Book: Dynamical Oceanography
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Chapter: 9, Exercise: 9.3: Basin modes
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Resonance phenomena can occur in an ocean basin that is forced by a time-dependent wind stress through so-called basin modes. In this exercise, we investigate the frequencies and patterns of these modes. Consider the unforced problem (9.34) with $\tau_p = L/U$, i.e.,

$$\frac{\partial}{\partial t}(\nabla^2\Psi - \Lambda\Psi) + \beta\frac{\partial\Psi}{\partial x} = 0$$

with $\Psi = 0$ at the boundaries ($x = 0, 1; y = 0, d$). We look for solutions of the form

$$\Psi(x, y, t) = \hat{\Phi}(x, y)\exp(-i\sigma t)$$

a. Show that the equation for $\Phi(x, y)$ is given by

$$-i\sigma(\nabla^2\hat{\Phi} - \Lambda\hat{\Phi}) + \beta\frac{\partial\hat{\Phi}}{\partial x} = 0$$

This easily follows from substitution of the Ansatz into the governing equation.

With the transformation $\hat{\Phi}(x, y) = \Phi(x, y)\exp(-i\beta x/(2\sigma))$ this equation becomes the eigenvalue problem

$$\nabla^2\Phi + \mu^2\Phi = 0$$

with

$$\mu^2 = \frac{\beta^2}{4\sigma^2} - \Lambda$$

and homogeneous boundary conditions.

b. Show that the eigenvalues σ_{nm} are given by

$$\sigma_{nm}^2 = \frac{\beta^2}{4}\frac{1}{n^2\pi^2 + m^2\pi^2/d^2 + \Lambda}$$

and the eigenfunctions through

$$\psi_{nm}(x, y, t) = A_{nm}\sin(n\pi x)\sin(\frac{m\pi y}{d})e^{-i(\sigma_{nm}t + \frac{x\beta}{2\sigma_{nm}})}$$

where A_{nm} is an arbitrary amplitude.

In order to obtain the equation given above, we substite

$$\hat{\Phi} = \Phi e^{-i\beta x/2\sigma},$$

into the result of a., using

$$\hat{\Phi}_x = \left(\Phi_x - \frac{i\beta}{2\sigma}\Phi\right)e^{-i\beta x/2\sigma}$$

$$\hat{\Phi}_{xx} = \left(\Phi_{xx} - \frac{i\beta}{\sigma}\Phi_x - \frac{\beta^2}{(2\sigma)^2}\Phi\right)e^{-i\beta x/2\sigma},$$

to find

$$-i\sigma\left(\nabla^2\Phi - \frac{i\beta}{\sigma}\Phi_x - \frac{\beta^2}{(2\sigma)^2}\Phi - \Lambda\Phi\right) + \beta\left(\Phi_x - \frac{i\beta}{2\sigma}\Phi\right) = 0.$$

This can indeed be simplified to

$$\nabla^2\Phi + \frac{\beta^2}{4\sigma^2}\Phi - \Lambda\Phi = \nabla^2\Phi + \mu^2\Phi = 0.$$

Applying separation of variables with $\Phi(x, y) = F(x)G(y)$ gives

$$\frac{F_{xx}}{F} + \frac{G_{yy}}{G} = -\mu^2$$

Let $F_{xx}/F = -\alpha^2$, then the solution for $F(x)$ is given by

$$F = A\sin\alpha x + B\cos\alpha x$$

As due to the boundary conditions $\Phi(x, y) = 0$ at $x = 0, 1$ we find

$$F(0) = F(1) = 0 \rightarrow B = 0 \text{ and } \alpha = n\pi; n = 1, 2, ...$$

Similarly, we put $G_{yy}/G = -\gamma^2$ and hence

$$G(y) = C\sin\gamma y + D\cos\gamma y$$

As $\Phi(x, y) = 0$ at $y = 0, d$ we find

$$G(0) = G(d) = 0 \rightarrow D = 0 \text{ and } \gamma = m\pi/d; m = 1, 2, ...$$

Finally, we find

$$-\alpha^2 - \gamma^2 = -\mu^2 \rightarrow \mu^2 = (n^2\pi^2) + (m^2\pi^2/d^2)$$

and hence

$$\sigma_{nm}^2 = \frac{\beta^2}{4}\left(n^2\pi^2 + \frac{m^2\pi^2}{d^2} + \Lambda\right)^{-1}.$$

The eigenfunctions are given by

$$\Phi_{nm} = F(x)G(y) = A_{nm}\sin(\alpha x)\sin(\gamma y) = A_{nm}\sin(n\pi x)\sin(m\pi y/d)$$

$$\Psi_{nm} = A_{nm}\sin(n\pi x)\sin(m\pi y/d)e^{-i(\sigma_{nm}t + \beta x/2\sigma_{nm})}$$

c. Sketch ψ_{11} for $\beta = 10^2, d = 1, \Lambda = 0$ and determine the dimensional frequency σ_{11*}.

For $n = 1$, $m = 1$ we find

$$\sigma_{11*}^2 = \frac{U^2}{L^2}\sigma_{11}^2 = \frac{U^2\beta^2}{4L^2}(\pi^2 + \pi^2)^{-1} = \frac{\beta_0^2 L^2}{8\pi^2} \rightarrow \sigma_{11*} = \frac{\beta_0 L}{2\sqrt{2}\pi}$$

d. Describe briefly what happens when the flow in the basin is forced by a wind stress with a time dependence $f(t) = \cos\sigma_{11}t$.

The forcing can excite the $(1, 1)$ basin mode due to linear resonance.