Notes

27/10/2020

$$\frac{dx}{dt} = -\lambda - x^2$$

fp: $\bar{x}^2 = -\lambda$

$\bar{x} = \pm\sqrt{-\lambda}\ , \quad \lambda < 0$

$$\lambda = \lambda_0 + \varepsilon t$$

$\tau = \varepsilon t \qquad y = \lambda$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{dx}{d\tau}\,\varepsilon \\[2ex] \dfrac{dy}{dt} = \varepsilon \dfrac{dy}{d\tau} = \varepsilon \end{cases}$$

$$\rightarrow \quad \begin{cases} \varepsilon \dfrac{dx}{d\tau} = \underbrace{-y - x^2} \\[2ex] \dfrac{dy}{d\tau} = 1 \end{cases}$$

$C: \quad -y - x^2 = 0$

Fast - slow systems

$$\begin{cases} dX_t = f(X_t, Y_t)\,dt + \sigma\, dW_t \\ dY_t = \varepsilon\, dt \end{cases}$$

$\varepsilon = 0$: FP eq

$$\frac{\partial p^y}{\partial t} = -\frac{\partial}{\partial x}\left(f\, p^y\right) + \frac{\sigma^2}{2}\frac{\partial^2 p^y}{\partial x^2}$$

eq pdf

$$\left(p^y\right)' - \frac{2}{\sigma^2} f\, p^y = 0$$

$$\frac{dp^y}{p^y} = \frac{2}{\sigma^2} f\, dx$$

$$\ln p^y = \frac{2}{\sigma^2}\int f\, dx$$

$$p^y = \frac{1}{N}\, e^{\frac{2}{\sigma^2}\int f\, dx}$$

Example : $f(x,y) = -y - x^2$

$\bar{x} = \sqrt{-y}$ stable

$J = f_x \big|_{\bar{x}} = -2\bar{x}$

$\bar{x} = -\sqrt{-y}$ unstable

$$p^y = \frac{1}{N} \exp\left\{ \frac{2}{\sigma^2} \int_{-\sqrt{-y}}^{x} f(s,y)\, ds \right\}$$

$$\int_{-\sqrt{-y}}^{x} \left(-y - s^2 \right) ds$$

$$= -yx - \frac{1}{3} x^2 + \frac{2}{3} \left(-y \right)^{3/2}$$

autonomous

1) $$dX_t = f(X_t)\,dt + \sigma\,dW_t$$

FP

$$\frac{\partial p}{\partial t} = \bar{L}(p)$$

$$\bar{L}(p) = \underbrace{-\frac{\partial}{\partial x}(f p)} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$$

Eq. $\bar{p}$

$$\bar{L}(\bar{p}) = 0$$

2) $$dX_t = (f + f_e)\,dt + \sigma\,dW_t$$

$$\frac{\partial p}{\partial t} = L(p)$$

$$L_e(\bar{p}) = -\frac{\partial}{\partial x}(f_e \bar{p})$$

$$L = \bar{L} + L_e$$

$$p = \bar{p} + p_e$$

$$\frac{\partial p}{\partial t} = L(p) \qquad p = \bar{p} + p_e$$

$$\rightarrow \quad \frac{\partial p_e}{\partial t} = \left(\bar{L} + L_e\right)\left(\bar{p} + p_e\right)$$

$$= \cancel{\bar{L}(\bar{p})} + L_e(\bar{p}) + \bar{L}(p_e) + \cancel{L_e(p_e)}$$

$$|p_e| < |\bar{p}|$$

$$\underbrace{\frac{\partial p_e}{\partial t} = \bar{L}(p_e)}_{} + \underbrace{L_e(\bar{p})}_{}$$

$$\underbrace{p_e(x,t) = \int_{-\infty}^{t} \underbrace{e^{\bar{L}(t-s)}}_{} L_e(\bar{p}(s))\, ds}_{}$$

$$E[T] = \int p\, T \, dx$$

$$E[\bar{T}] = \int \bar{p}\, T \, dx$$

$$E[T - \bar{T}] = \int (p - \bar{p})\, T \, dx$$

$$= \int_{-\infty}^{\infty} p_e\, T \, dx$$

Assumption: $p_e = G(x)\, F(t)$

$$\underline{E[\Delta T]} = -\int_{-\infty}^{\infty} T \int_{-\infty}^{t} e^{\bar{L}(t-s)} \frac{\partial}{\partial x} (G F \bar{p})\, ds$$

$$= \int_{-\infty}^{t} \underbrace{G_A (t - s)}_{\text{kernel}} F(s)\, ds$$