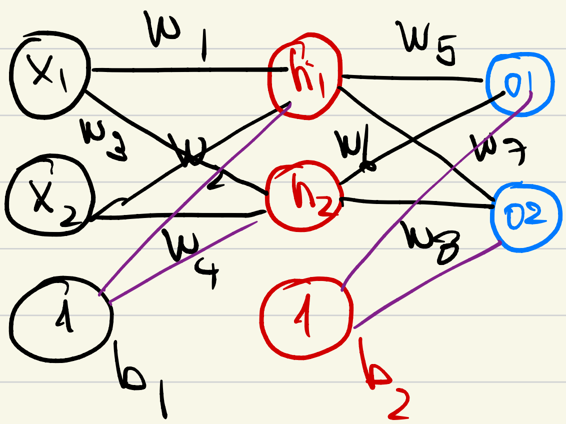
Example of backpropagation



data

y_1

y_2

Hidden:

$$z_1 = w_1 x_1 + w_2 x_2 + b_1$$

$$h_1 = \phi(z_1) = \frac{1}{1+e^{-z_1}}$$

$$z_2 = w_3 x_1 + w_4 x_2 + b_1$$

$$h_2 = \phi(z_2) = \frac{1}{1+e^{-z_2}}$$

Output:

$$z_3 = w_5 h_1 + w_6 h_2 + b_2$$

$$o_1 = \phi(z_3)$$

$$z_4 = w_7 h_1 + w_8 h_2 + b_2$$

$$o_2 = \phi(z_4)$$

$$E(\underline{w}) = \frac{1}{2}\left\{ \left(y_1 - o_1 \right)^2 + \left(y_2 - o_2 \right)^2 \right\}$$

$$\underline{w} = \left(w_1, \ldots, w_8, b_1, b_2 \right)^T$$

1) Consider first the weights of links connecting the hidden and the output layer, e.g. w_5.

$$\frac{\partial E}{\partial w_5} = \frac{\partial E}{\partial o_1} \frac{\partial o_1}{\partial z_3} \frac{\partial z_3}{\partial w_5}$$

$$\frac{\partial E}{\partial o_1} = -\left(y_1 - o_1 \right)$$

$$\frac{\partial o_1}{\partial z_3} = \phi'(z_3) = \phi(z_3)\left(1 - \phi(z_3)\right)$$

$$= o_1 \left(1 - o_1 \right)$$

$$\frac{\partial z_3}{\partial w_5} = h_1$$

hence

$$\frac{\partial J}{\partial w_5} = -(y_1 - o_1)\, o_1 (1 - o_1)\, h_1$$

$$\equiv \delta_{o_1} h_1$$

Hence, $\hat{w}_5 = w_5 - \eta \frac{\partial J}{\partial w_5}$ update

The same can be done for w_6, w_7, w_8.

2) Now consider the update for w_1:

$$\frac{\partial J}{\partial w_1} = \frac{\partial J}{\partial h_1} \frac{\partial h_1}{\partial z_1} \frac{\partial z_1}{\partial w_1}$$

$$\frac{\partial J}{\partial h_1} = \frac{\partial J}{\partial o_1} \cdot \frac{\partial o_1}{\partial h_1} + \frac{\partial J}{\partial o_2} \frac{\partial o_2}{\partial h_1}$$

$$\frac{\partial h_1}{\partial z_1} = h_1 (1 - h_1)$$

$$\frac{\partial z_1}{\partial w_1} = x_1$$

and $$\frac{\partial O_1}{\partial h_1} = O_1(1 - O_1)\, w_5$$

$$\frac{\partial O_2}{\partial h_1} = O_2(1 - O_2)\, w_7$$

hence we can write:

$$\frac{\partial E}{\partial w_1} = \partial_{h_1} \cdot x_1 \quad , \text{ where}$$

$$\partial_{h_1} = -\Big\{ (y_1 - O_1)\, O_1(1 - O_1)\, w_5 + (y_2 - O_2)\, O_2(1 - O_2)\, w_7 \Big\} * h_1(1 - h_1)$$

and this can also be done for the other weights w_2, w_3 and w_4.