y) = A(x)e^{\eta} + c_1 e^{-\eta}$$

Because the solution has to be bounded for $\eta \to \infty$, $A(x) = 0$ and hence we find

$$\phi_N(x, y) = c_1 e^{-(1-y)/\delta}$$

When the same procedure is applied to $y = -1$, we find (just reflection through $y = 0$)

$$\phi_S(x, y) = c_2 e^{-(1+y)/\delta}$$

and finally the total solution is given by

$$\psi_B(x, y) = \psi_{WE}(x, y) + c_1 e^{-(1-y)L/\delta_I} + c_2 e^{-(y+1)L/\delta_I}$$

The constants $c_1$ and $c_2$ are determined from mass conservation. Over a meridional section, the integrated transport

$$\int_{-1}^{1} u_B \; dy = 0 \to \psi_B(x, 1) = \psi_B(x, -1)$$

as $u_B = -\partial\psi_B/\partial y$. At $x = 1$ and $x = -1$ this relation must also hold and because $\psi_B = 0$ there, we finally find $\psi_B(x, 1) = \psi_B(x, -1) = 0$.

Near the northern boundary we have

$$\psi_B(x, 1) = \psi_{WE}(x, 1) + c_1 = 0 \to c_1 = -\psi_{WE}(x, 1) = -1 + e^{-x/\delta} + e^{-(1-x)/\delta}$$

and near the southern boundary

$$\psi_B(x, -1) = \psi_{WE}(x, -1) + c_2 = 0 \to c_2 = -\psi_{WE}(x, -1) = 1 - e^{-x/\delta} - e^{-(1-x)/\delta}$$

*f. Make a sketch (or plot) of the resulting flow under e. This is the Fofonoff inertial flow.*