Notes

29/9/2020

slide 14

$$\rho c_p \int_{-h}^{0} \frac{\partial T}{\partial t}\, dz = k \frac{\partial T}{\partial z}\Big|_{-h}^{0}$$

$$\bar{T} = \frac{1}{h}\int_{-h}^{0} T\, dz$$

$$\rightarrow \quad \rho c_p h \frac{d\bar{T}}{dt} = Q_{oa} = \alpha\left(T_a - \bar{T}\right)$$

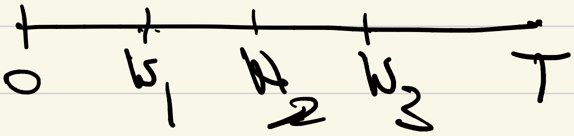
$$\alpha = 50 \ \frac{W}{m^2 K}$$

$$\rightarrow \quad \frac{d\bar{T}}{dt} = \frac{\alpha}{\rho c_p h}\left(T_a - \bar{T}\right)$$

$$\underbrace{\frac{\alpha}{\rho c_p h}}_{= \gamma} \qquad \gamma \sim \frac{1}{100 \text{ days}}$$

$$\rightarrow \quad \frac{d\tilde{T}}{dt} = \gamma\left(\tilde{T}_a - \tilde{T}\right)$$

f smooth



$$f(W_t + dW_t) - f(W_t) =$$

$$f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,(dW_t)^2 + \dots$$

rhs : $\displaystyle \int_0^T f'(W_t)\,dW_t + \frac{1}{2}\int_0^T f''(W_t)\,dt$

$$\|$$

lhs : $f(W_t) - f(W_0)$

$$=$$

$f(t) = t^2 \qquad f' = 2t \qquad f'' = 2$

$\rightarrow \displaystyle \int_0^T 2W_t\,dW_t + \underbrace{\frac{1}{2}\int_0^T 2\,dt}_{=T}$

$= W_T^2 - \underbrace{W_0^2}_{=0}$

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$$f(t,x) = e^{\gamma t} x$$

$$f_1 = \gamma e^{\gamma t} x$$

$$f_2 = e^{\gamma t} \qquad f_{22} = 0$$

$$\rightarrow \quad e^{\gamma t} X_t - X_0 = \int_0^t \Big[\overset{f_1}{\cancel{\gamma e^{\gamma s} X_s}} +$$

$$+ \; e^{\gamma s} \cancel{\left(-\gamma X_s \right)} \Big] ds \; +$$

$$+ \int_0^t e^{\gamma s} \sigma \, dW_s$$

$$X_t = e^{-\gamma t} \left(X_0 + \sigma \int_0^t e^{\gamma s} dW_s \right)$$

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$\boxed{4}$

$$0 = (\gamma x p)' + \frac{\sigma^2}{2} p''$$

$$\frac{\sigma^2}{2} p' + \gamma x p = c_1 \overset{b.c.}{=} 0$$

$$p' = -\frac{2\gamma x}{\sigma^2} p$$

$$p_e(x) = C e^{-\frac{\gamma x^2}{\sigma^2}}$$

slide 29

$$E[X_t X_{t+s}] =$$

$$E\Big[\Big(X_0 e^{-\gamma t} + \sigma e^{-\gamma t}\int_0^t e^{\gamma t'}\, dW_{t'}\Big) *$$

$$* \Big(X_0 e^{-\gamma(t+s)} + \sigma e^{-\gamma(t+s)}\int_0^{t+s} e^{\gamma t''}\, dW_{t''}\Big)\Big]$$

$$E[dW_t] = 0$$

$$= X_0^2 e^{-\gamma(2t+s)} + \underbrace{\sigma^2 e^{-\gamma(2t+s)}} *$$

$$I\Big\{ E\Big[\int_0^t dW_{t'} \int_0^{t+s} e^{\gamma(t''+t')}\, dW_{t''}\Big]\Big\}$$

$\xi(t')\xi(t'')$

$$dW_{t'} = \xi(t')\, dt'$$

$$dW_{t''} = \xi(t'')\, dt''$$

$$E[\xi(t')\xi(t'')] = \delta(t'-t'')$$

$$I = \int_0^t e^{2\gamma t'}\, dt' = \underbrace{\frac{1}{2\gamma}\left(e^{2\gamma t} - 1\right)}$$

$$E[X_t X_{t+s}] = \qquad s>0 \qquad \boxed{6}$$

$$X_0^2 e^{-\gamma(2t+s)} + $$

$$+ \sigma^2 e^{-\gamma(2t+s)} \frac{1}{2\gamma}\left(e^{2\gamma t} - 1\right)$$

$t \to \infty$

$$E[X_t X_{t+s}] \longrightarrow \frac{\sigma^2}{2\gamma} e^{-\gamma s}$$

Spectrum :

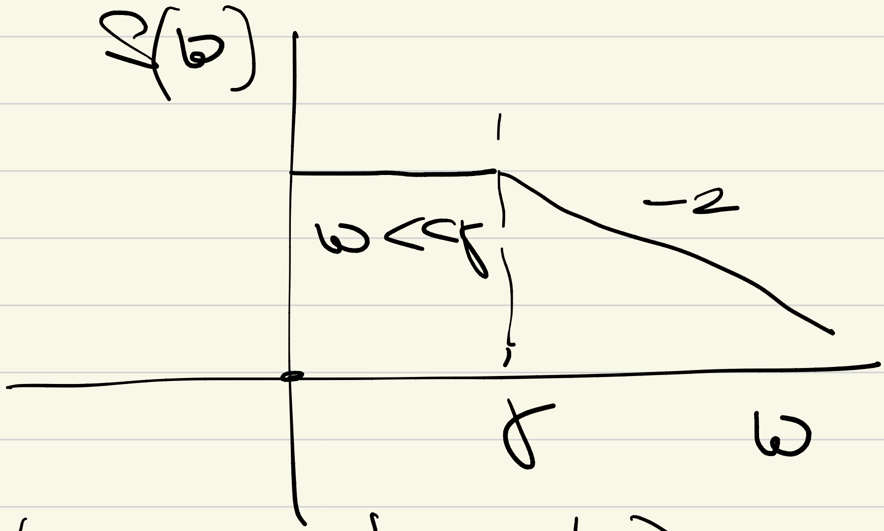
$$S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}\left[e^{-\gamma s}\right]$$

$$= \frac{\sigma^2}{\cancel{2\gamma}} \frac{\cancel{2\gamma}}{\omega^2+\gamma^2} = \frac{\sigma^2}{\omega^2+\gamma^2}$$

$$S(\omega) = \frac{\sigma^2}{\omega^2 + \gamma^2}$$

$\omega \ll \gamma$

$$S(\omega) \simeq \frac{\sigma^2}{\gamma^2}$$



Lorentzian

$\omega \gg \gamma$

$$S(\omega) \simeq \sigma^2 / \omega^2$$

$$X_{t_{n+1}} = \underline{e^{-\gamma t_{n+1}}} X_0 + \sigma e^{-\gamma t_{n+1}} \int_0^{t_{n+1}} e^{\gamma s}\, dW_s$$

$$* \left[X_{t_n} = e^{-\gamma t_n} X_0 + \sigma e^{-\gamma t_n} \int_0^{t_n} e^{\gamma s}\, dW_s \right]$$

$$* \; e^{-\gamma (\underbrace{t_{n+1} - t_n}_{\Delta t})}$$

$$X_{t_{n+1}} = X_{n+1}$$

$$\rightarrow \; X_{n+1} - e^{-\gamma \Delta t} X_n =$$

$$\sigma \underline{e^{-\gamma t_{n+1}}} \int_{t_n}^{t_{n+1}} e^{\gamma s}\, dW_s$$

$$\| \; \underbrace{\sigma\, dW_{n+1}}$$

$$\rightarrow \; \boxed{X_{n+1} = \alpha X_n + Z_{n+1}}$$

red noise

AR(1)

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t$$

$b(t)$: additive noise

$b(X_t)$: state-dep noise

multiplicative