CriticalEarth EW1: Project AMOC

In this project, you will practice with the solution of nonlinear (stochastic) differential equations using a conceptual model of the AMOC. The consequences of this nonlinear coupling of the temperature, salinity and AMOC can be studied in its simplest form using a two-box model. A sketch of a variant of this model is shown in Fig. 1.

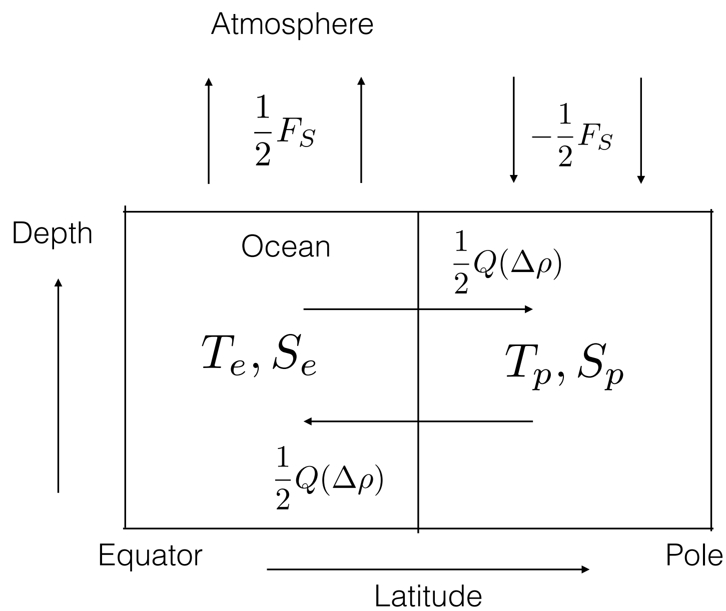

Figure 1: *A Stommel-type two-box model of the Atlantic MOC.*

A polar box (with temperature T_p and salinity S_p) and an equatorial box (with temperature T_e and salinity S_e) having the same volume V_0 are connected by an overturning flow and exchange heat and fresh water with the atmosphere. The heat and salt balances are

$$\frac{dT_e}{dt} = -\frac{1}{t_r}(T_e - (T_0 + \frac{\theta}{2})) - \frac{1}{2}Q(\Delta\rho)(T_e - T_p), \tag{1a}$$

$$\frac{dT_p}{dt} = -\frac{1}{t_r}(T_p - (T_0 - \frac{\theta}{2})) - \frac{1}{2}Q(\Delta\rho)(T_p - T_e), \tag{1b}$$

$$\frac{dS_e}{dt} = \frac{F_S}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_e - S_p), \tag{1c}$$

$$\frac{dS_p}{dt} = -\frac{F_S}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_p - S_e), \tag{1d}$$

where F_S is the fresh water flux, H the ocean depth, t_r is the surface temperature restoring time scale and θ is the equator-to-pole atmospheric temperature difference. The equation of state is linear and given by

$$\rho(T,S) = \rho_0(1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)) \tag{2}$$

where T_0, S_0 and ρ_0 are reference values. The transport function Q is chosen as

$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q_0}{V_0}(\frac{\Delta\rho}{\rho_0})^2, \tag{3}$$

where q_0 is a transport coefficient, t_d a diffusion time and $\Delta\rho = \rho_p - \rho_e$.

Subtracting (1b) from (1a) and (1d) from (1c) and introducing $\Delta T = T_e - T_p$ and $\Delta S = S_e - S_p$ leads to

$$\frac{d\Delta T}{dt} = -\frac{1}{t_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T, \tag{4a}$$

$$\frac{d\Delta S}{dt} = \frac{F_S}{H}S_0 - Q(\Delta\rho)\Delta S. \tag{4b}$$

When non-dimensional quantities x and y are introduced according to $\Delta T = x\ \theta$, $\Delta S = y\ \alpha_T\theta/\alpha_S$ and time is scaled with t_d, the non-dimensional system of equations (4) becomes

$$\frac{dx}{dt} = -\alpha(x-1) - x(1 + \mu(x-y)^2), \tag{5a}$$

$$\frac{dy}{dt} = F - y(1 + \mu(x-y)^2), \tag{5b}$$

where $\alpha = t_d/t_r$ and

$$\mu = \frac{q_0 t_d (\alpha_T\theta)^2}{V_0}, \tag{6}$$

is the ratio of the diffusion time scale t_d and the advective time scale $t_a = V_0/(q_0(\alpha_T\theta)^2)$. Finally, the dimensionless freshwater flux parameter F is given by

$$F = \frac{\alpha_S S_0 t_d}{\alpha_T \theta H} F_S. \tag{7}$$

Typical values of the non-dimensional parameters are $\alpha = 130$, $F = 1.1$ and $\mu = 6.2$. The volume V_0 is for example based on the area of transport near the western boundary in the Atlantic Ocean and q_0 is determined from the strength of the southward branch of the AMOC.

Part A: Mandatory

(i) Argue that, because $\alpha \gg 1$, the equations (5) can be well approximated by

$$\frac{dy}{dt} = F - y(1 + \mu(1-y)^2) \tag{8}$$

(ii) Write a Python program (template provided) to integrate the equations (8) in time, given initial conditions $y(t=0) = y_0$.

(iii) Show numerically that for $F = 1.1$, there exist at least two steady states of the model (8). Give a physical interpretation of these states (in terms of the AMOC, salinity and temperature).

(iv) Compute the bifurcation diagram for (8) with F as control parameter and determine the values of F at the saddle-node bifurcations, for example using the PyDSTool package.

Part B: optional

We next consider the extended nonlinear stochastic Stommel model where the freshwater forcing has a transient and stochastic component given by the equations

$$
\begin{aligned}
dY_t &= (\bar{F} - Y_t(1 + \mu(1 - Y_t)^2))dt + \bar{F} dZ_t && \text{(9a)} \\
dZ_t &= g'(t)dt + \sigma dW_t, && \text{(9b)}
\end{aligned}
$$

where $g(t)$ describes a time dependence of the freshwater forcing, $\bar{F}$ the reference value of F, σ represents the noise amplitude and W_t is a Wiener process.

(v) Write a Python program to integrate the system of SDEs (9) using the Euler-Maruyama (EM) scheme. Determine a solution of the equations for $\bar{F} = 1.1$, $\sigma = 0.1$, $\mu = 6.2$ and $g(t) = 0$. Make a plot of the probability density function for this case.

(vi) Determine the equilibrium solution of the Fokker-Planck equation for this case (so $g(t) = 0$) analytically, and compare the result with that in (v).

(vii) Next, consider the stochastic case with $g(t) = \epsilon t$ and $\epsilon = 0.001$. Study the behavior of the model for increasing noise amplitude (again with $\bar{F} = 1.1$ and $\mu = 6.2$). What type of tipping occurs when σ increases?