EW1 CriticalEarth: Project El Niño-Southern Oscillation (ENSO)

A 'toy' model of the ENSO wave oscillator is the delay equation for the spatially averaged eastern Pacific temperature

$$\frac{dT(t)}{dt} = aT(t) - bT(t-d) - cT^3(t) \tag{1}$$

Here a represents the growth rate of the temperature disturbance T in the eastern Pacific. The quantity d is the delay time due to the propagation of equatorial waves, and b measures its influence with respect to the local feedbacks. The nonlinear term with coefficient c is needed for equilibration of the temperature to finite amplitude.

Part A: Mandatory

(i) Apply a scaling in temperature and time to derive the equation

$$\frac{dT(t)}{dt} = T(t) - \alpha T(t-\delta) - T^3(t) \tag{2}$$

with $\alpha = b/a$ and $\delta = ad$.

(ii) Determine (analytically) the fixed points (steady states) of this model.

(iii) Consider the stability of the fixed points in both regimes $\alpha < 1$ and $\alpha > 1$. Which regime is more realistic for ENSO dynamics (and why)?

(iv) Write a Python program to integrate equation (2). Make an appropriate choice for the initial conditions (over the interval $[0, \delta]$ or $[-\delta, 0]$). Provide a check of the code using the analytical results of (ii) and (iii).

(v) Explore the solutions of (2) for a range of values of δ and identify the different dynamical regimes.

Part B: Optional

Next, we consider the stochastic version given by

$$dT_t = (T_t - \alpha T_{t-\delta} - T_t^3)dt + \sigma dW_t \tag{3}$$

where additive noise is assumed with variance σ.

(vi) Write an Euler-Maruyama scheme to integrate equation (3) for different σ.

(vii) Fix $\alpha = 0.9$. Determine the probability density function of the temperature variability, both in the (deterministically) subcritical (e.g. $\delta = 1$) and supercritical case (e.g. $\delta = 2$) for different values of σ. Does a stochastic Hopf bifurcation occur at the transition from steady to oscillatory behavior?

(viii) What is the underlying dynamical mechanism (in a stochastic Hopf bifurcation) explaining how noise excites an oscillatory mode under (deterministic) subcritical conditions?