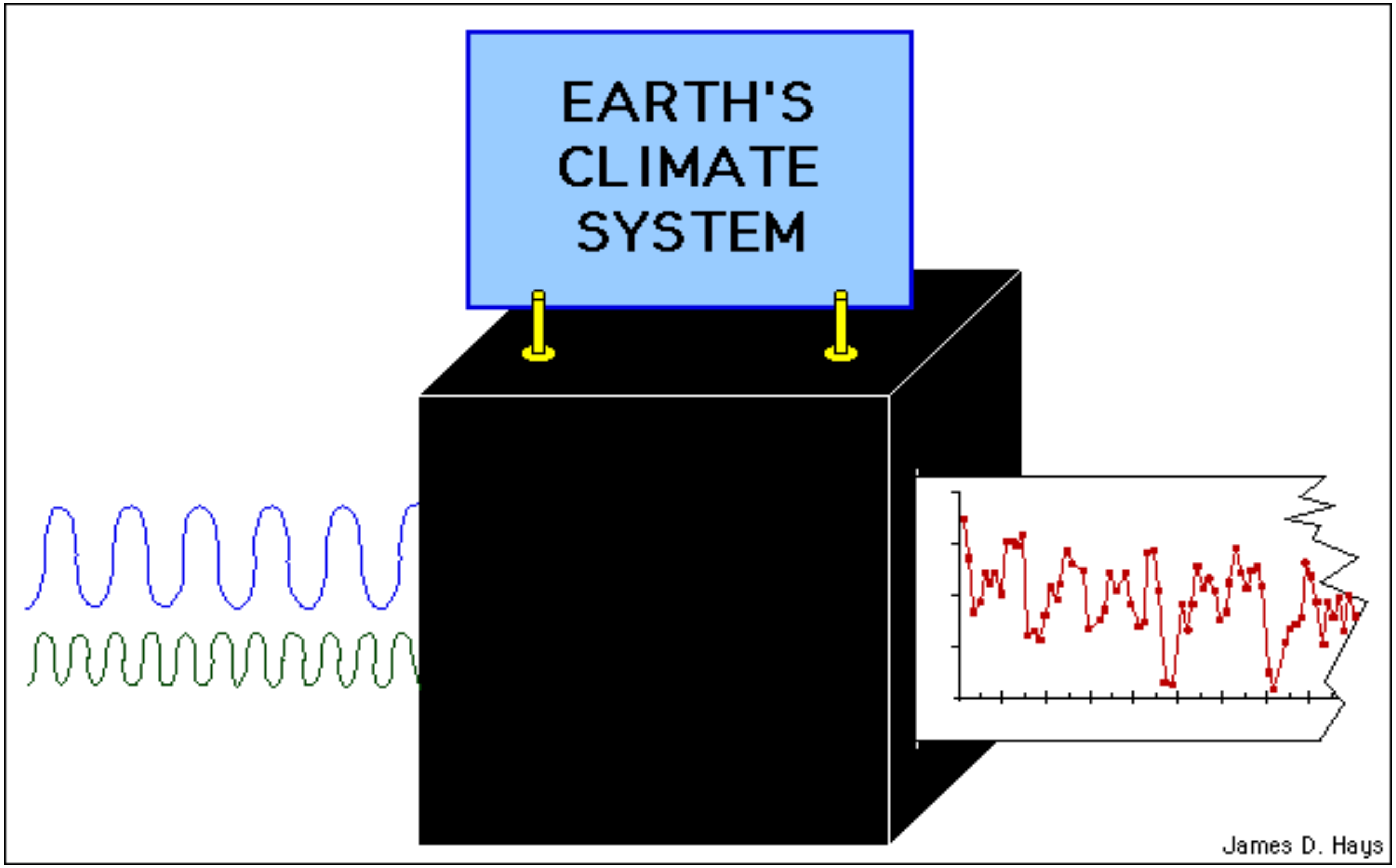
Stochastic Climate Dynamics



Henk Dijkstra, IMAU & CCSS
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https://webspace.science.uu.nl/~dijks101/styled-6/

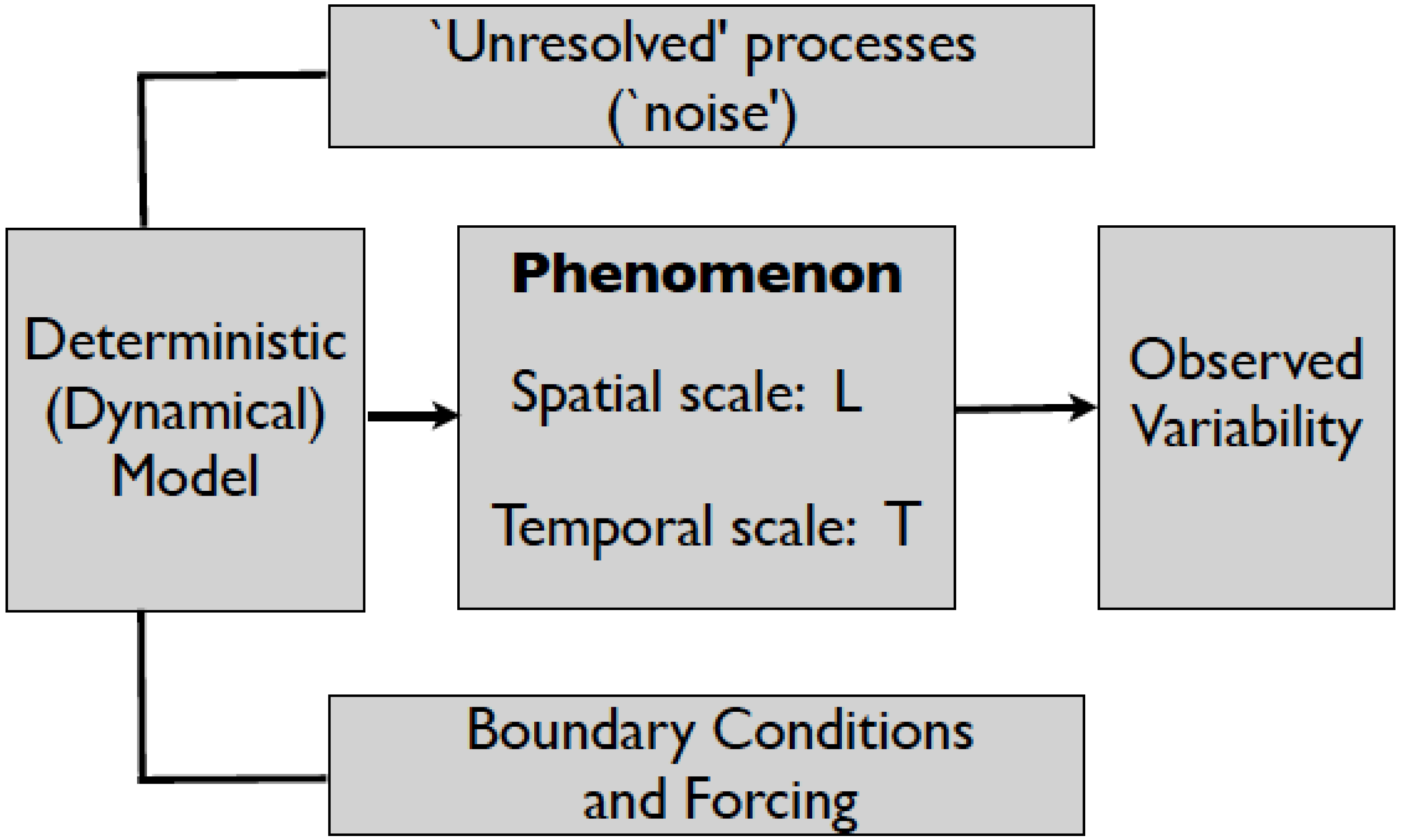
Stochastic dynamical systems approach

Hierarchy of models

Behavior (statistics) over the full parameter space

Geometrical view of motion

Dynamical system



Summary day 1 + 2 + 3

Stochastic linear dynamical systems

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Hasselmann, null-hypothesis (SST)

Deterministic nonlinear systems

$$\frac{dx}{dt} = \lambda - x^2$$

bifurcations, transitions, chaos

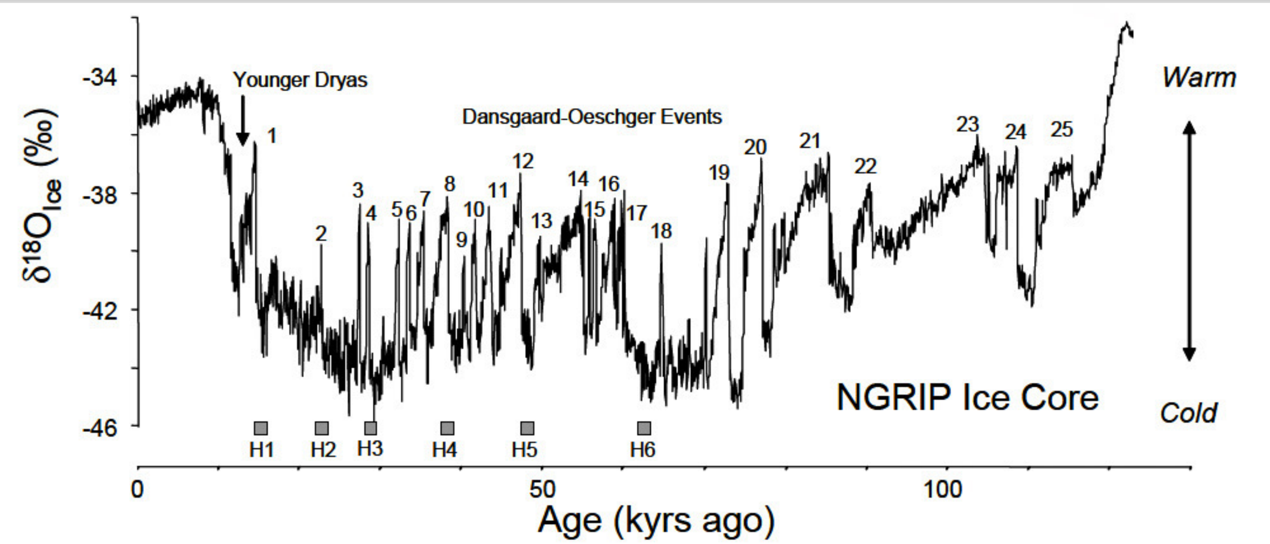
Interaction noise and multiple equilibria

$$\frac{dy}{dt} = F - y(1 + \mu^2(1-y)^2)$$

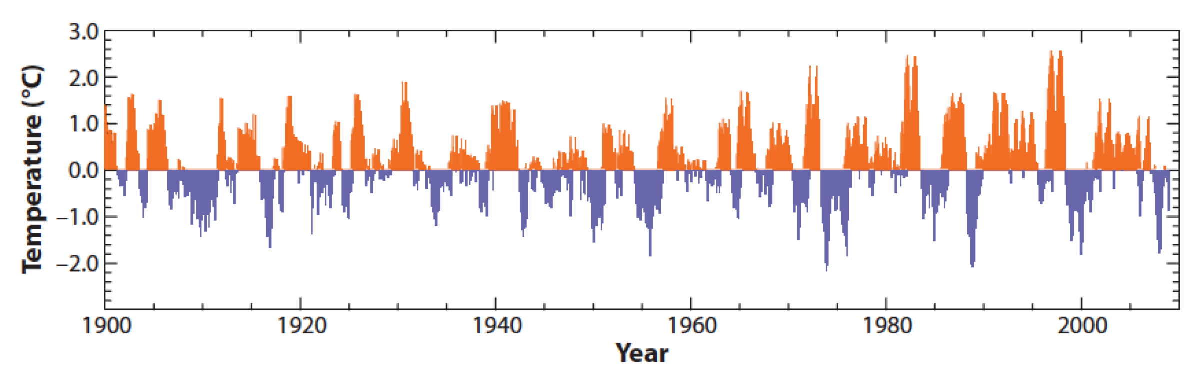
$$F(t) = \bar{F} + \sigma\xi(t)$$

Interaction noise and internal oscillations

$$\begin{aligned} dX =& (\lambda X - \omega Y - X(X^2+Y^2))dt + \sigma dW_1 \\ dY =& (\lambda Y + \omega X - Y(X^2+Y^2))dt + \sigma dW_2 \end{aligned}$$

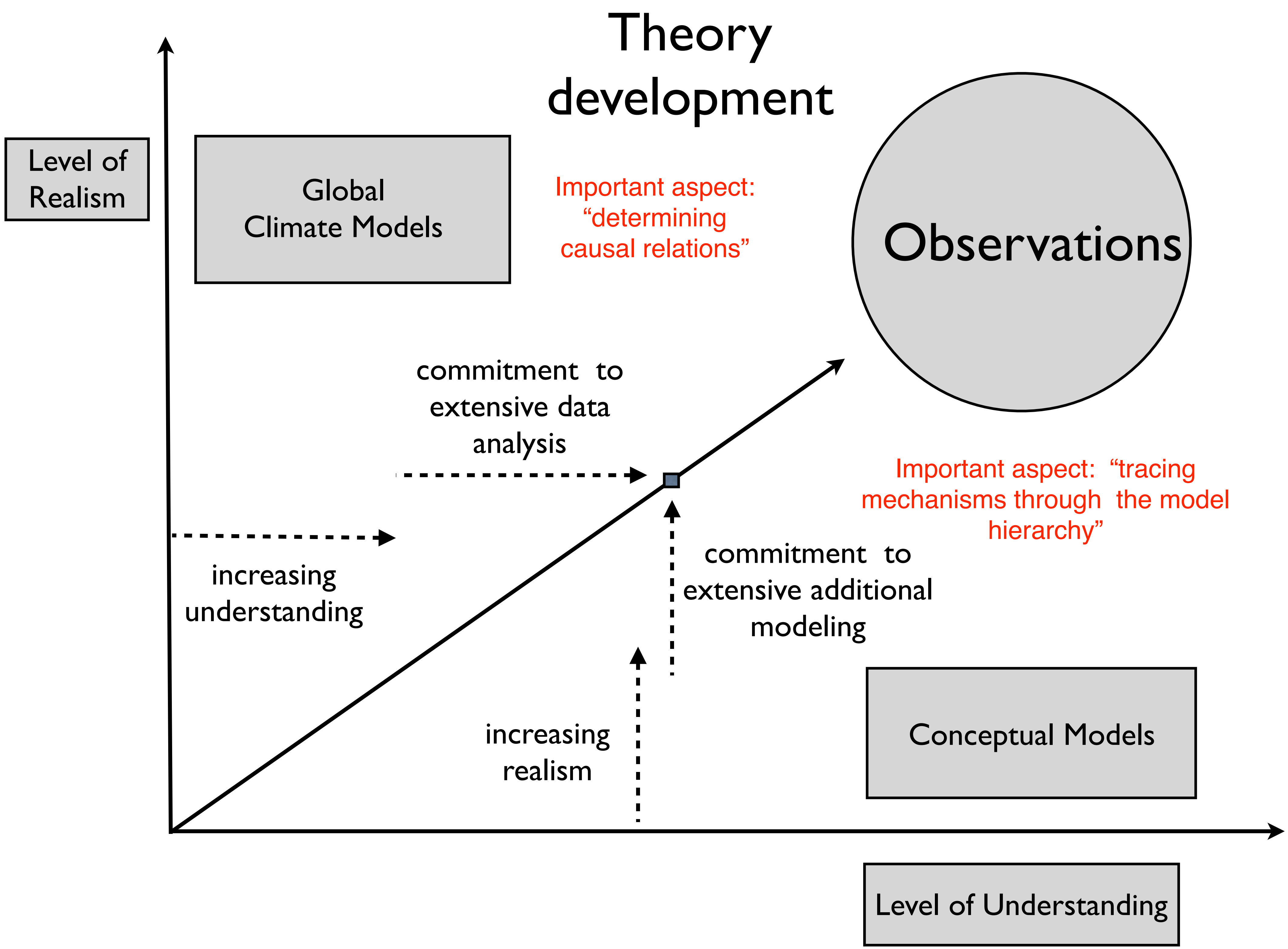


Dansgaard-Oeschger events



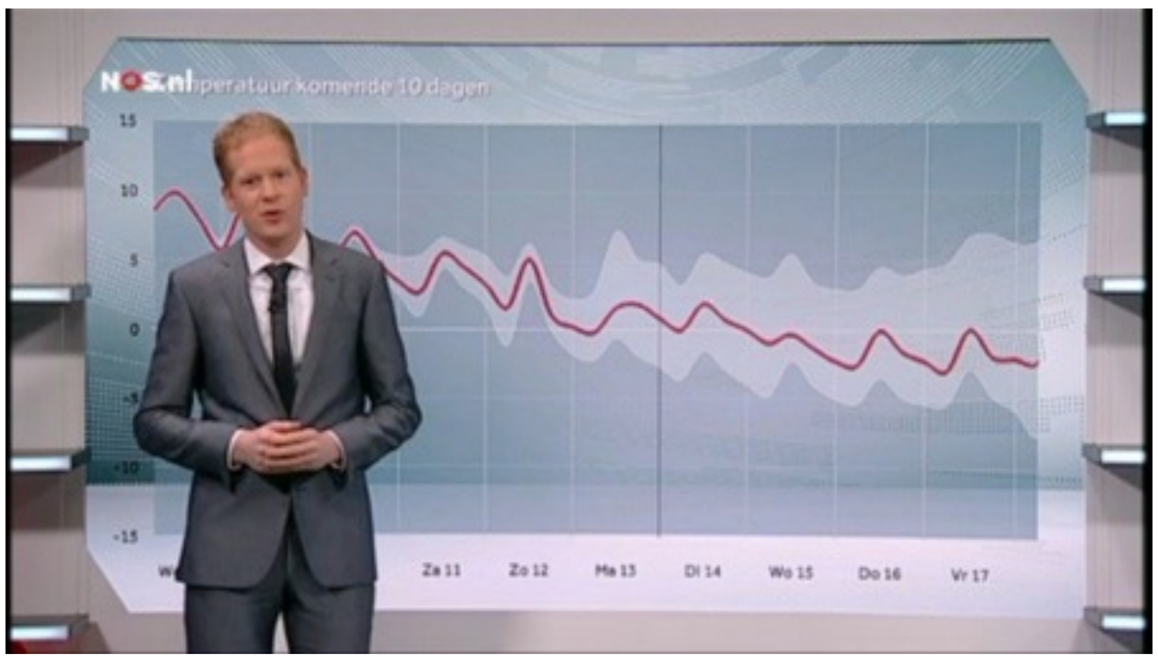
ENSO variability

Towards understanding …



Stochastic DS Approach & Predictability

Weather



ENSO

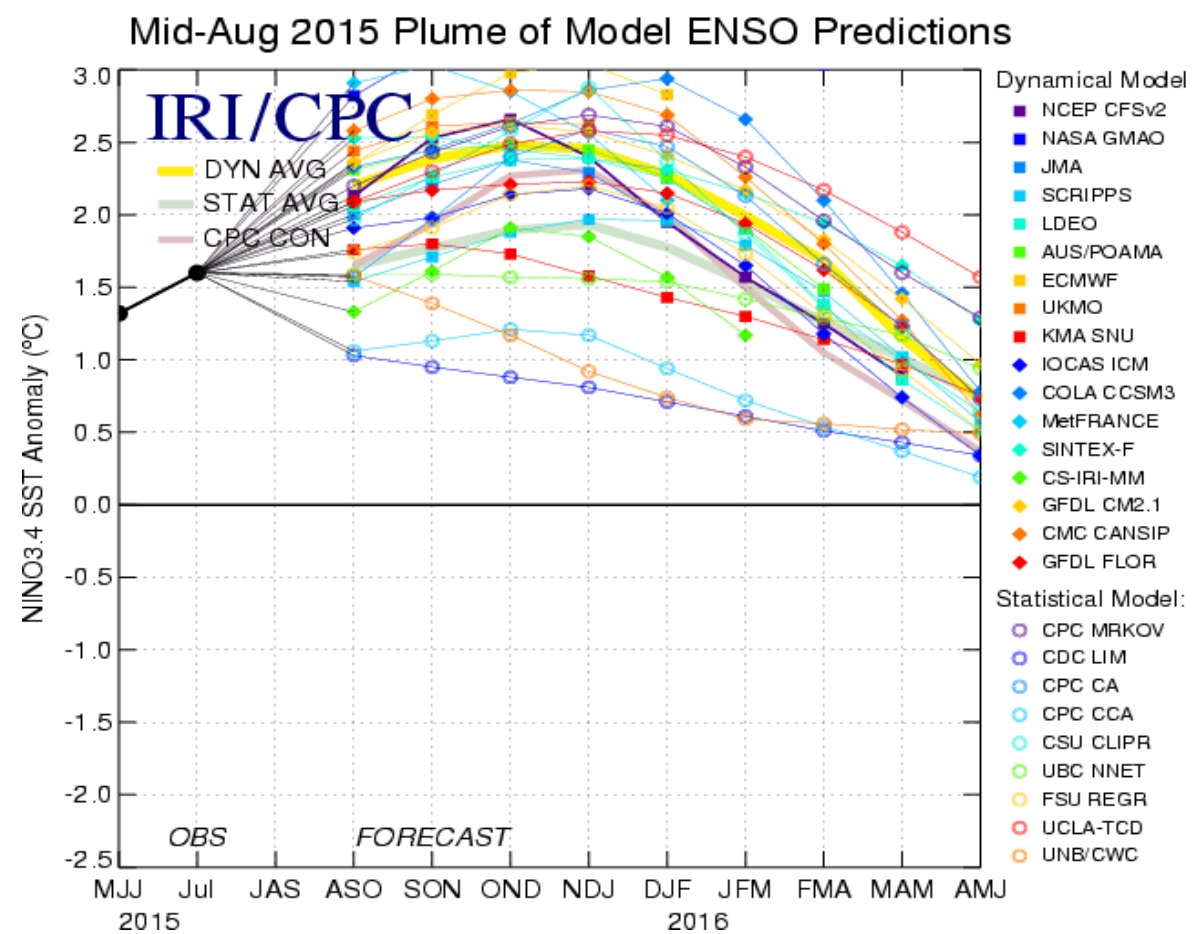
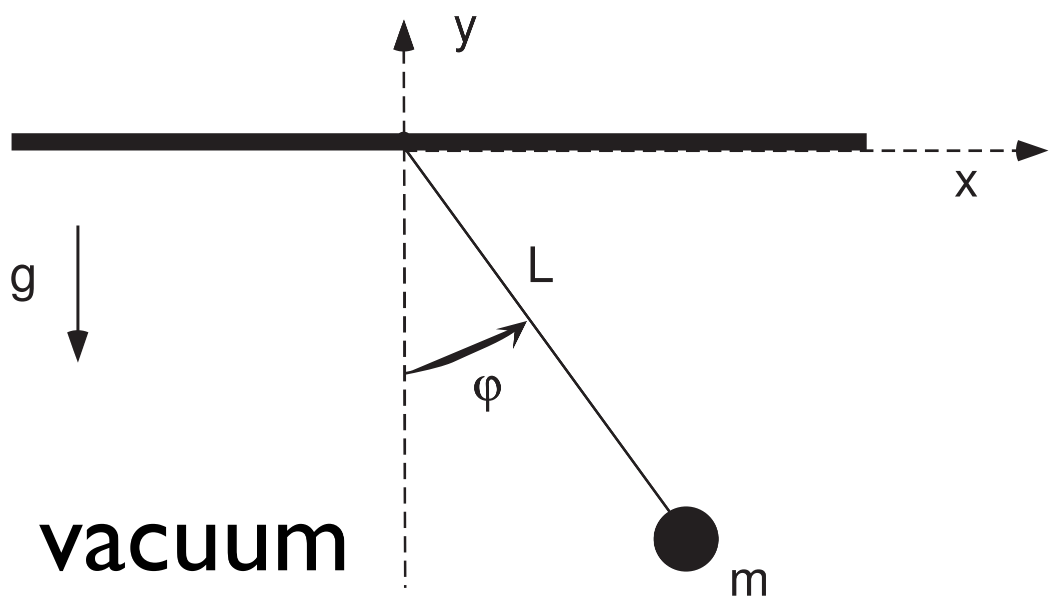


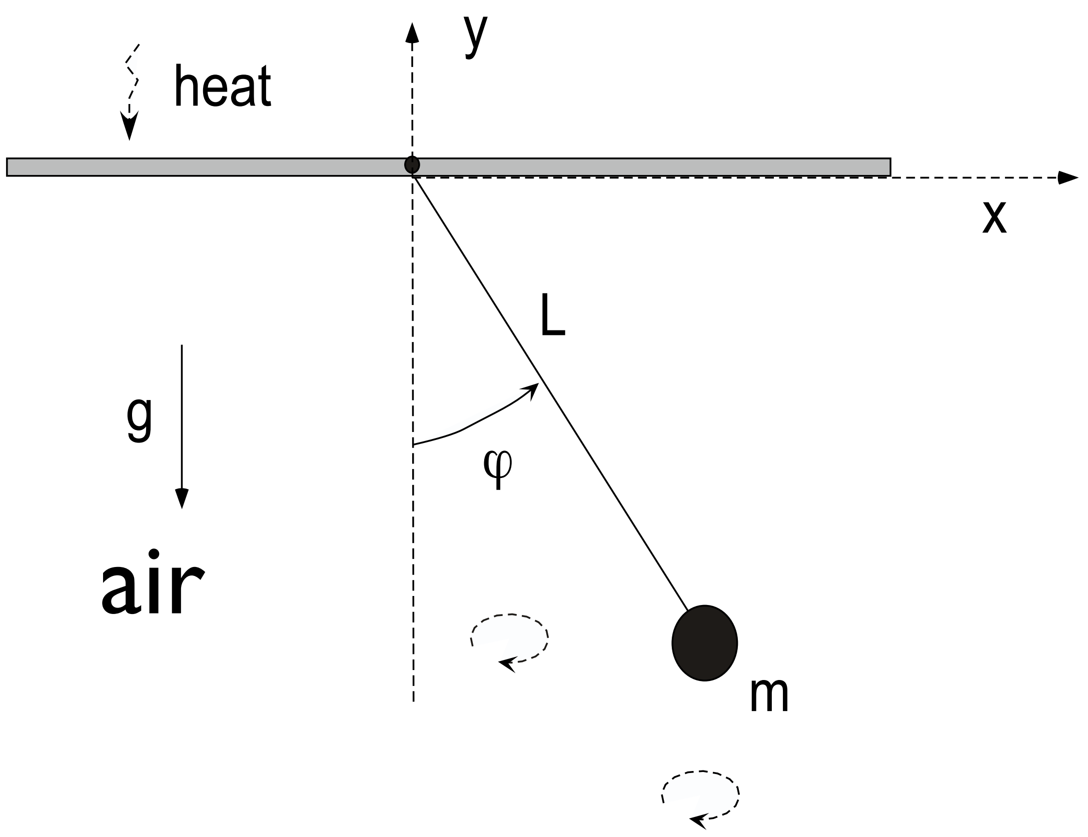
Figure provided by the International Research Institute (IRI) for Climate and Society (updated 18 August 2015).

What determines the skill of these forecasts?

Question time

Prediction problem: pendulum



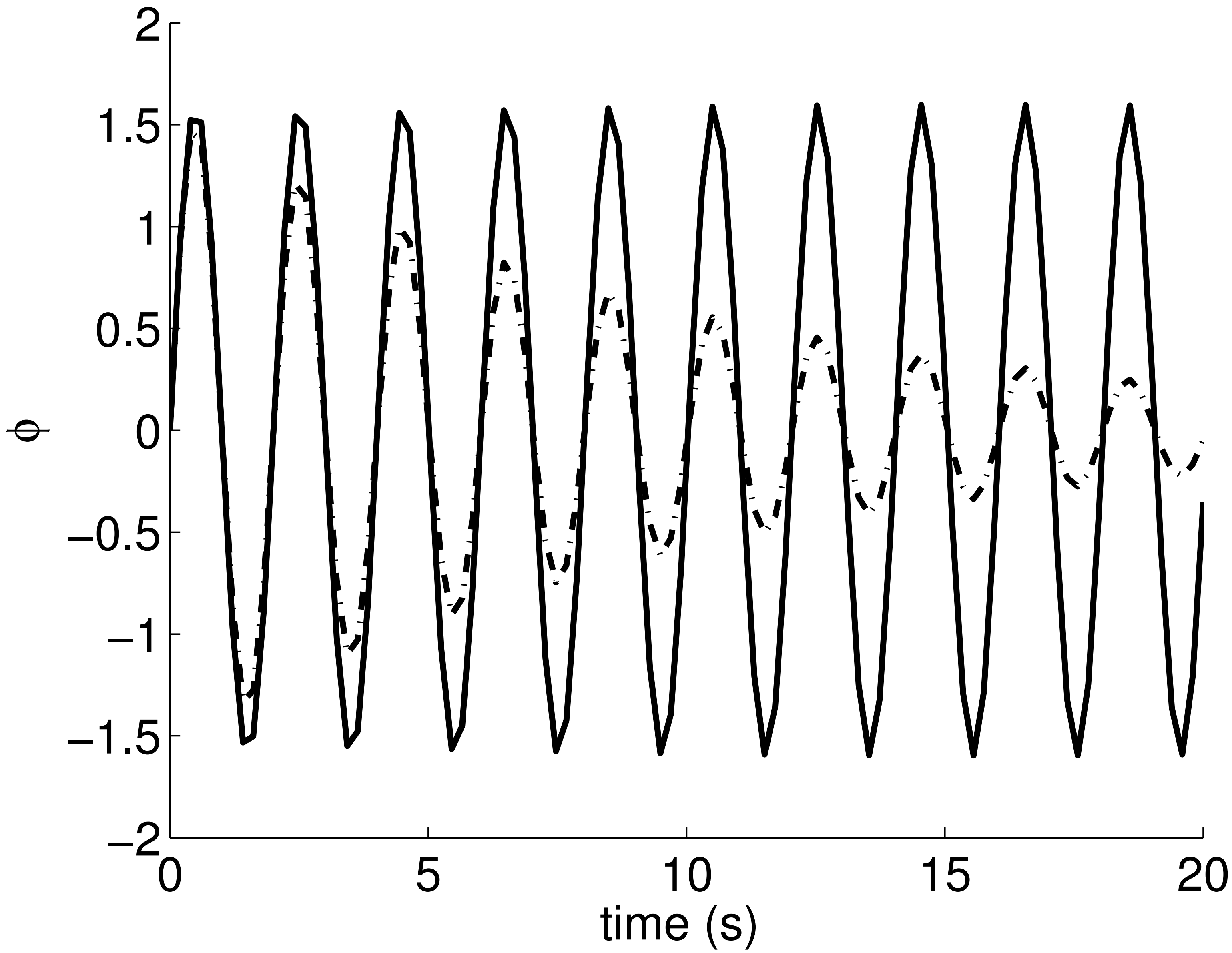


$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0,$$

$$\phi(0)\ ;\ \frac{d\phi}{dt}(0)$$

$$\frac{d^2\phi}{dt^2} + \gamma\frac{d\phi}{dt} + \frac{g}{L}\phi = 0.$$

Results: pendulum

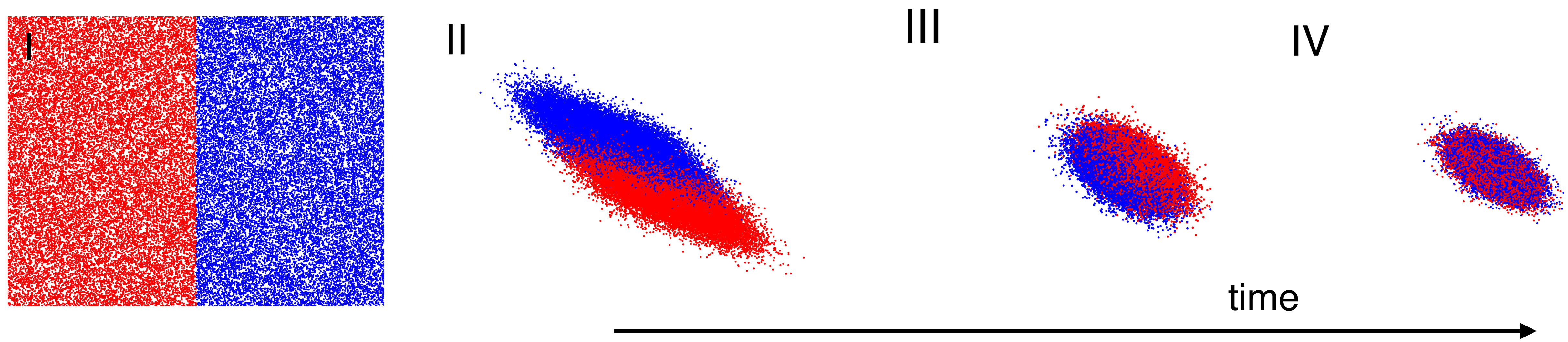


Predictability study of the first kind: effect of initial condition uncertainty

Stochastic linear systems: mixing

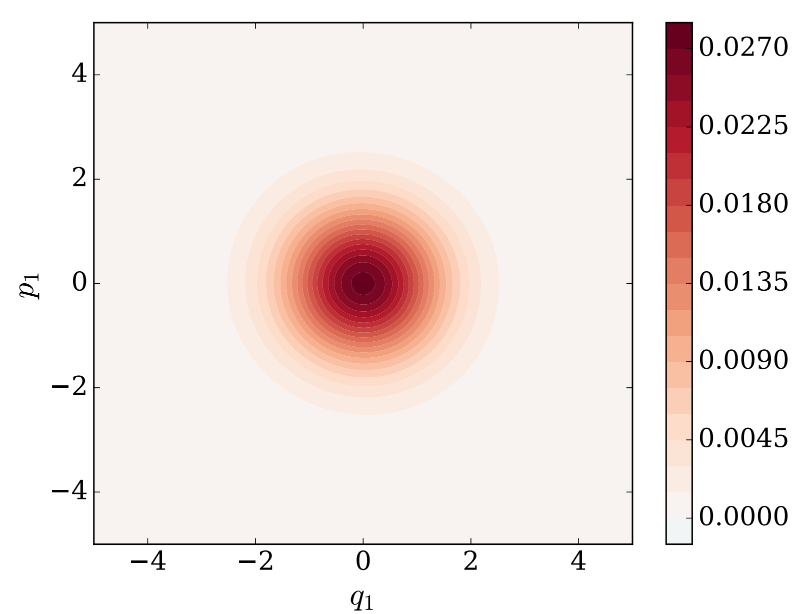
$$dq_1 = \frac{p_1}{m_1} dt$$

$$dp_1 = -(k_1 q_1 + \gamma p_1) dt + \sqrt{2\frac{\gamma}{\beta}} dW_1$$



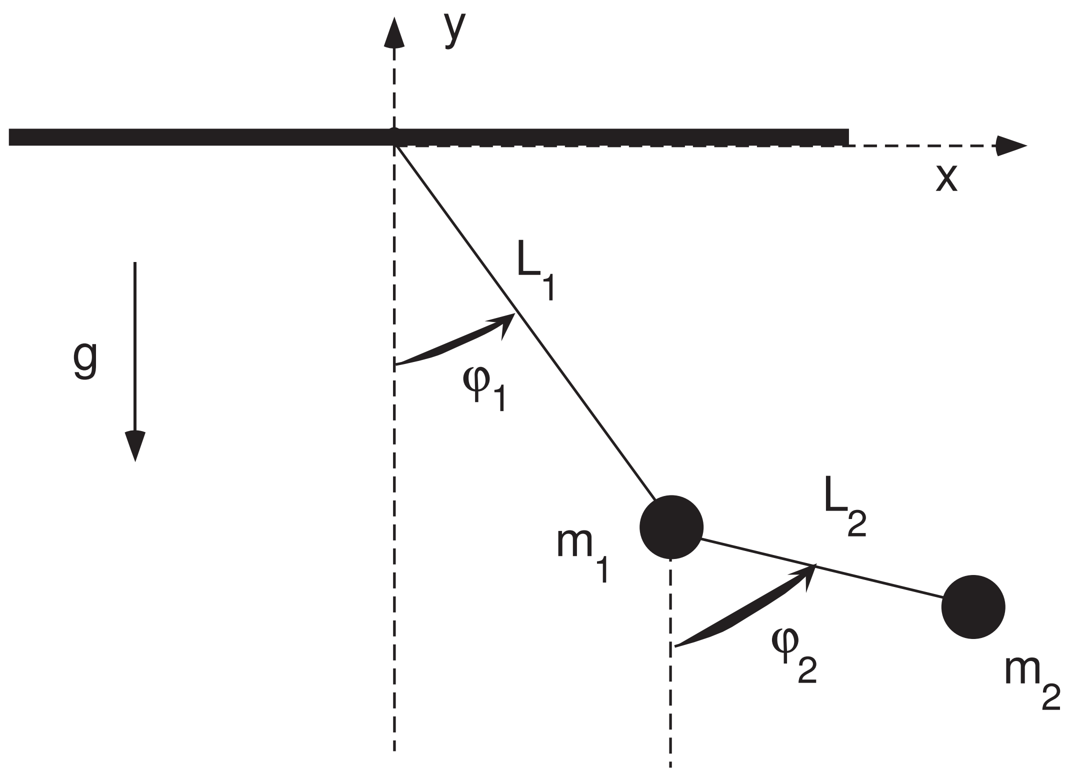
Equilibrium density

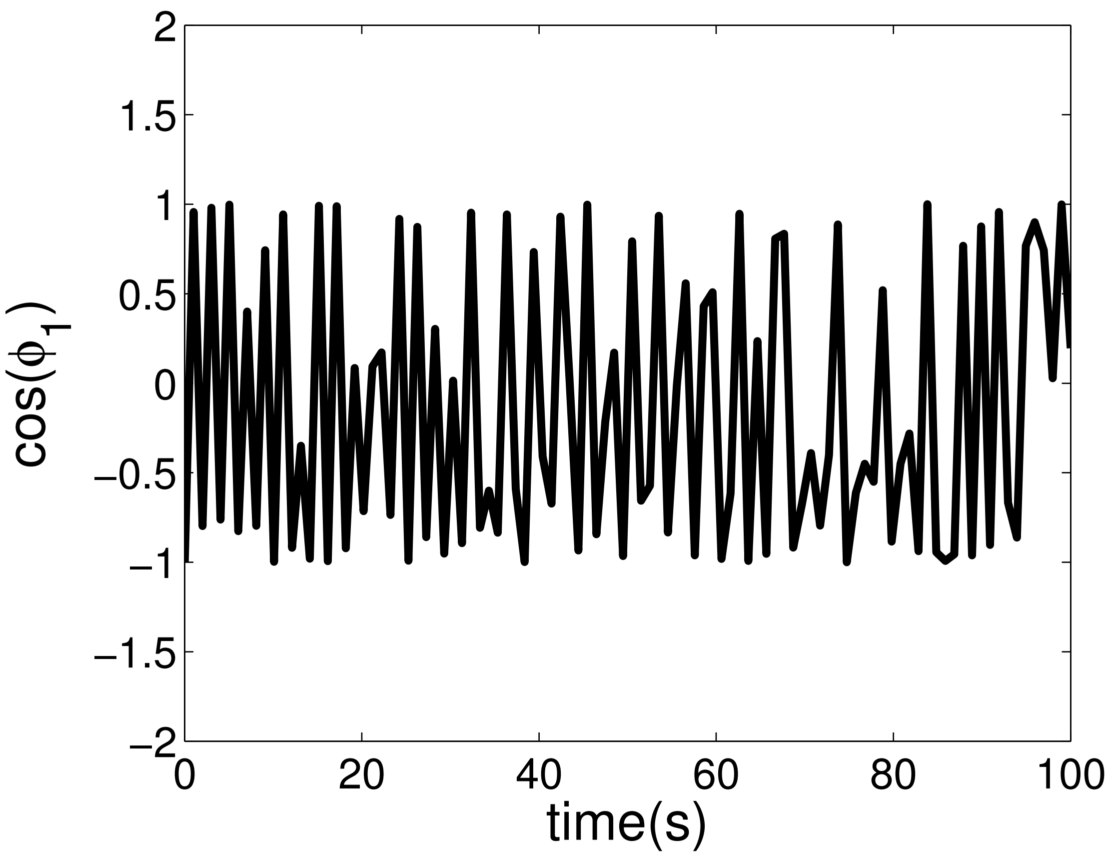
$$\lim_{t\to\infty} \rho(x,t) = \rho_\infty(x)$$



Deterministic nonlinear systems: mixing

Double Pendulum





Formal solution

Trajectories $x(t), t \geq 0$, governed by SDE:

$$dx = F(x,t)dt + G(x,t)dW$$

Conservation of Probabilities:

$$\frac{\partial \rho}{\partial t} = K\rho = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F_i \rho) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}(D_{ij}\rho)$$

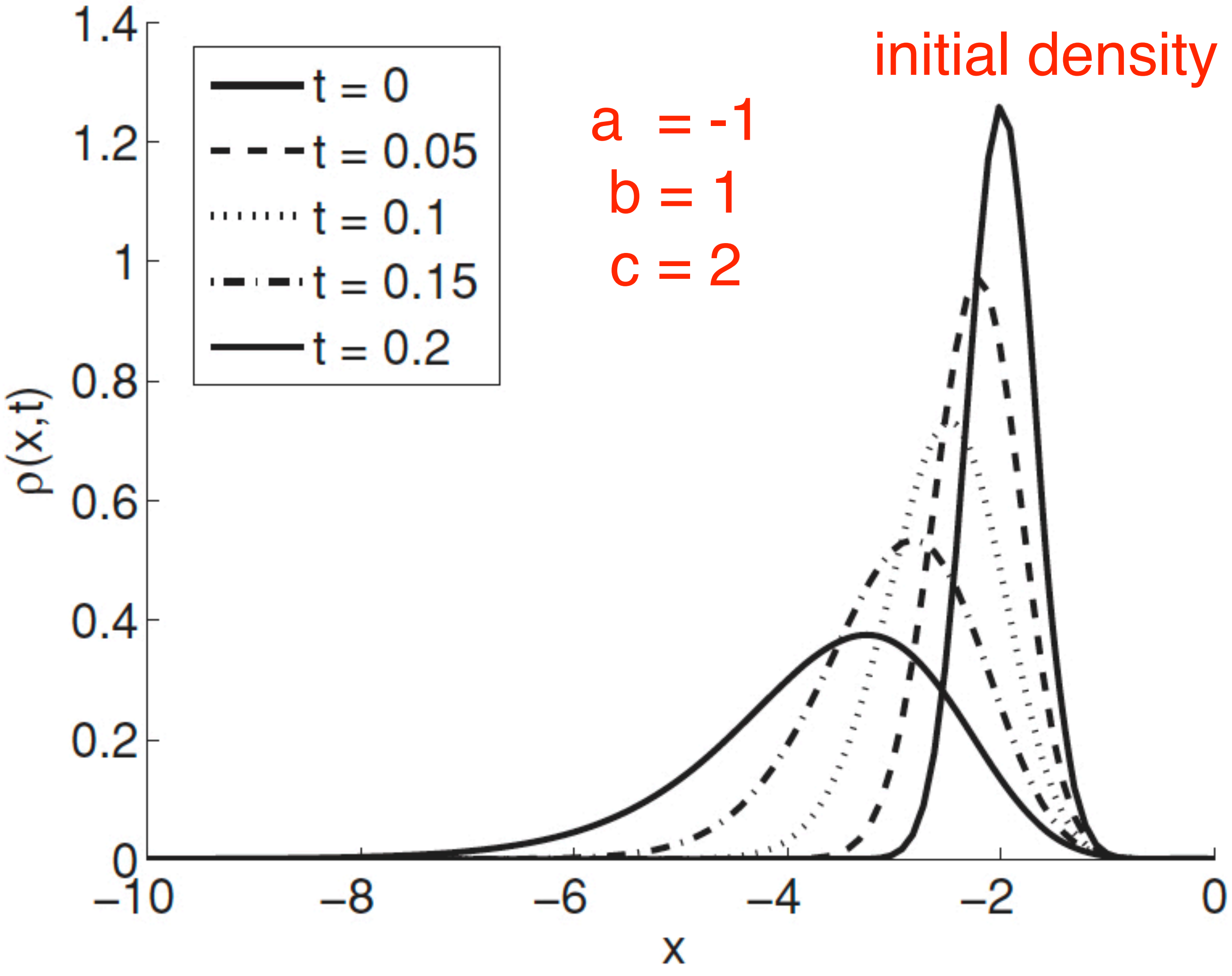
with $$D_{ij} = \sum_{k=1}^{N} G_{ik}G_{jk}$$ (diffusion tensor)

deterministic (G = 0): Liouville equation
stochastic: Fokker-Planck equation

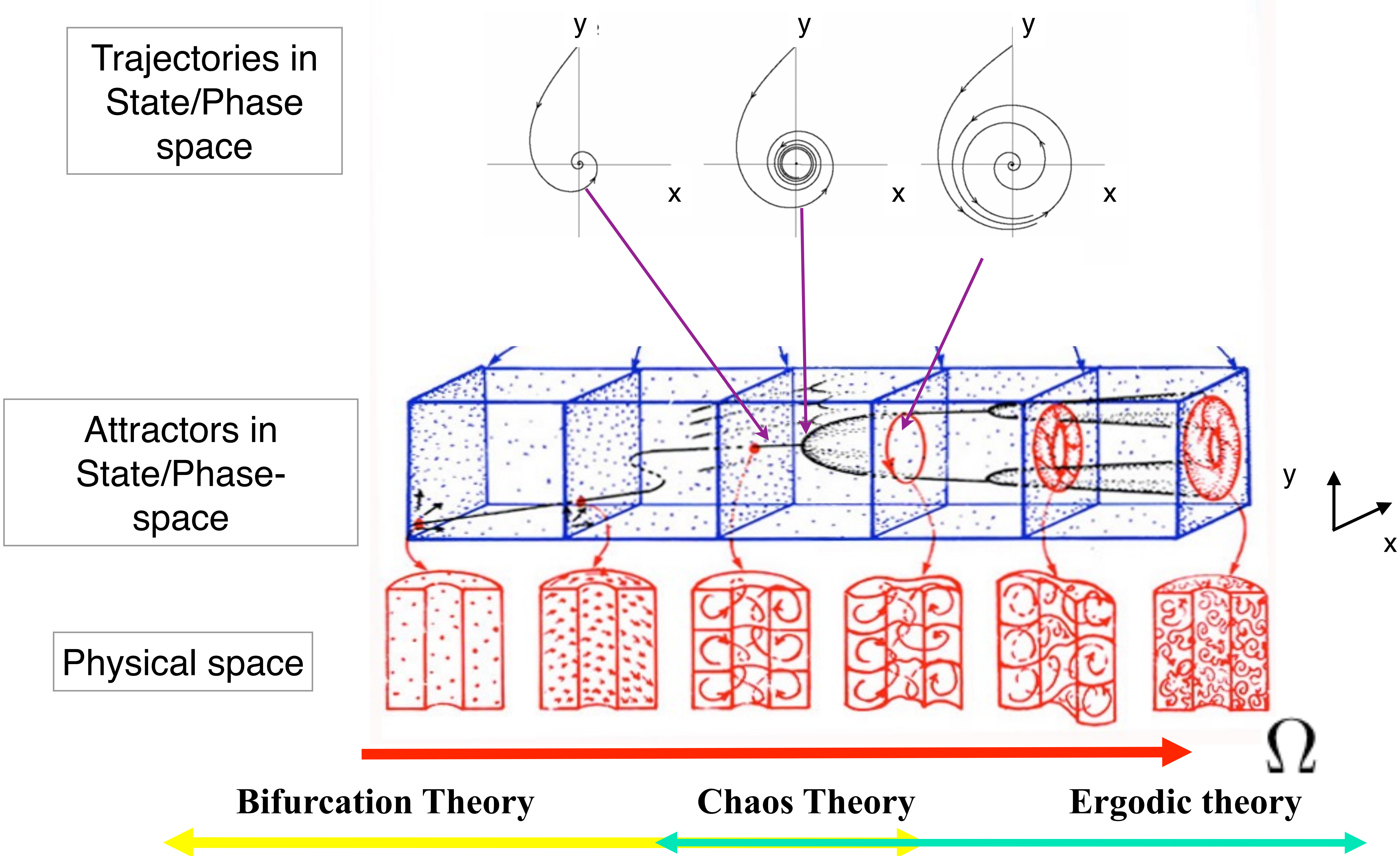
Example solution Liouville equation

$$\frac{dx}{dt} = ax^2 + bx + c\ ,\ \ \Delta = \frac{b^2}{4} - ac > 0,$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial((ax^2 + bx + c)\rho)}{\partial x} \qquad \rho(x,0) = \rho_0(x)$$

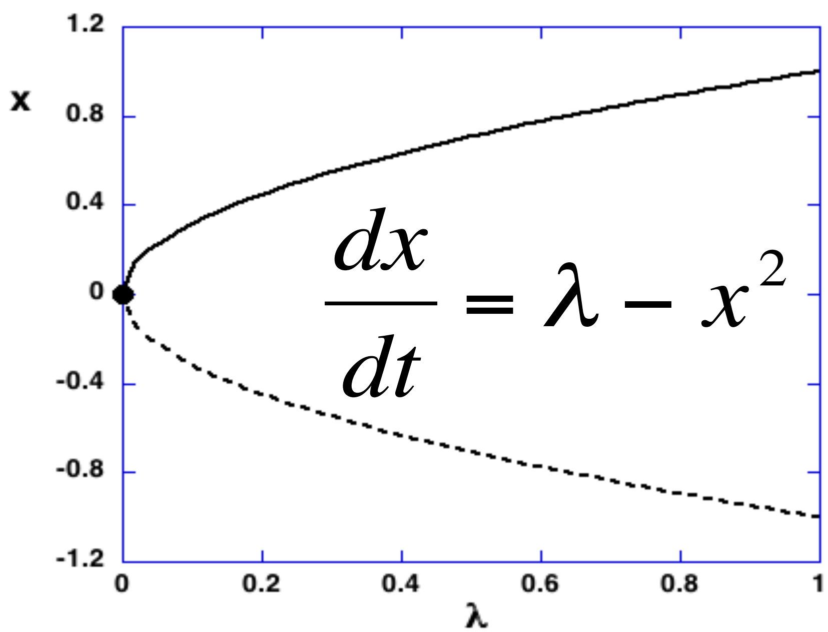


Deterministic mixing



Steady --> Periodic --> Quasi-periodic --> … --> Irregular (Chaotic) … -> Turbulent

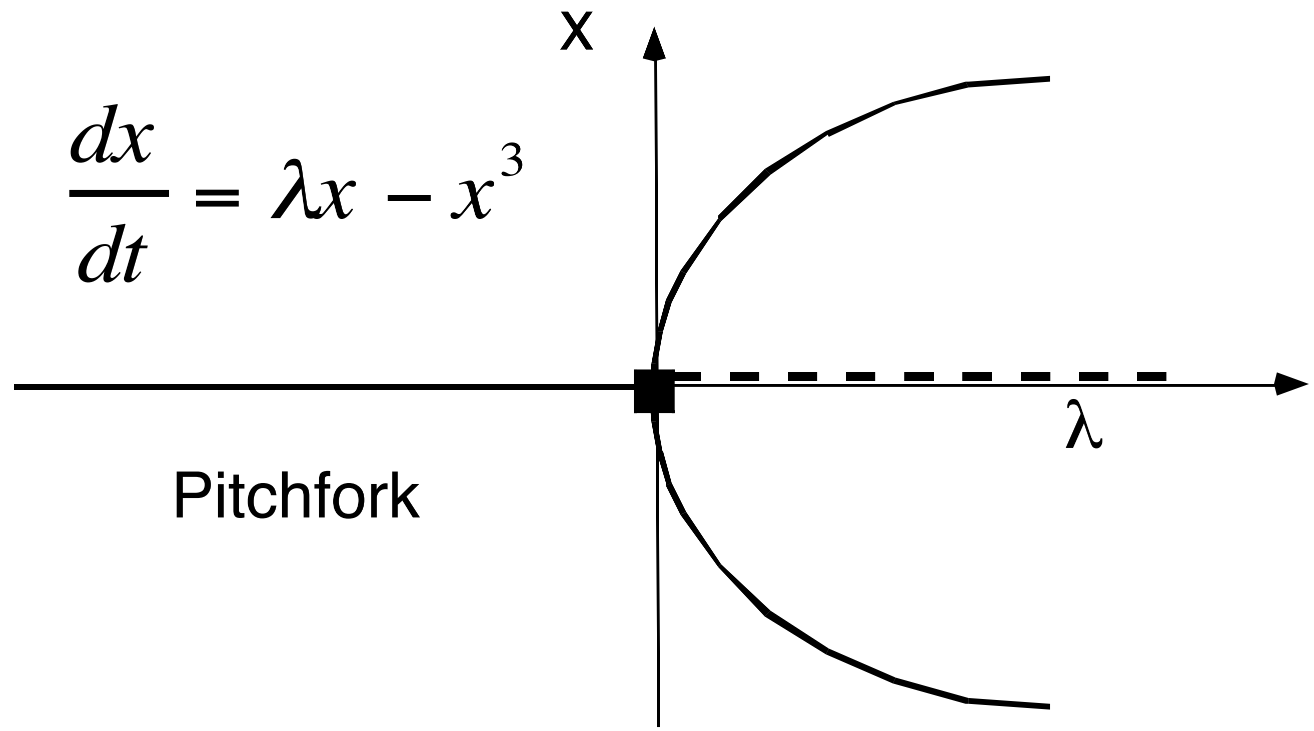
Bifurcation theory (one control parameter)



$$\frac{dx}{dt} = \lambda - x^2$$

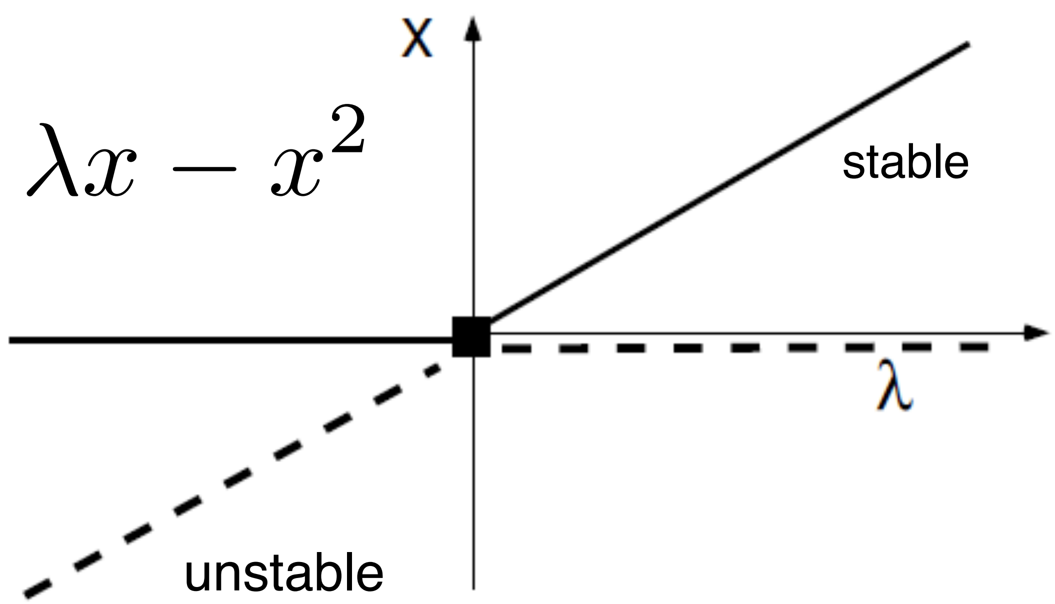
Saddle-node

$$\frac{dx}{dt} = \lambda x - x^3$$



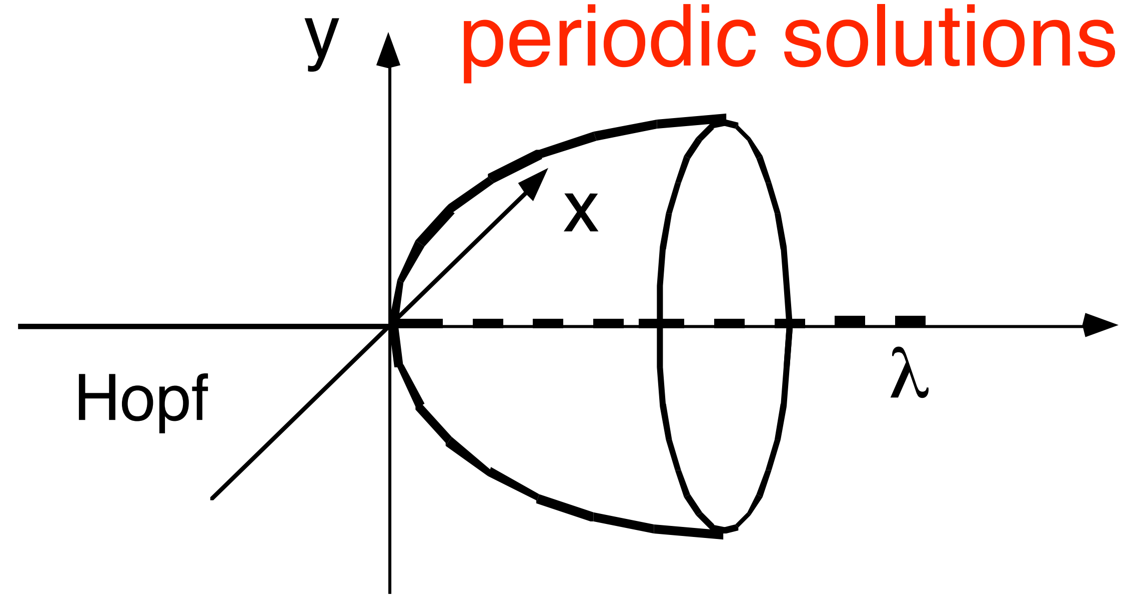
Pitchfork

$$\frac{dx}{dt} = \lambda x - x^2$$



Transcritical

periodic solutions

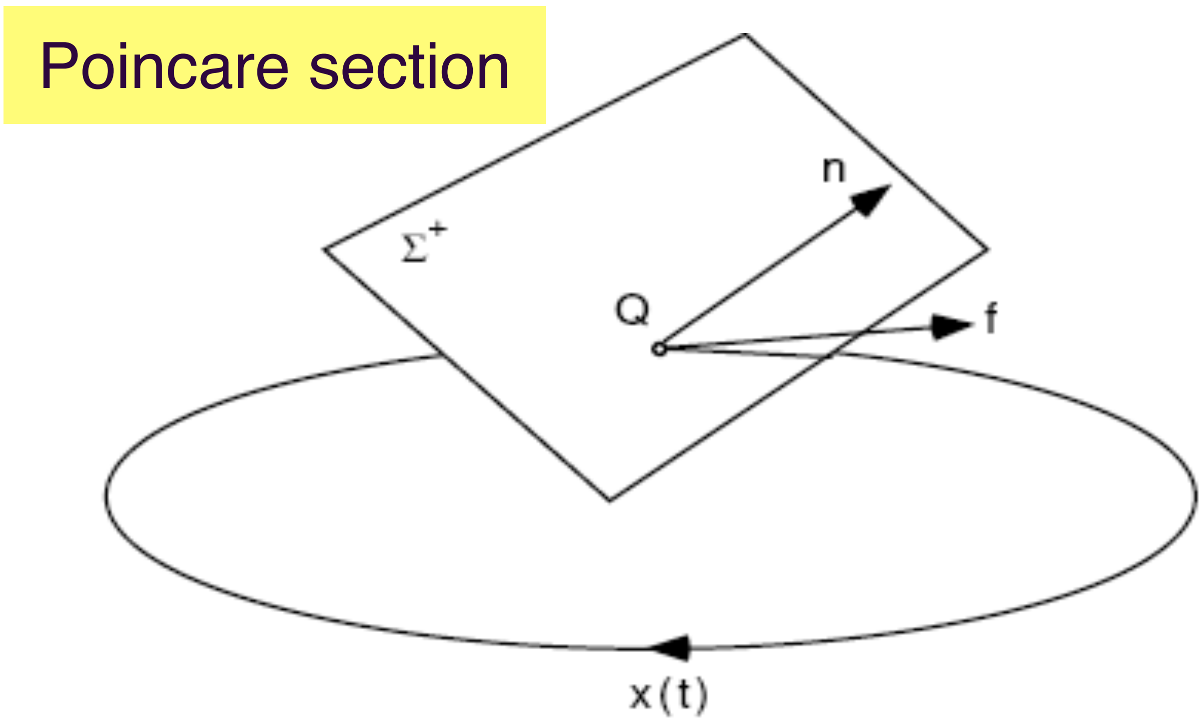


Hopf

$$\dot{x} = \lambda x - \omega y - x(x^2 + y^2)$$

$$\dot{y} = \lambda y + \omega x - y(x^2 + y^2)$$

Poincare section & map



period: p

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, p)$$

Fixed points of Poincare map: $\mathbf{x} - \mathbf{F}(\mathbf{x}, p) = 0$

Stability of Periodic Orbits: I

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$$

Fixed point (periodic orbit)

$$\bar{\mathbf{x}} = \mathbf{F}(\bar{\mathbf{x}})$$

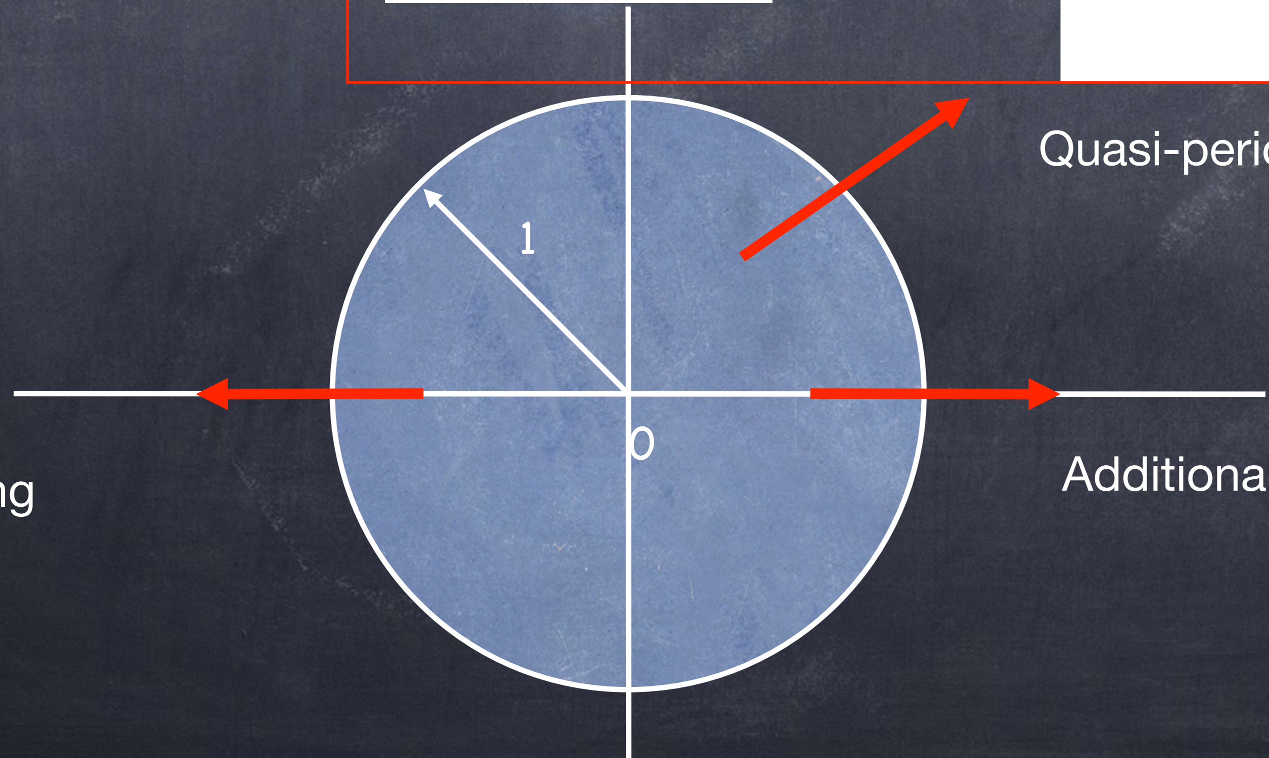
Linear stability

$$\mathbf{x}_{k+1} = \bar{\mathbf{x}} + \mathbf{y}_{k+1}$$

$$\mathbf{y}_{k+1} = J\mathbf{y}_k$$

Example: $\sigma_i \ real$

$$\mathbf{z}_k = T\mathbf{y}_k \longrightarrow \mathbf{z}_{k+1} = \begin{pmatrix} \sigma_1 & \ldots & 0 \\ 0 & \sigma_i & 0 \\ 0 & \ldots & \sigma_n \end{pmatrix} \mathbf{z}_k$$



Quasi-periodic behavior

C

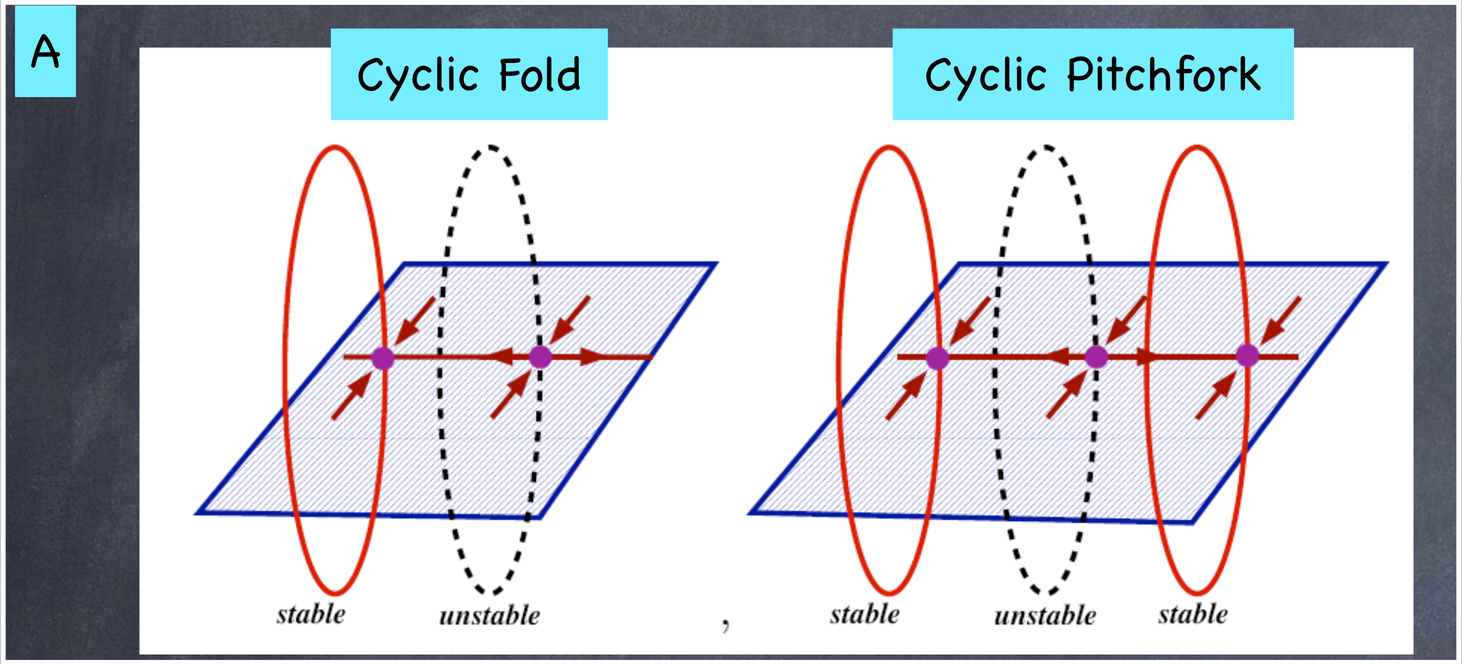
Period doubling

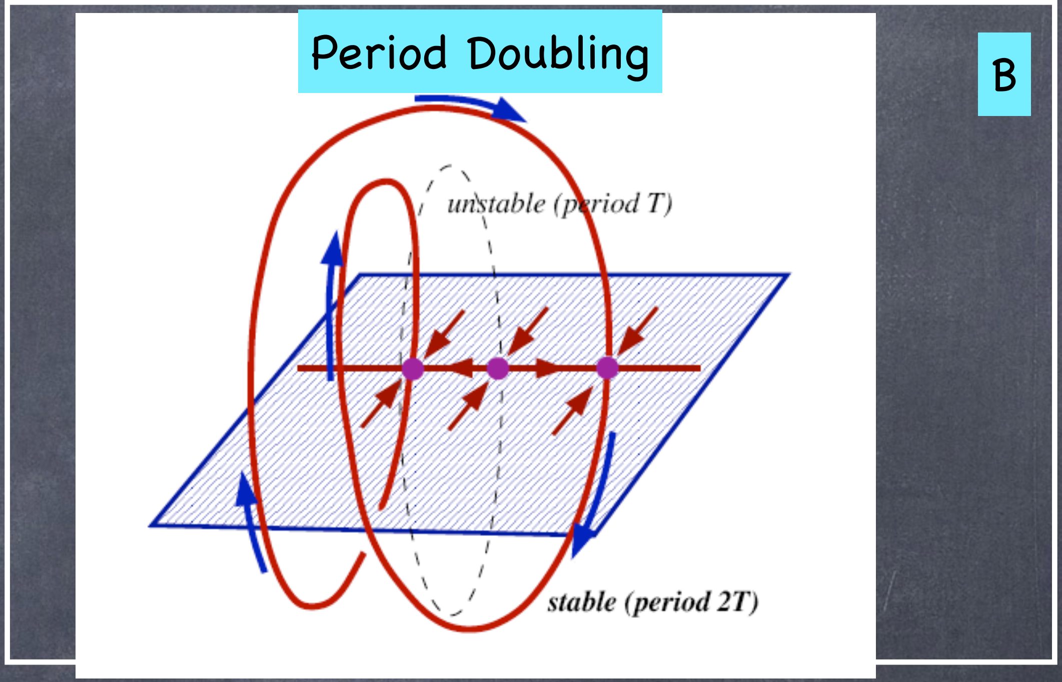
B

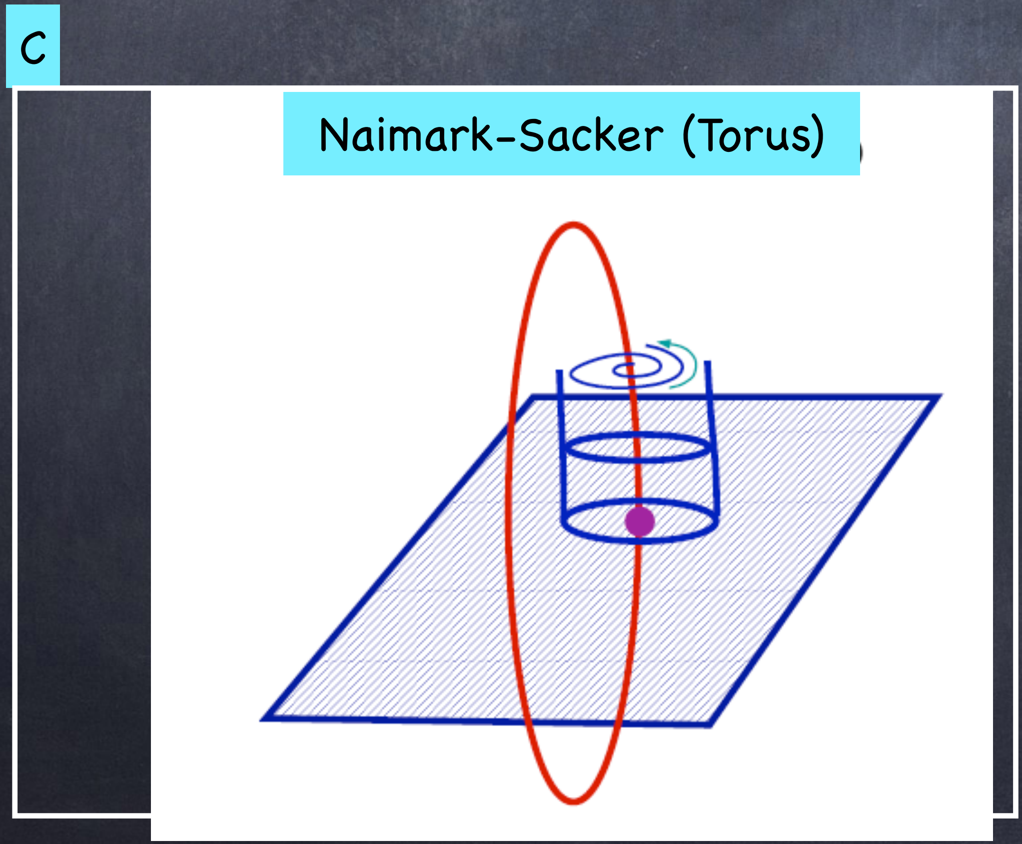
Additional periodic orbits

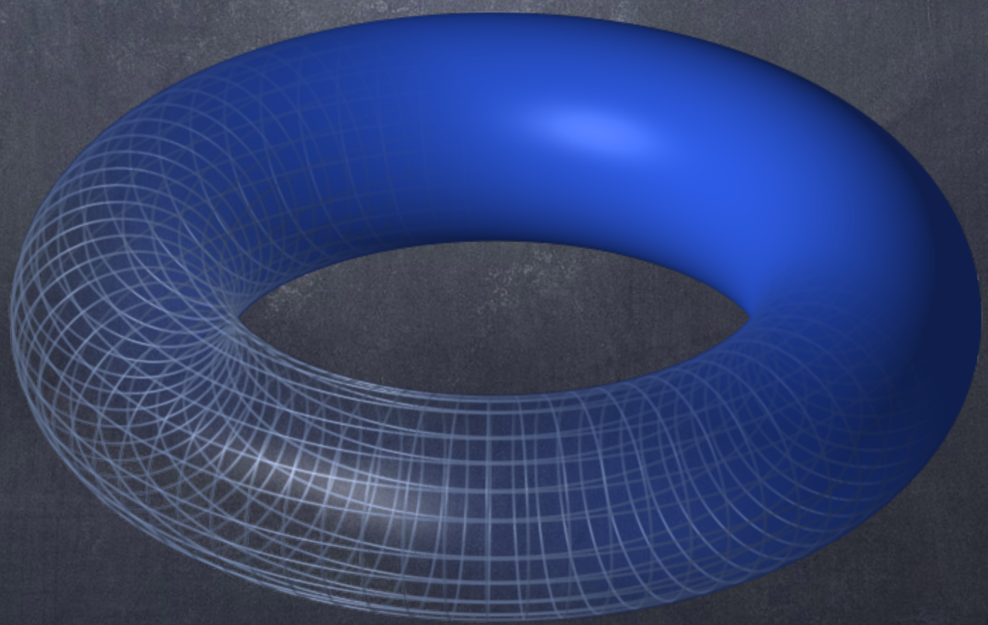
A

Stability of Periodic Orbits: II

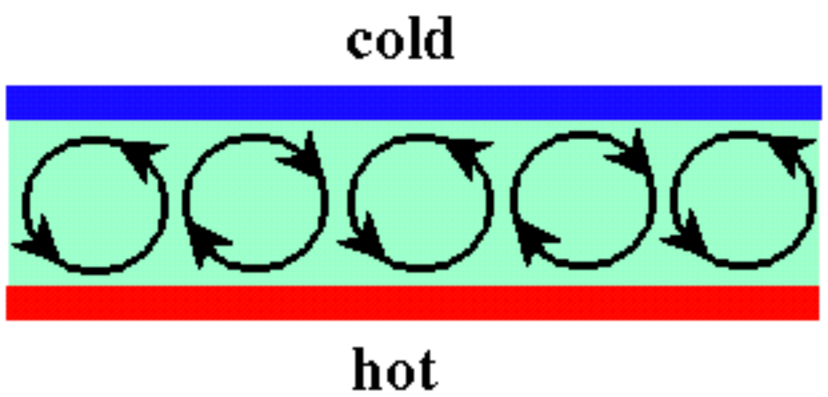








Idealized atmospheric flow



r: vertical temperature difference

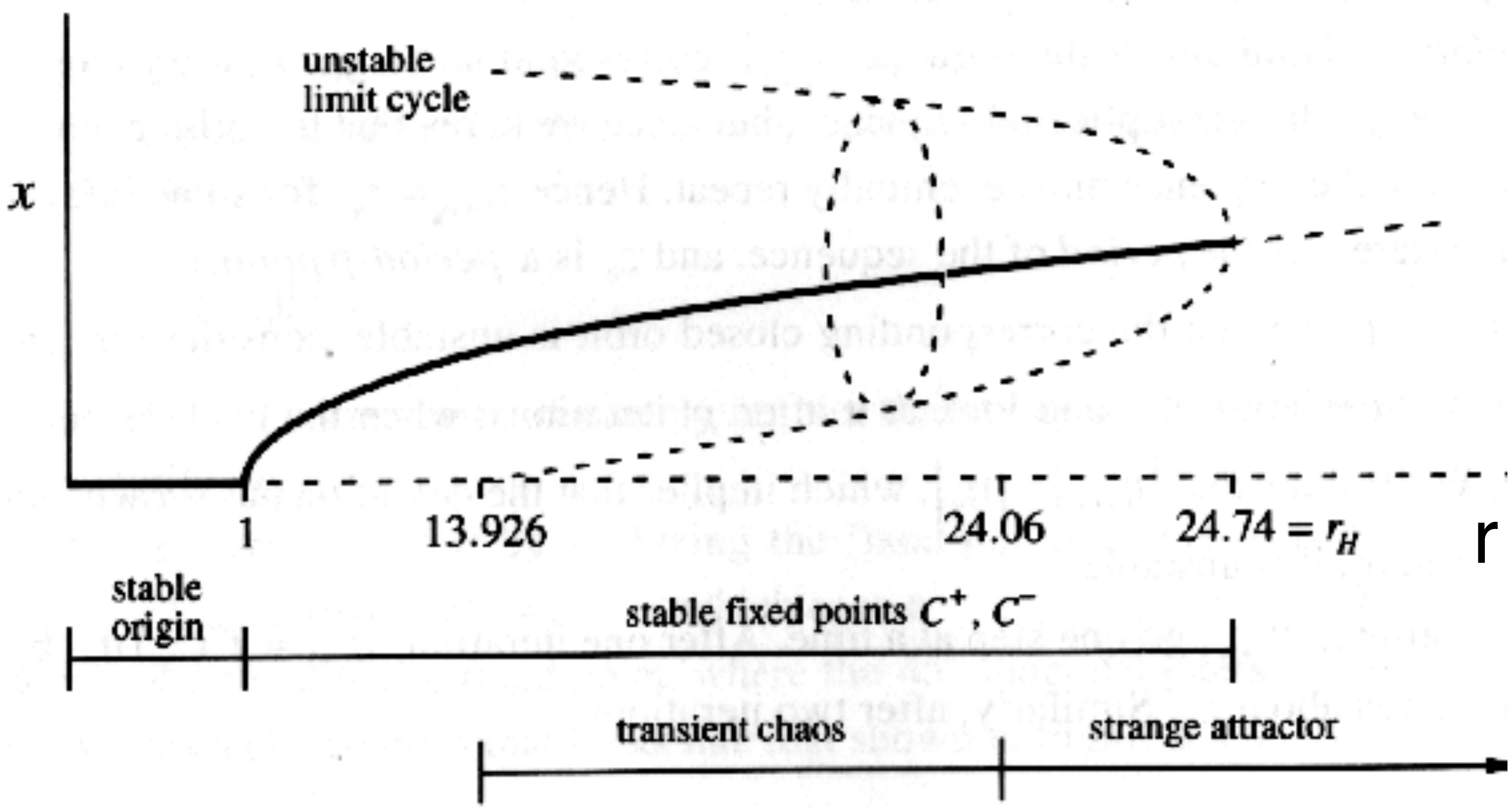
Edward Lorenz (1917-2008)

$$\dot{x} = s(y - x)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

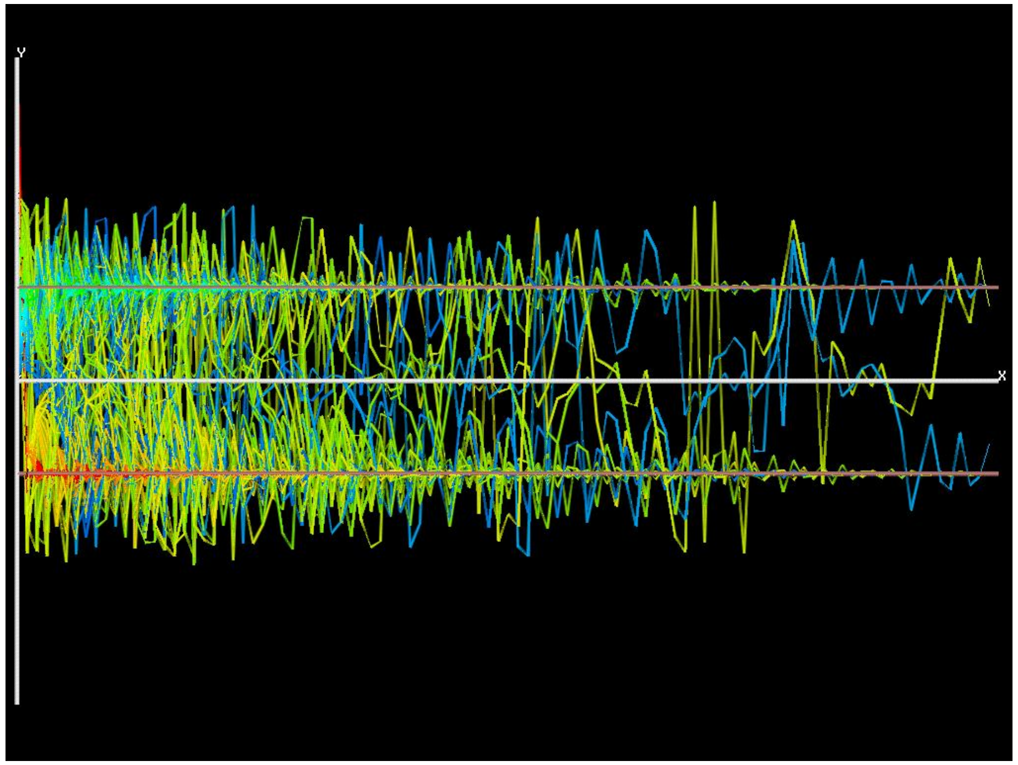
The Lorenz system

s = 10, b = 10/3, r: control parameter

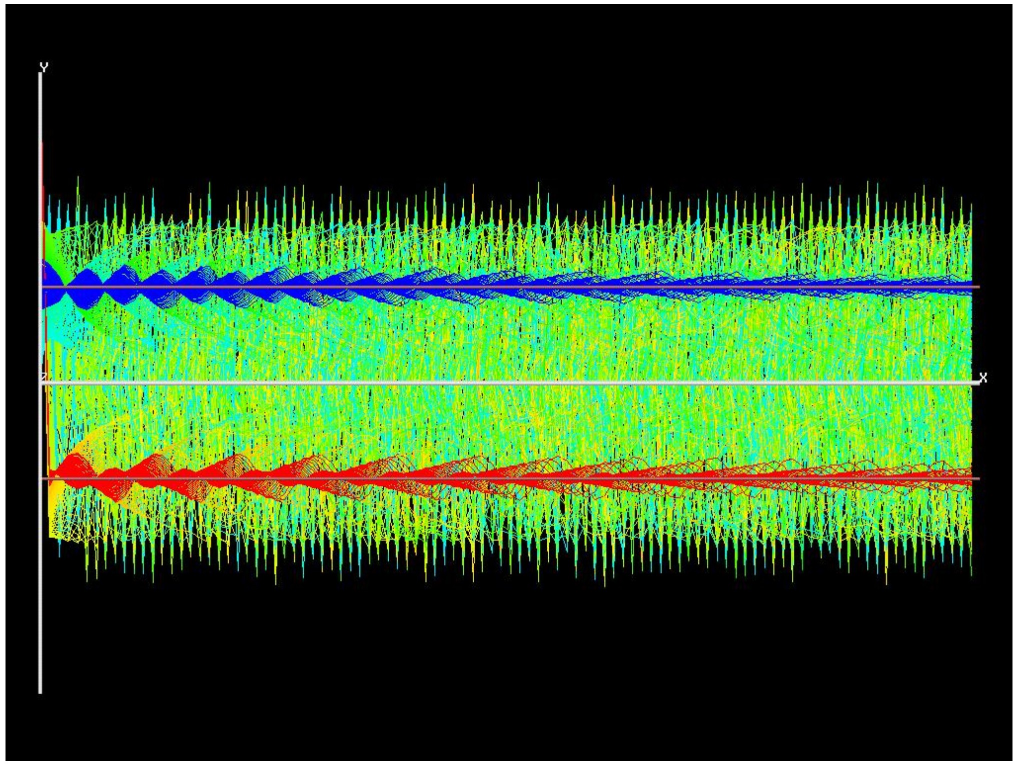
Bifurcation diagram



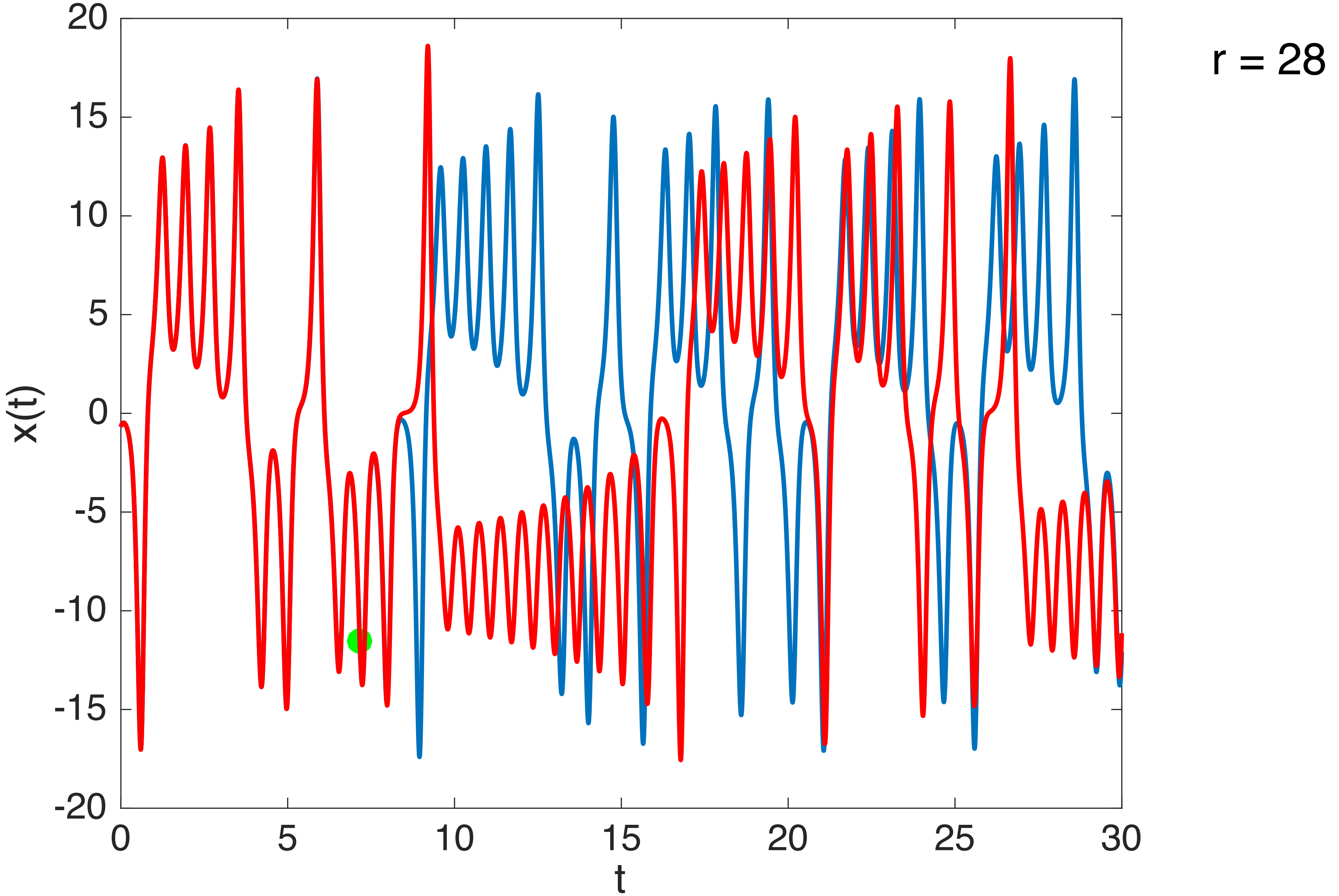
r = 21



r = 24.1

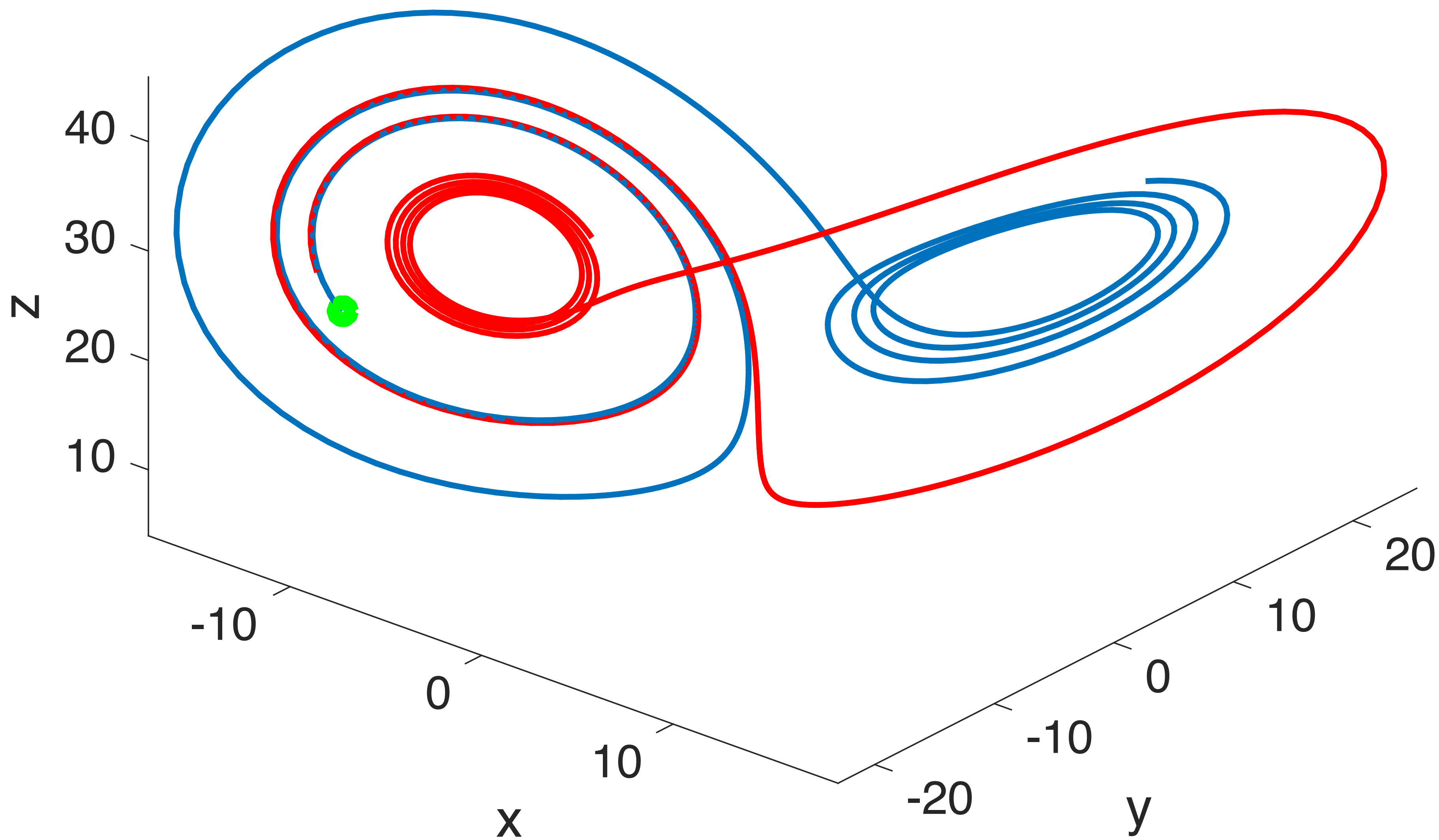


Behavior of x(t)

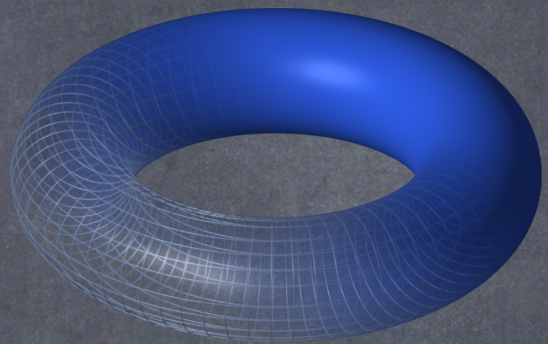


Chaotic behavior: sensitivity to <u>initial</u> conditions

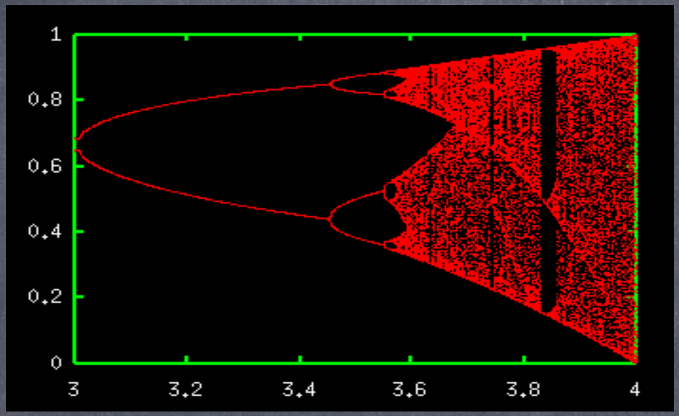
Lorenz attractor



Routes to Chaotic behavior

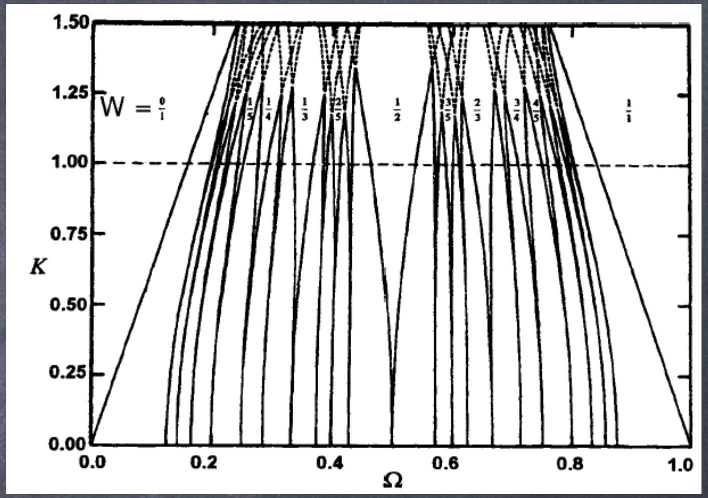


1. Three-frequency route

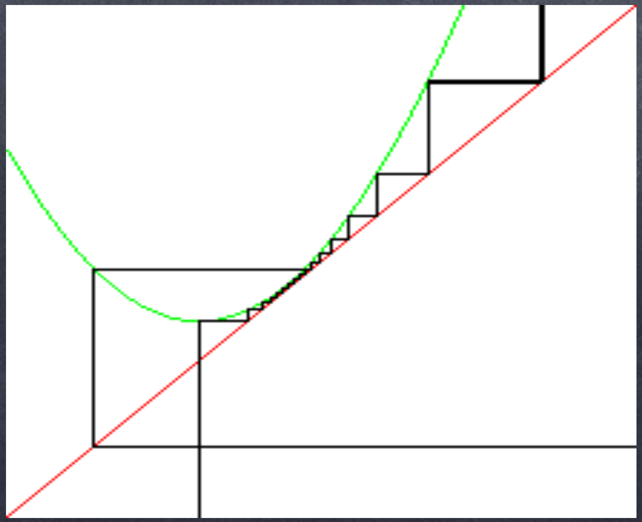


2. Period-doubling route

3. Quasi-periodicity route



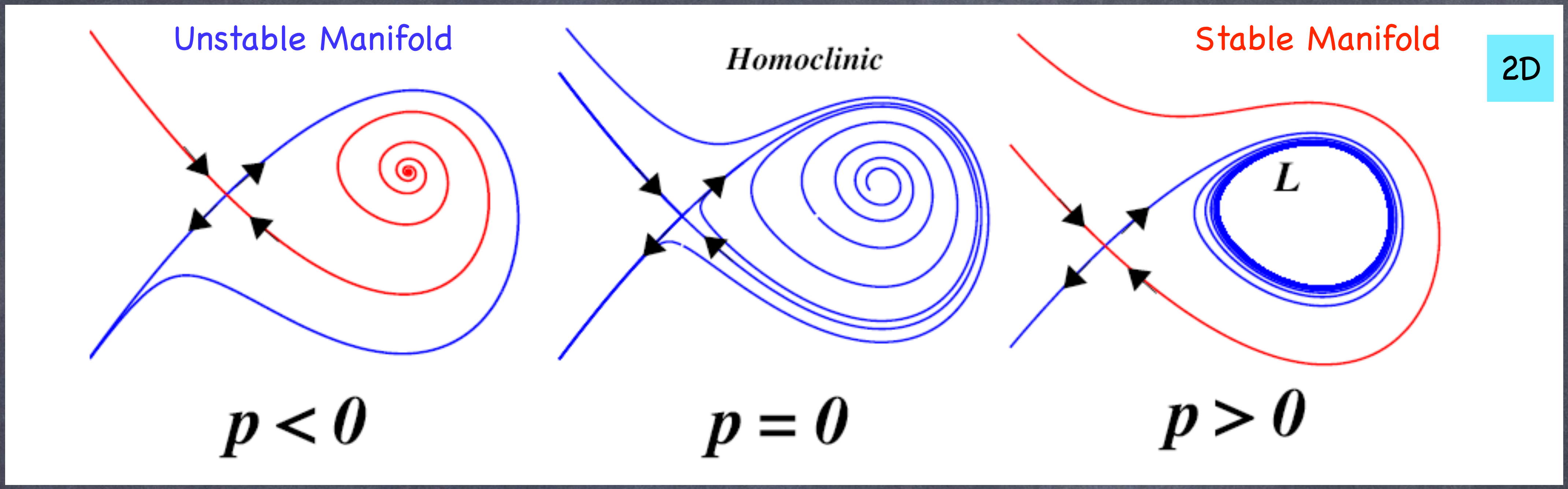
4. Global bifurcations
   (e.g. Homoclinic/Heteroclinic)
   connections



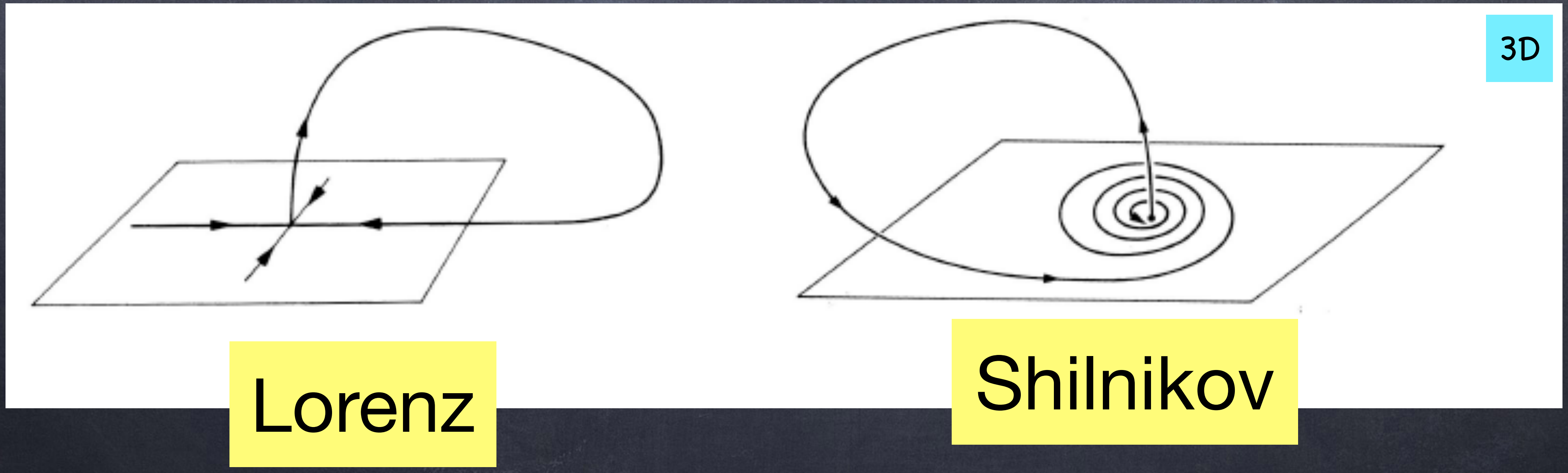
5. Intermittency

6. other (e.g. crisis)

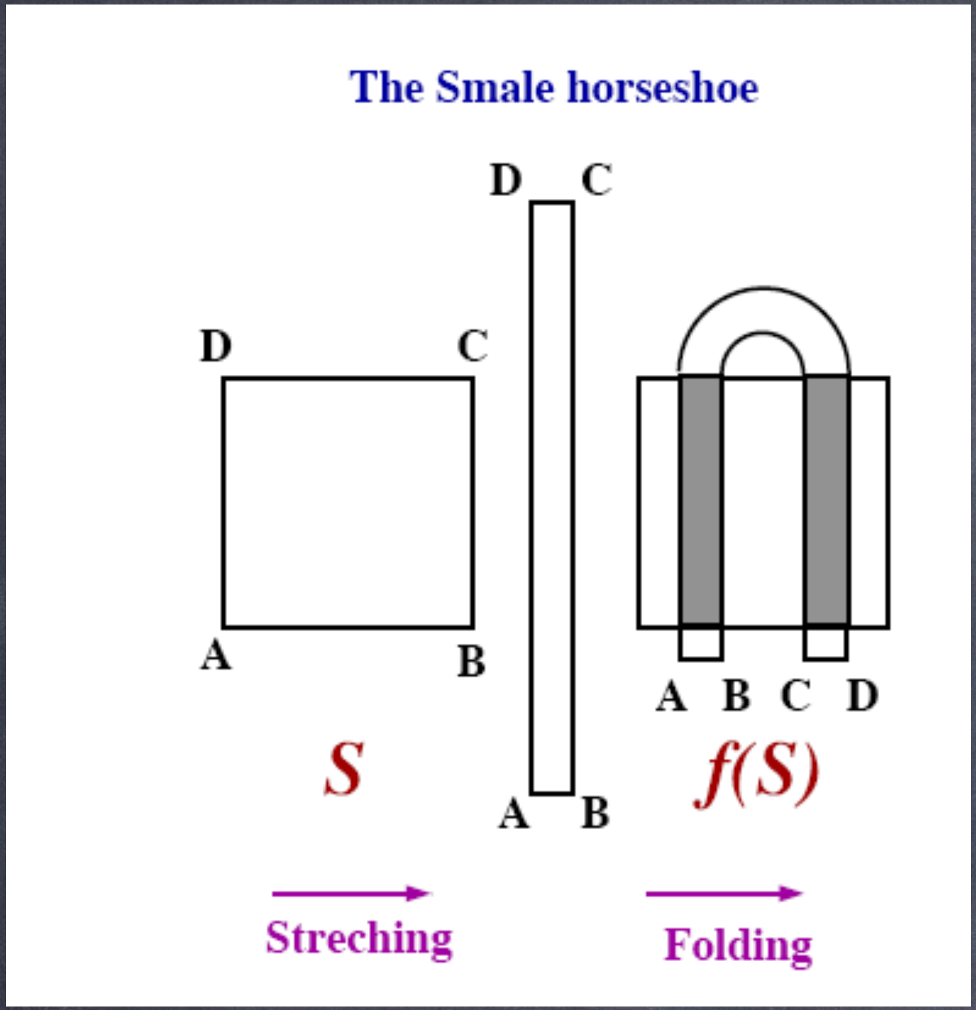
Global bifurcations: homoclinic orbits

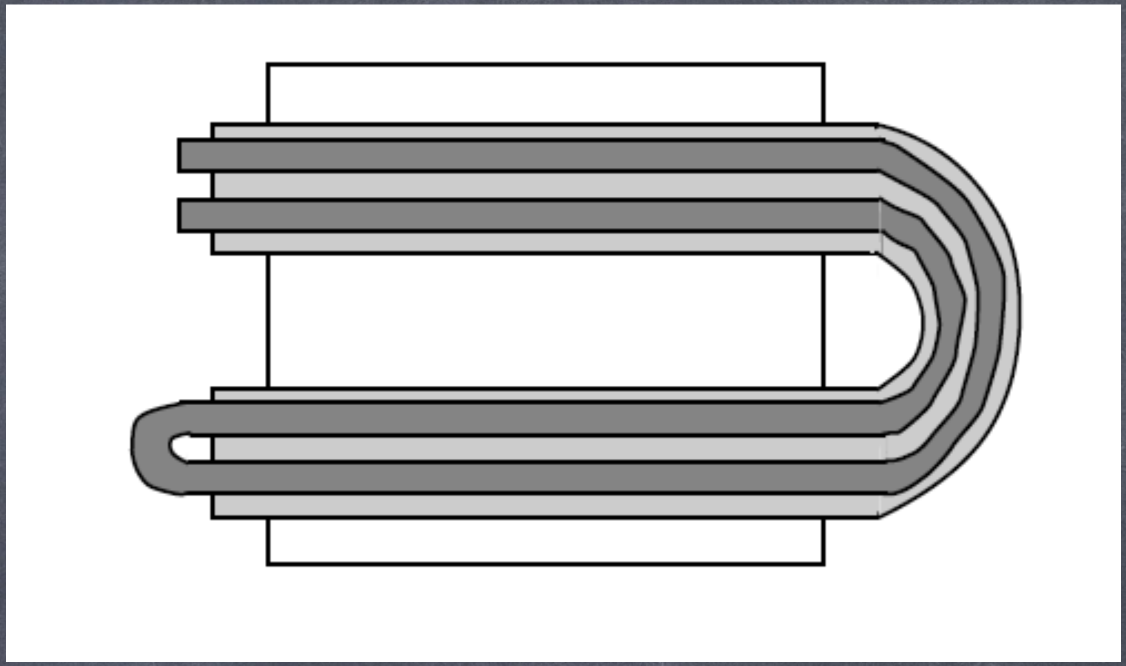


Ex:



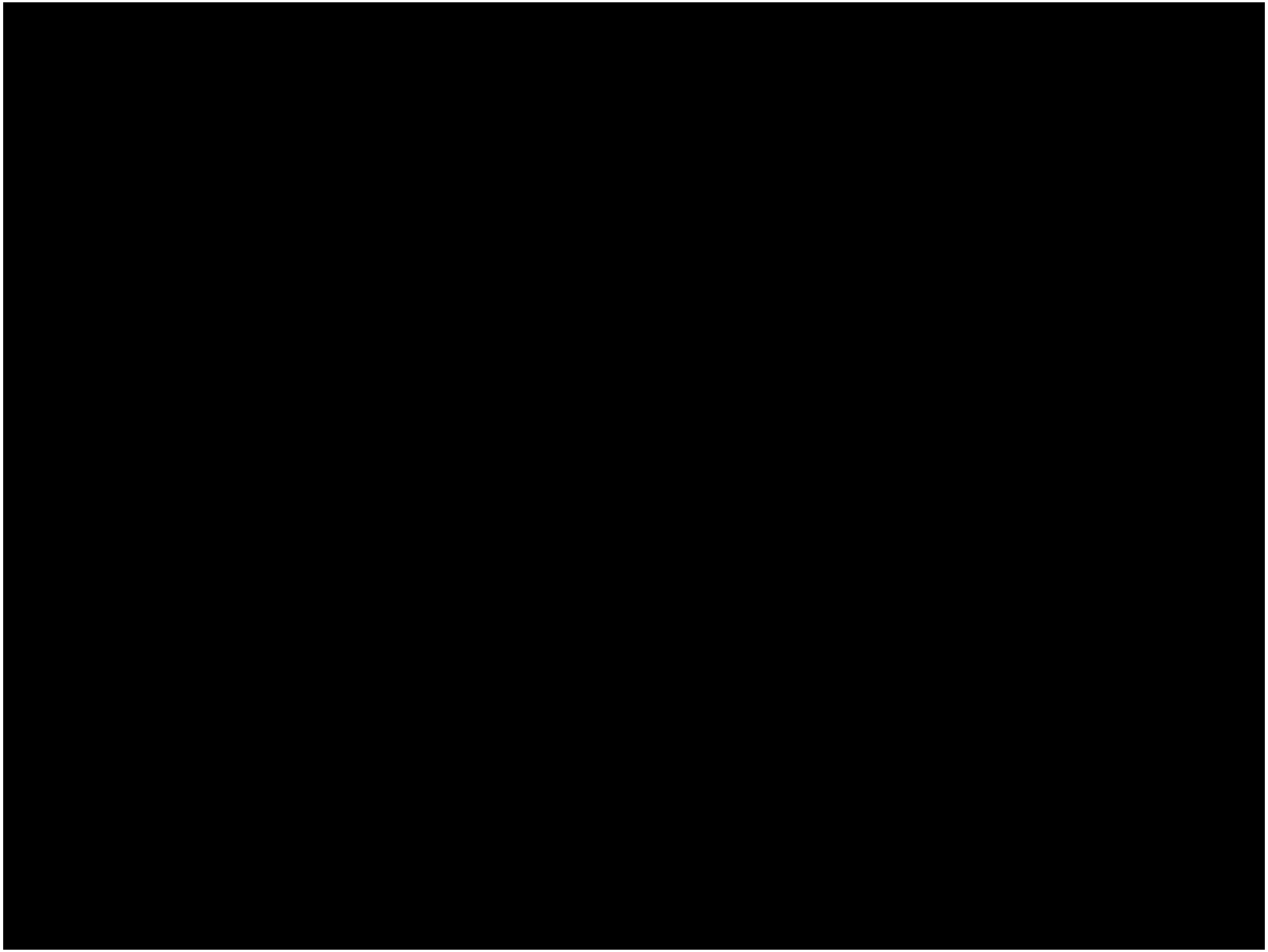
Horseshoes



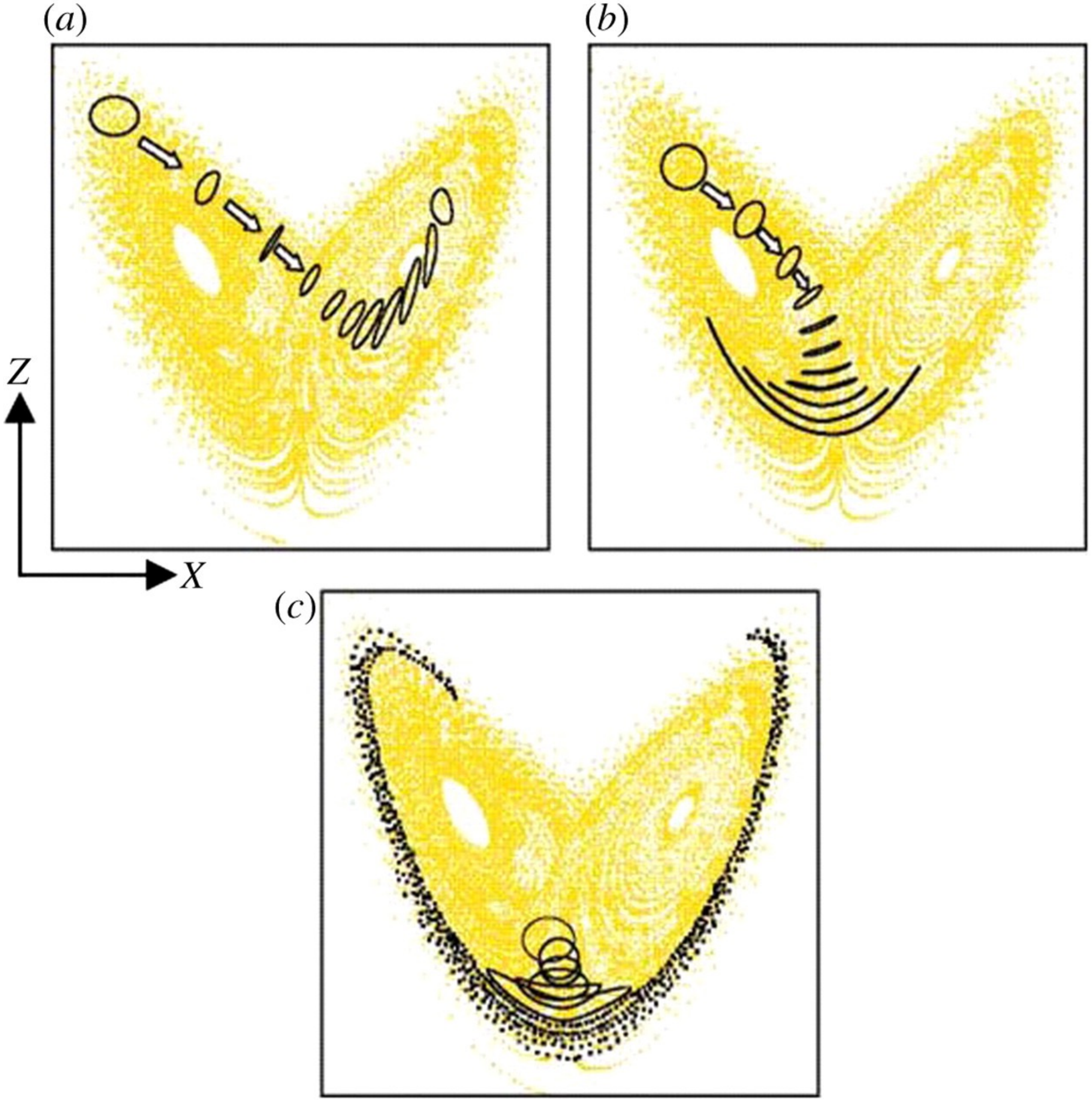


Cantor - set

Spread of trajectories: Lorenz

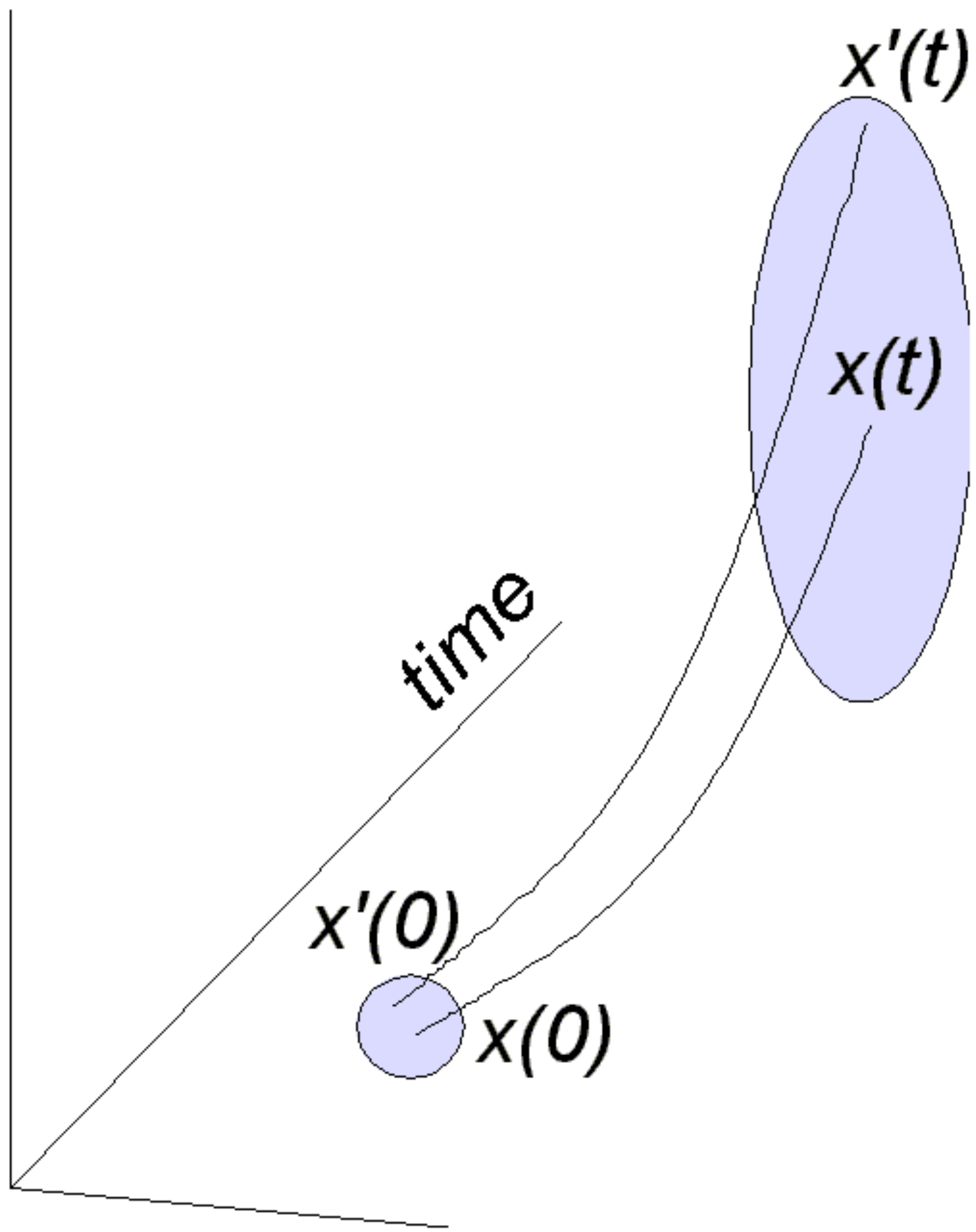


Error growth in the Lorenz model



Examples of finite-time error growth on the Lorenz attractor for three probabilistic predictions starting from different points on the attractor.

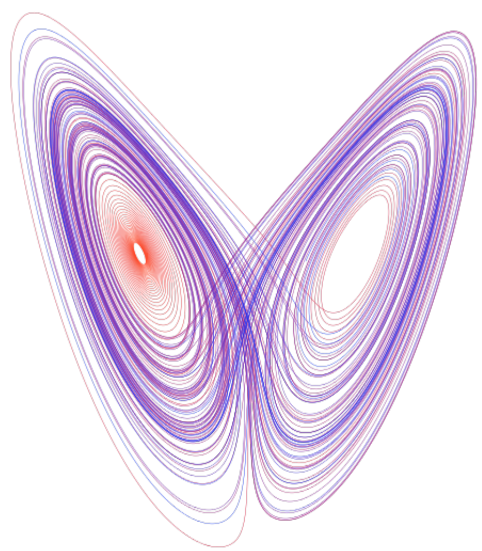
Lyapunov exponent



$$d(t) = x'(t) - x(t)$$

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{d_i(t)}{d_i(0)} \right|$$

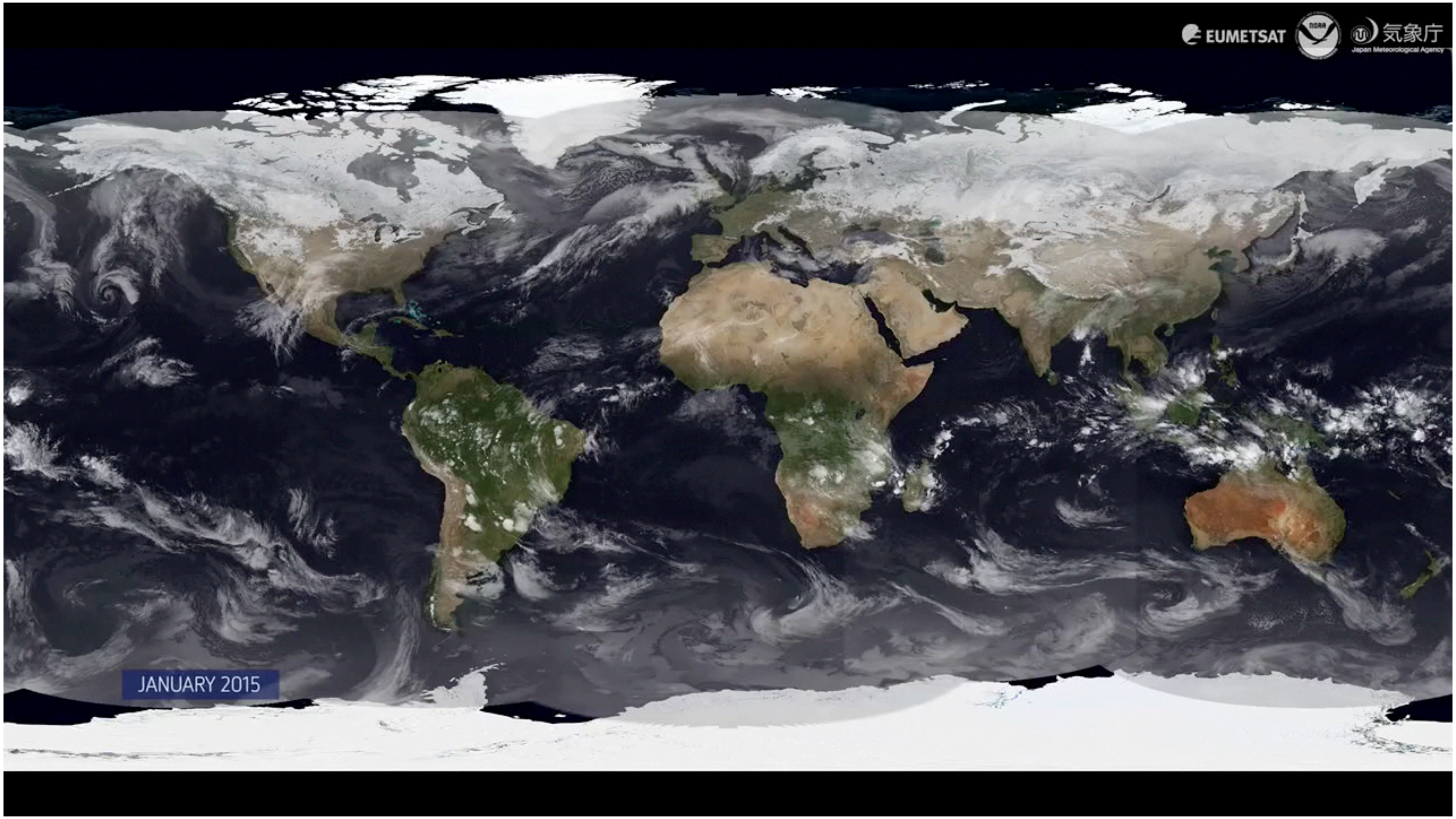
Lorenz equations: 0.9056, 0, -14.5723



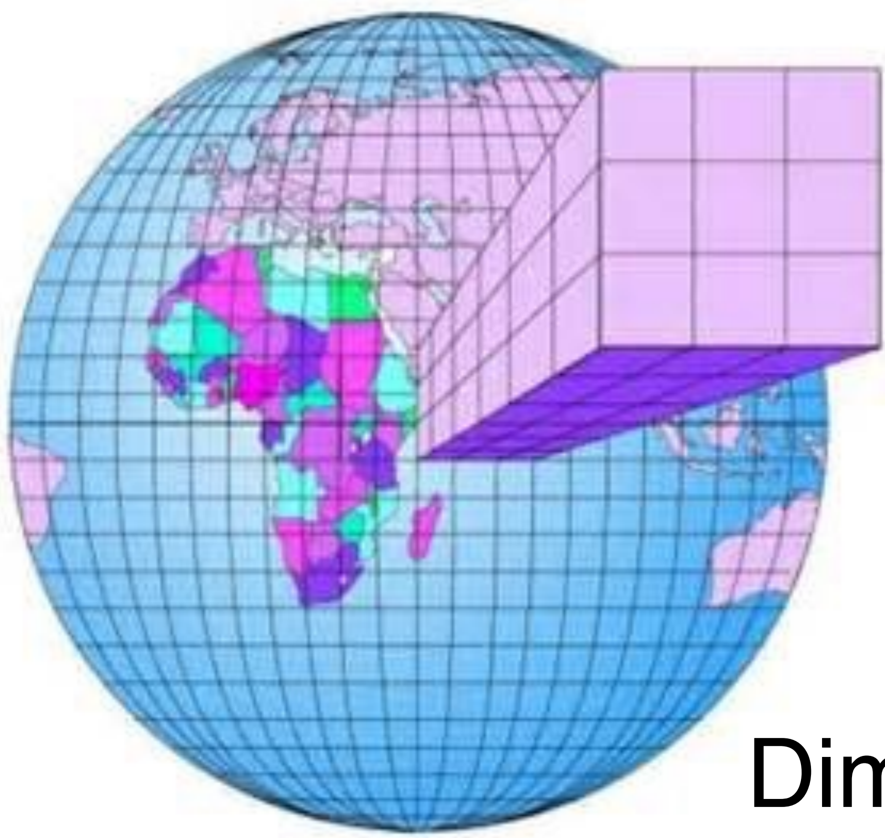
$\lambda > 0 \rightarrow$ chaotic motion

Question time

Atmospheric flow



Numerical Weather Prediction Model



Grid: N x M x L

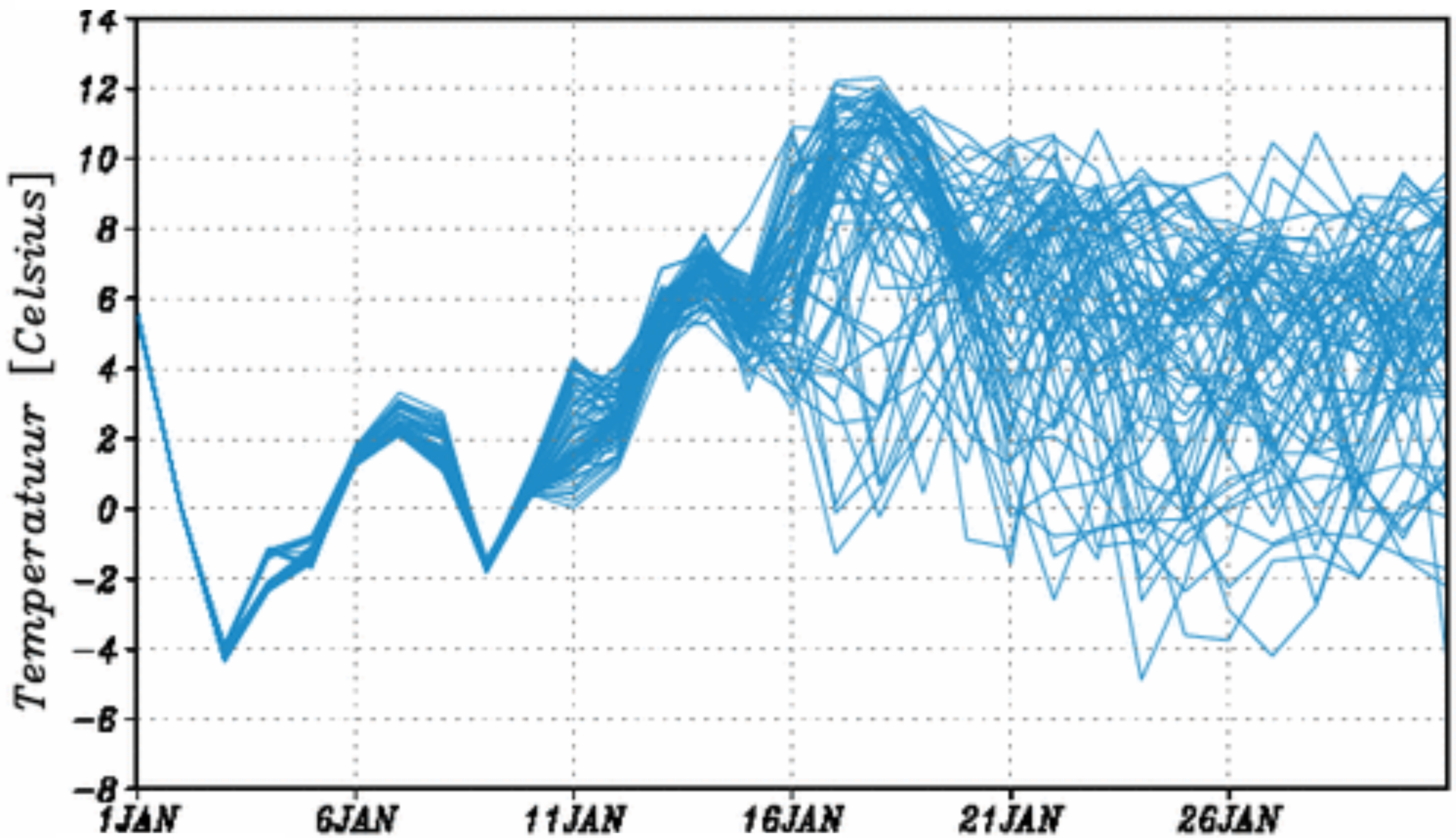
Observables (temperature, humidity, velocities, etc.): k

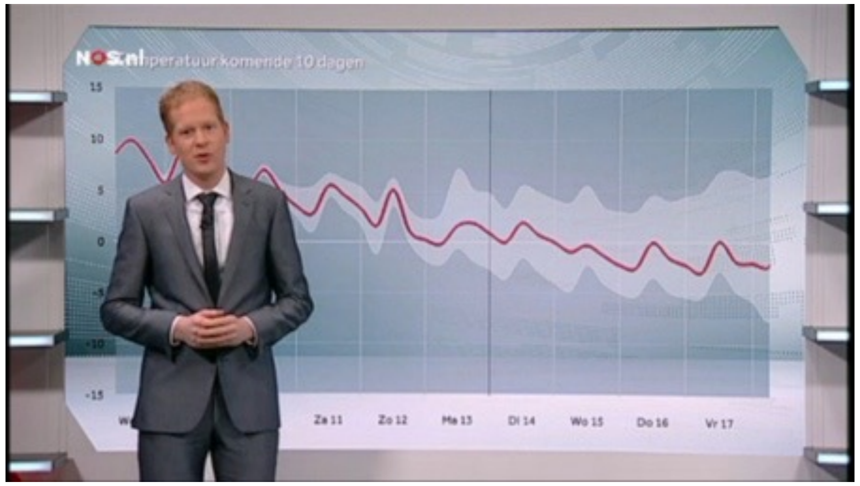
Dimension phase space: d = k x N x M x L

Typically: d = $10^5 - 10^9$

State of the art: 10 km horizontal resolution

Origin of the ‘plume’ in weather forecasts



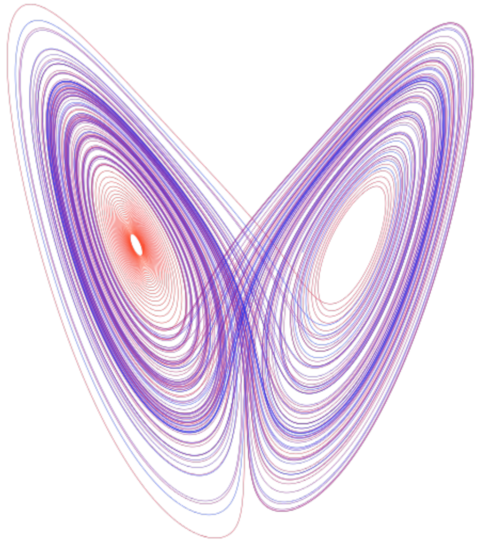


Numerical weather prediction models:
many Lyapunov exponents > 0

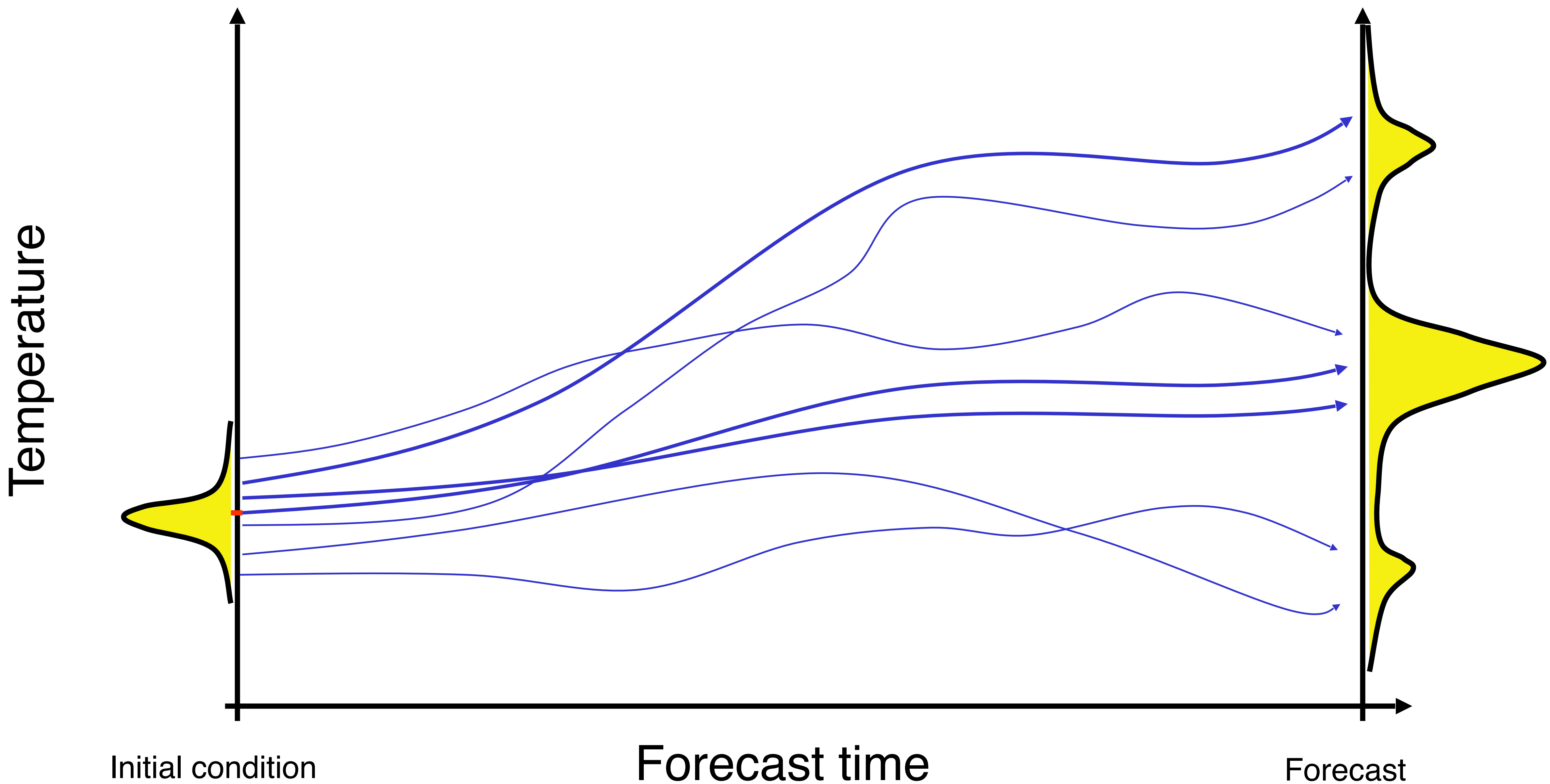
Weather prediction

Lorenz (1969):

… one flap of a sea-gull's wing may forever change the future course of the weather



Ensemble forecasting



How to choose the initial conditions of the ensemble members (the initial PDF)?

Optimal modes

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x}(t) = \mathcal{M}(\mathbf{x}(t_0)),$$

$$\mathcal{M}(\mathbf{x}(t_0) + \mathbf{y}(t_0)) = \mathcal{M}(\mathbf{x}(t_0)) + \frac{\partial \mathcal{M}}{\partial \mathbf{x}}\mathbf{y}(t_0) + \mathcal{O}(\epsilon^2) \approx \mathbf{x}(t) + \mathbf{y}(t),$$

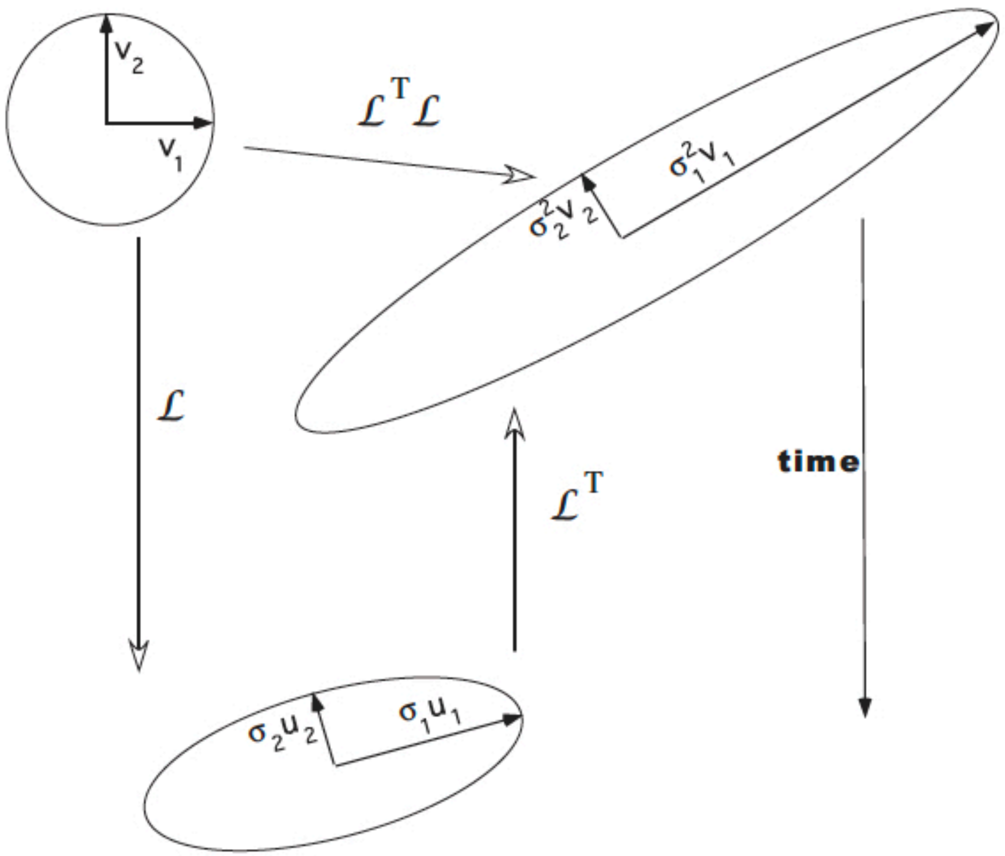
$$\mathbf{y}(t) = \mathcal{L}(t_0, t)\mathbf{y}(t_0),$$

tangent linear model

$$\mathbf{L} \equiv \mathcal{L}(t_0, t_1)$$

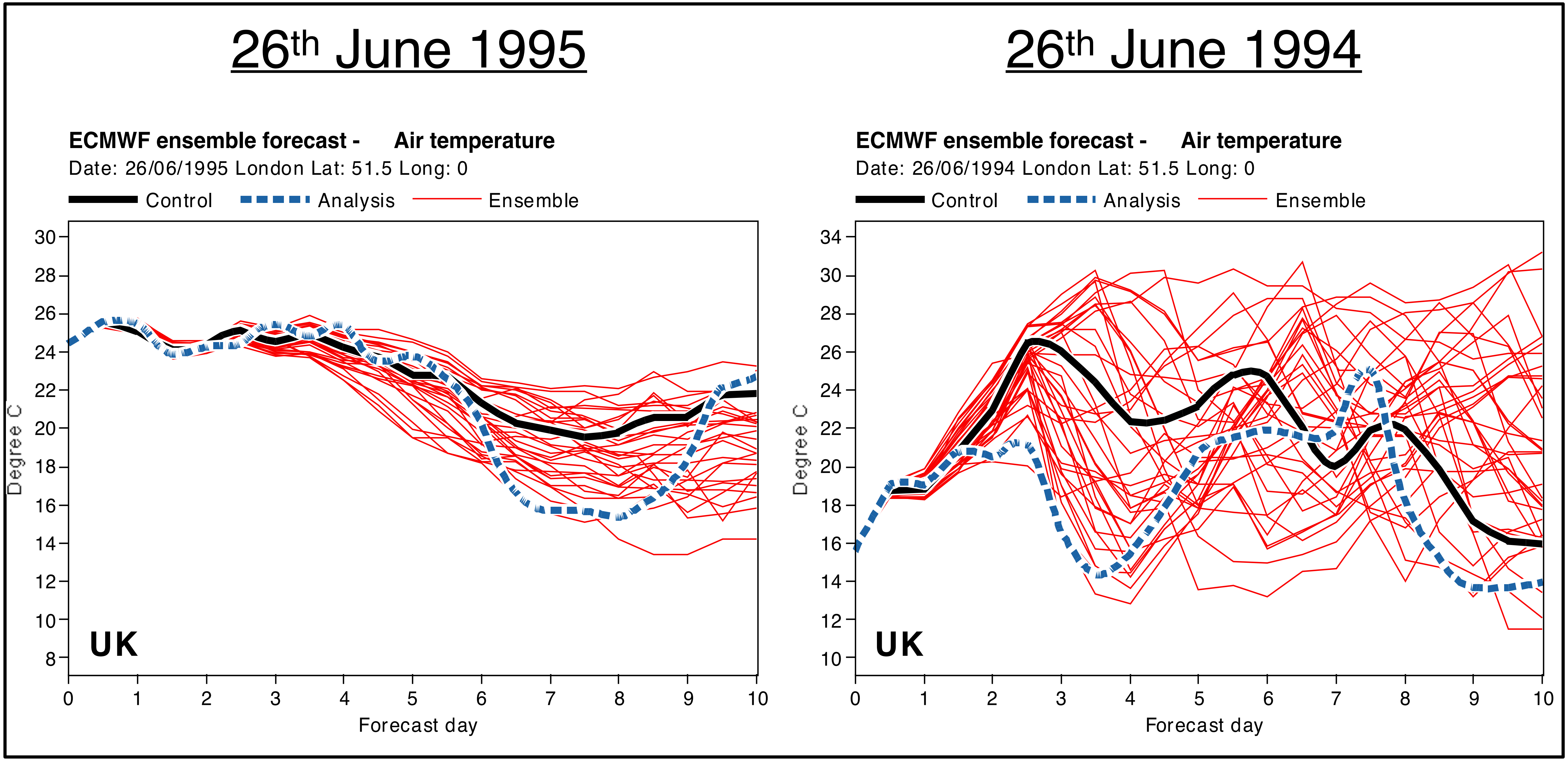
$$\|\mathbf{y}(t_1)\|^2 = \langle \mathbf{L}\mathbf{y}(t_0), \mathbf{L}\mathbf{y}(t_0)\rangle = \langle \mathbf{L}^T\mathbf{L}\mathbf{y}(t_0), \mathbf{y}(t_0)\rangle,$$

$$\mathbf{L}^T\mathbf{L}\mathbf{v}_i = \sigma\mathbf{v}_i$$



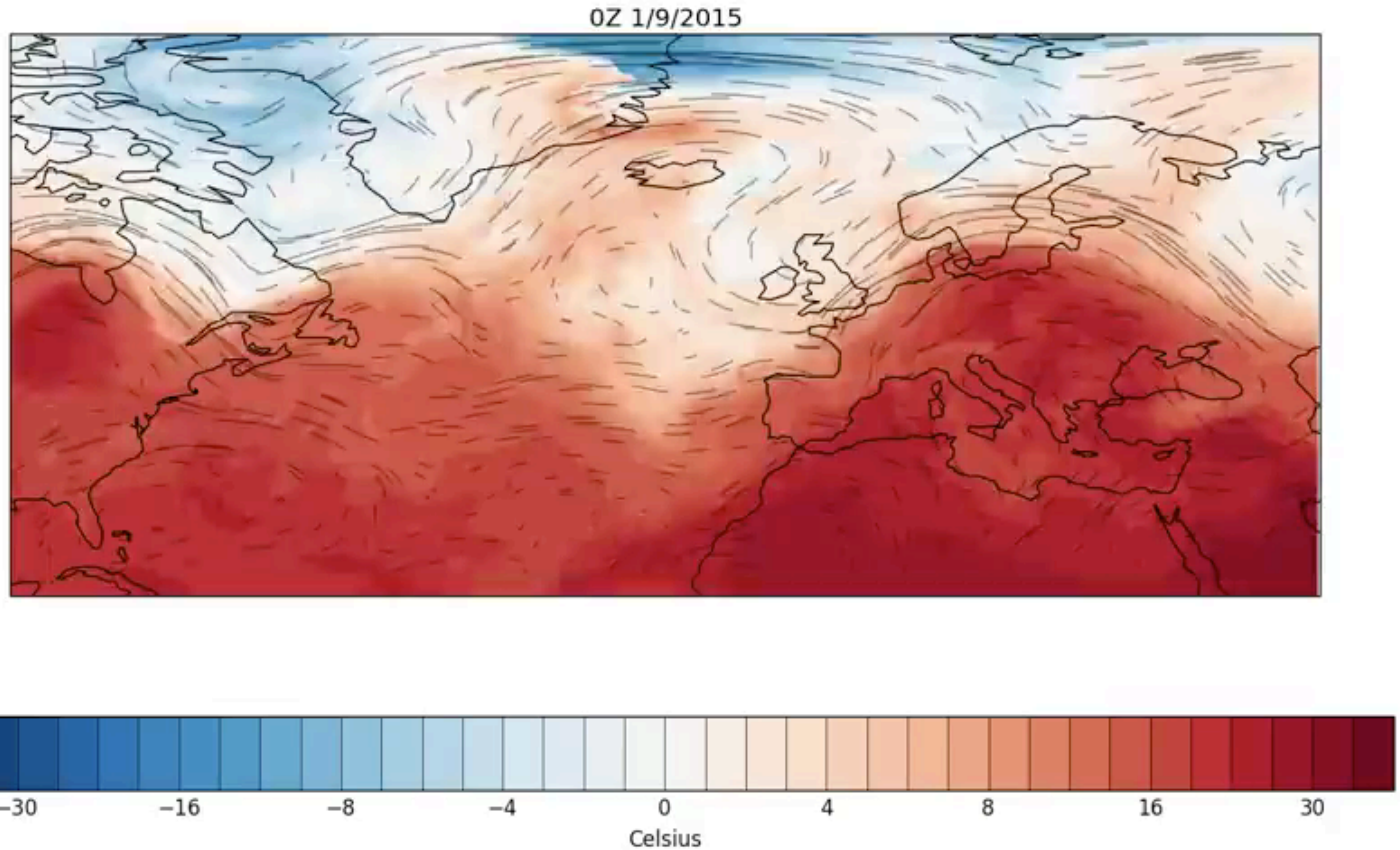
singular values/vectors -> most expanding directions

Flow dependence of forecast errors



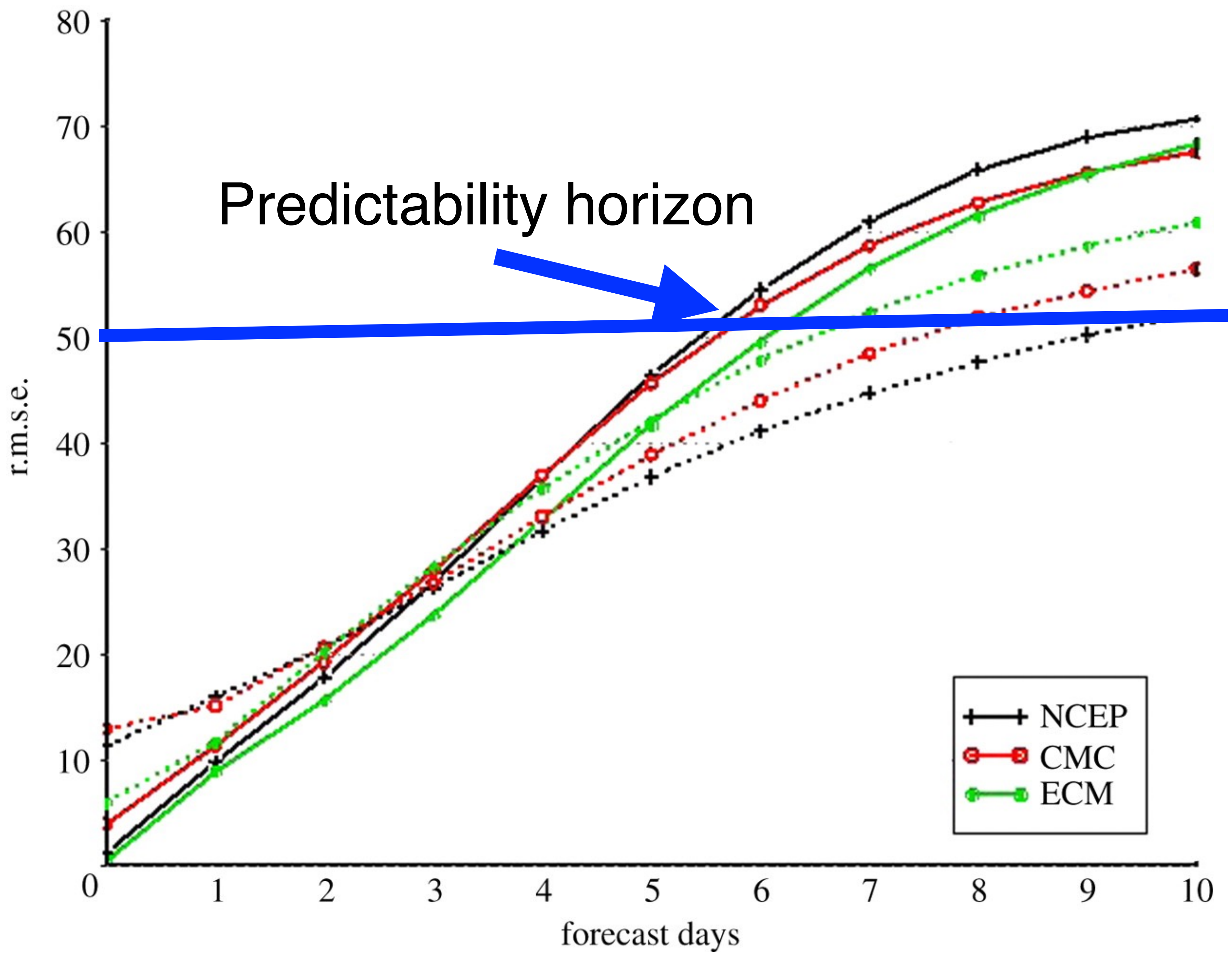
If the forecasts are coherent (small spread) the atmosphere is in a more predictable state than if the forecasts diverge (large spread)

Processes limiting predictability: formation of High and Low pressure systems and their interaction



Positive Feedback - > Instability of the Jet Stream

Predictability limits



Statistics of ensemble mean forecast error (r.m.s.e.; solid line) and ensemble spread (dotted line) in Northern Hemisphere systems

Data based methods: Machine Learning

"Learning is any process by which a system improves performance from experience."

- Herbert Simon

Definition by Tom Mitchell (1998):

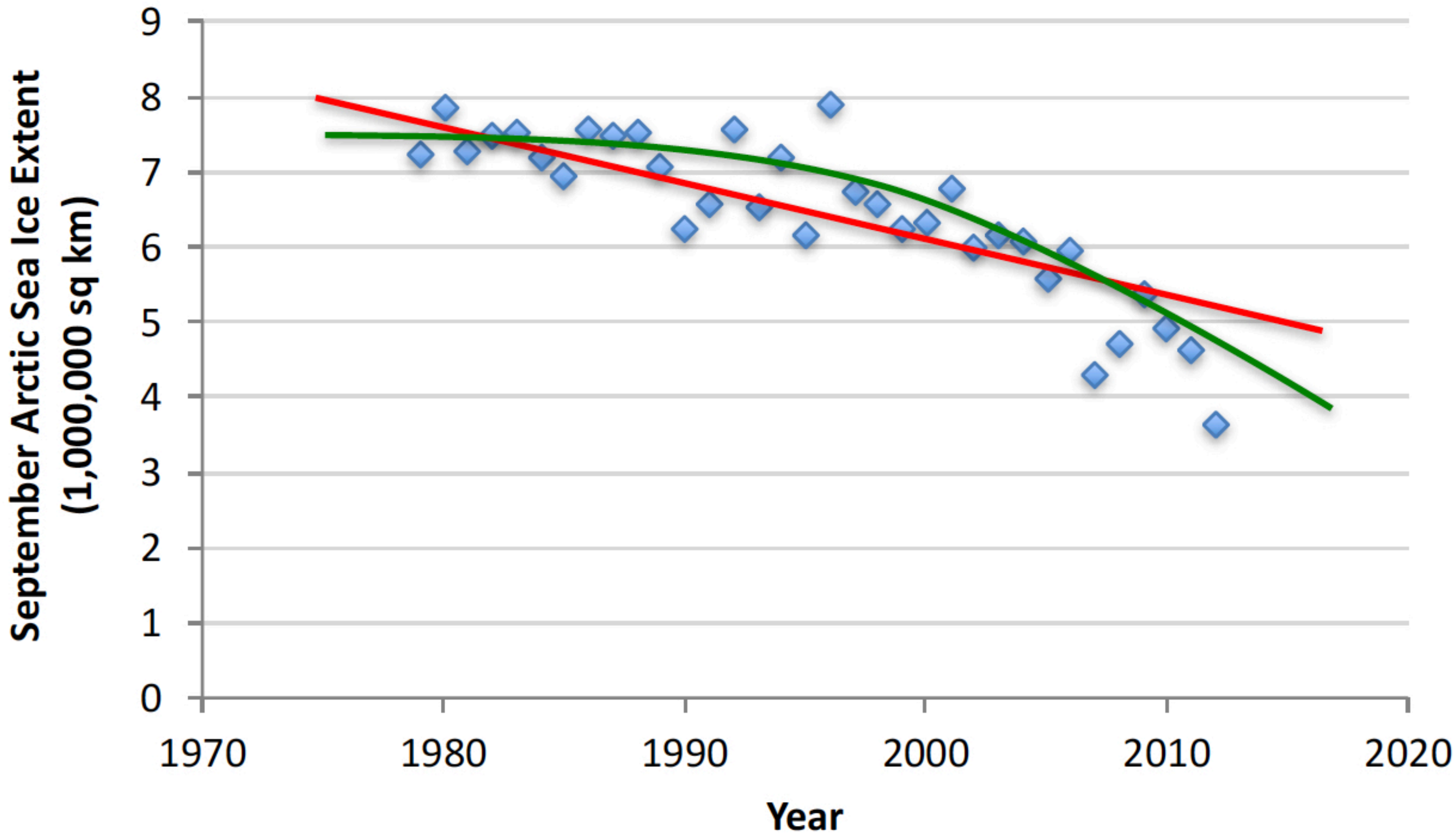
Machine Learning is the study of algorithms that

- improve their performance P
- at some task T
- with experience E.

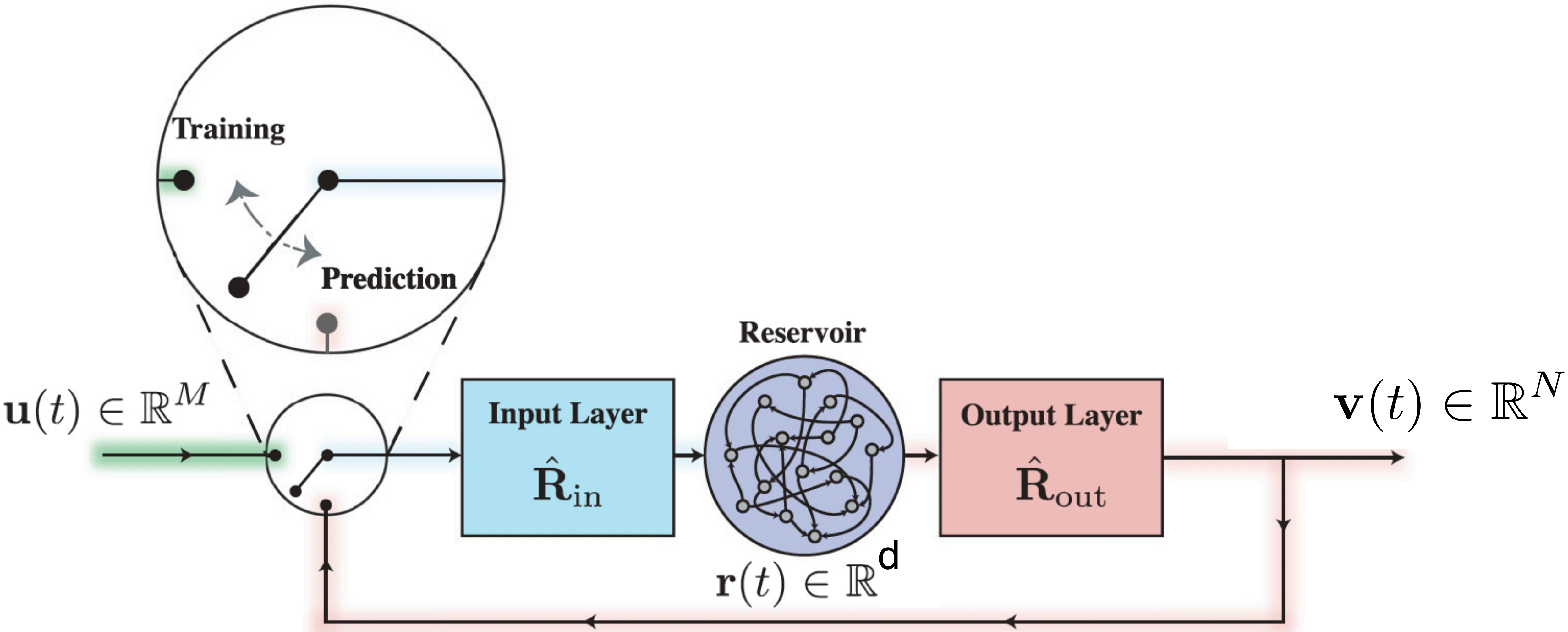
A well-defined learning task is given by <P, T, E>.

Supervised Learning: regression

- Given $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Learn a function $f(x)$ to predict y given x
  - y is real-valued == regression

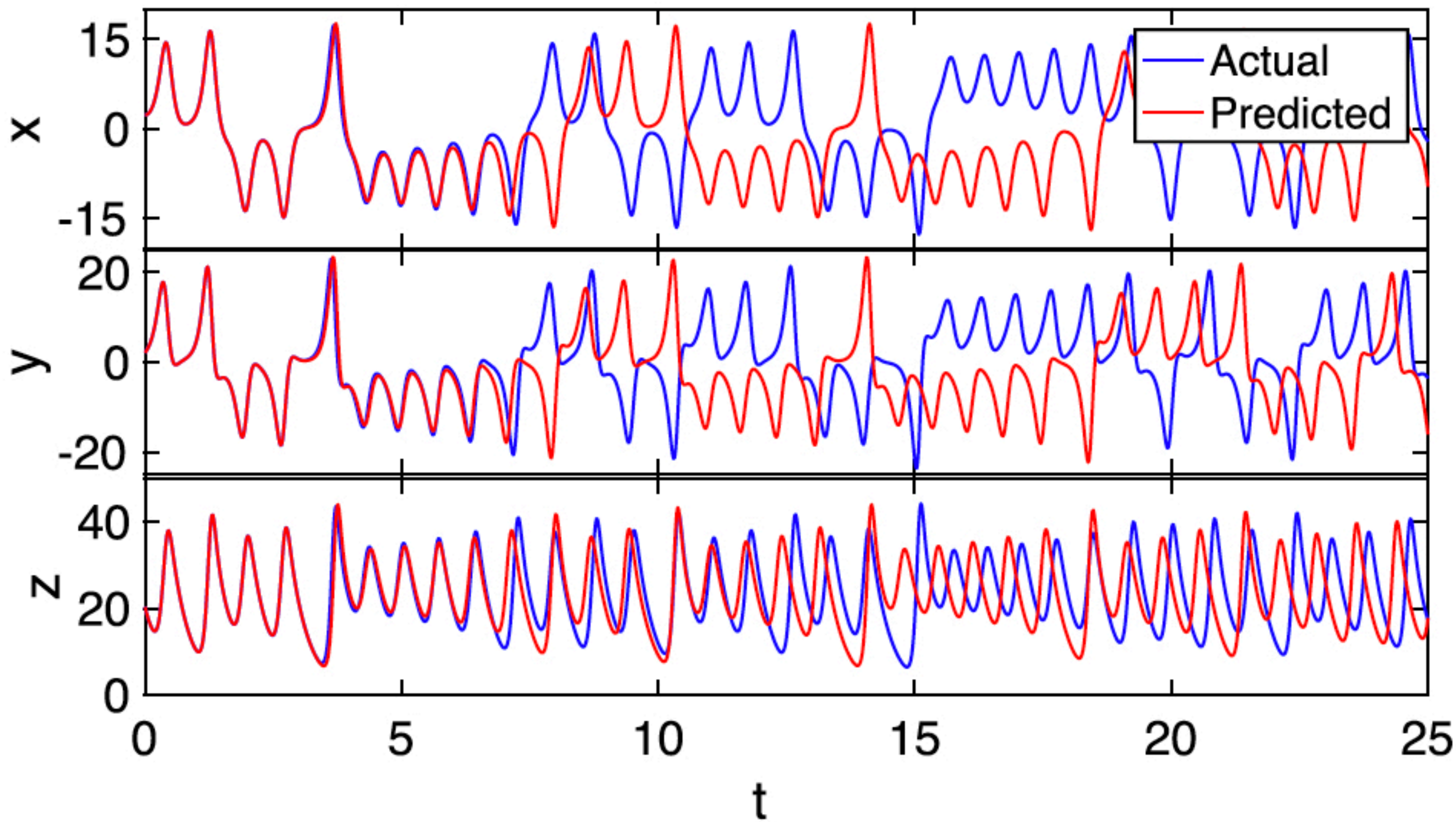


Reservoir Computing



Pathak et al., Chaos, (2017, 2018)

Results for the Lorenz'63 model



Summary: Numerical Weather Prediction

Chaotic behavior plays a very different role in uncertainties of weather and climate forecasting

Sensitivity to initial conditions; Lyapunov exponent

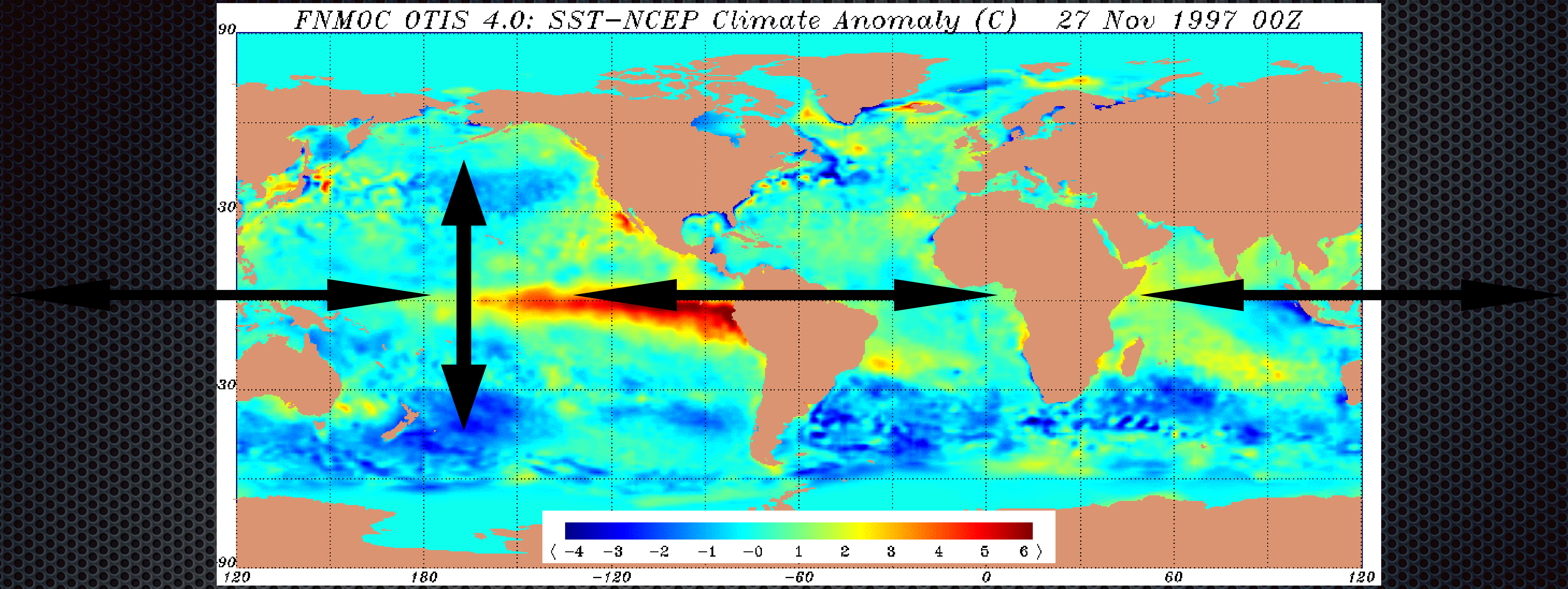
Future weather forecasts:

- relevant processes are instabilities of the large-scale atmospheric circulation with typical time scales of up to 5 days
- limited prediction skill beyond 10 days
- Reservoir Computing methods are promising to extend this horizon

Ensemble forecasting using Singular Vectors

Question time

Aspects of ENSO predictability

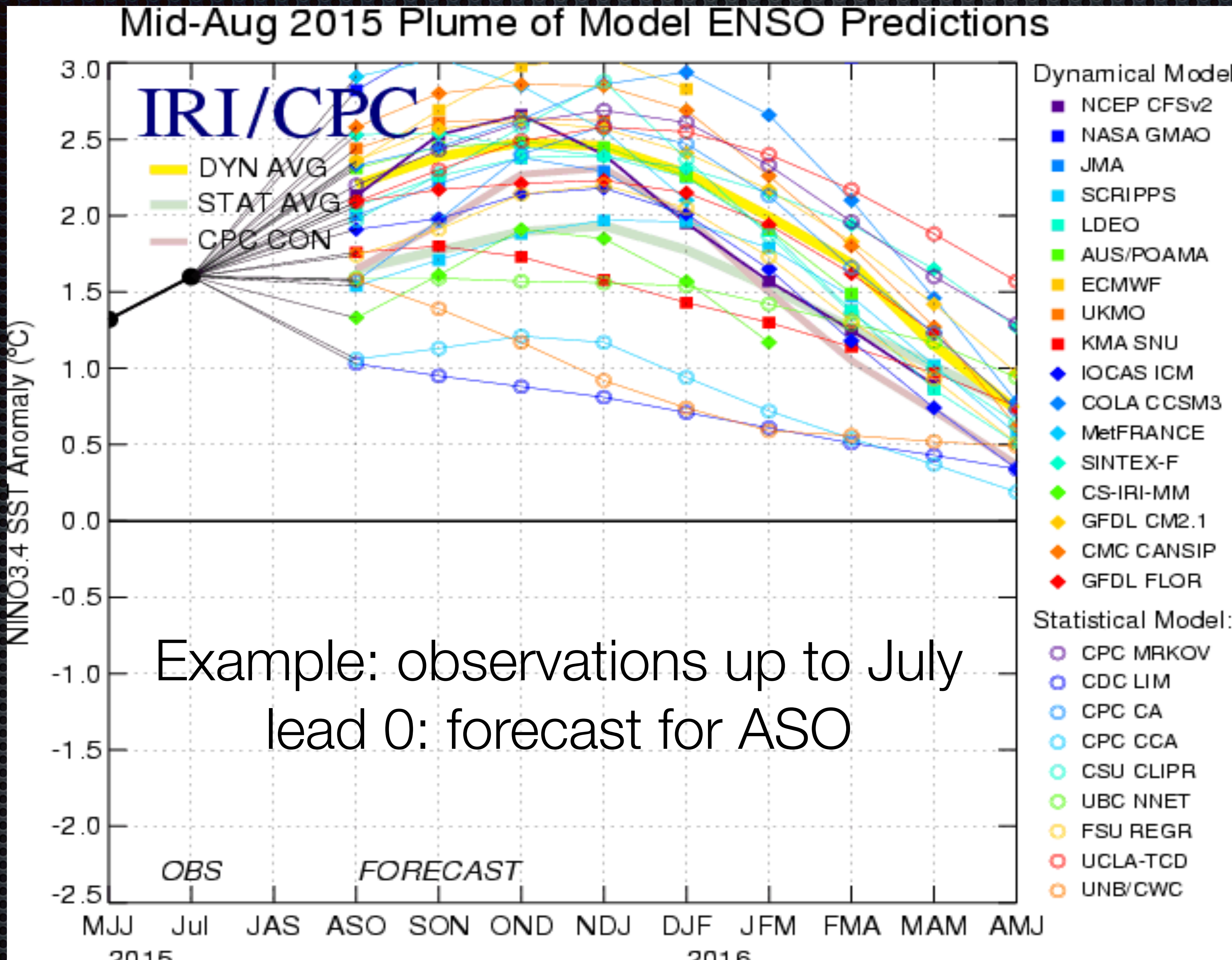


Internal (possibly chaotic) variability
Memory: ocean adjustment
Strong atmospheric high frequency variability

`External' variation of the background climate
Effects of Indian Ocean & Atlantic Ocean
Tropical - Extratropical interactions in the Pacific

Procedure

NINO3.4



Example: observations up to July
lead 0: forecast for ASO

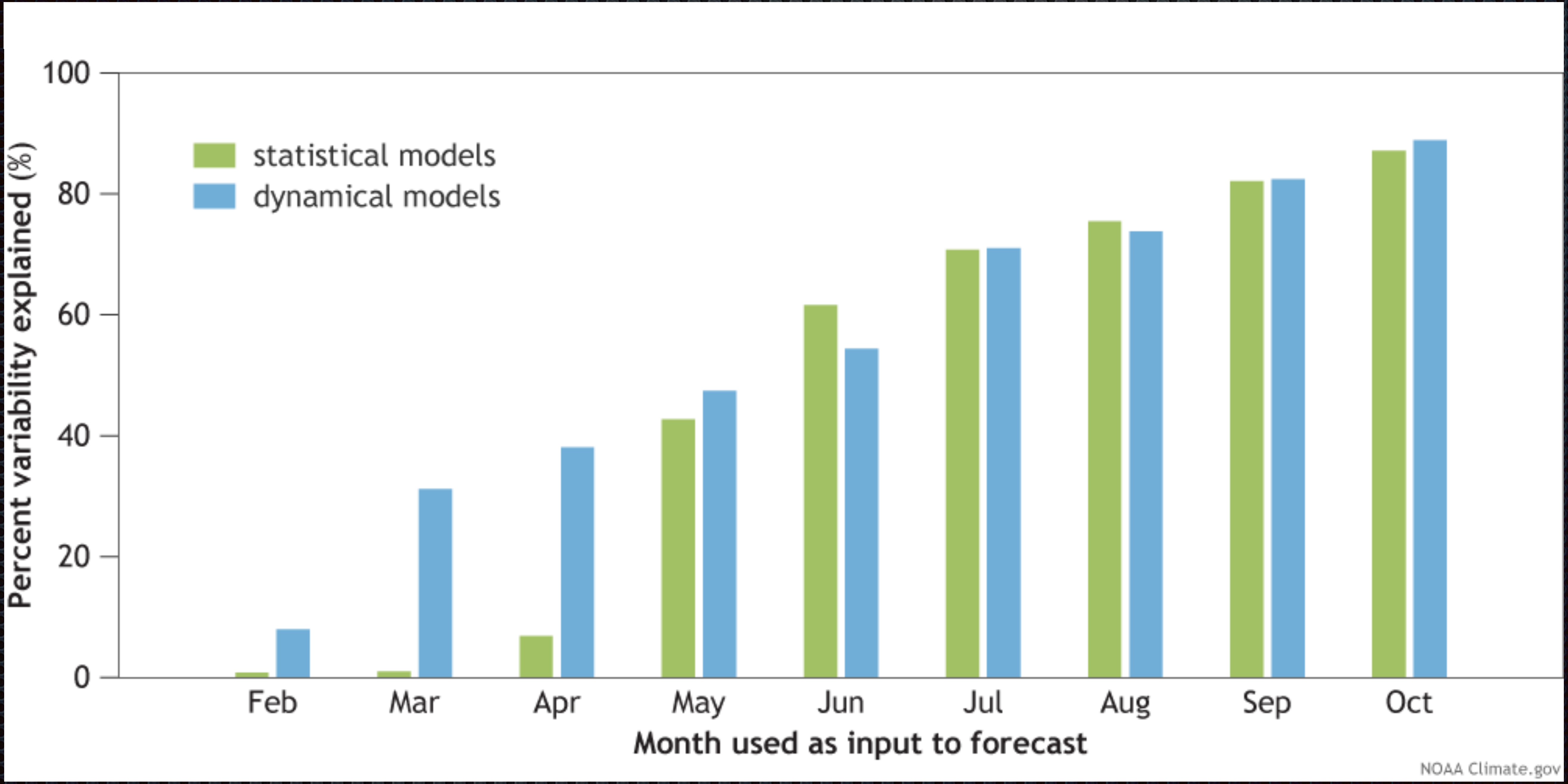
Figure provided by the International ... and Society (updated 18 August 2015).

target season

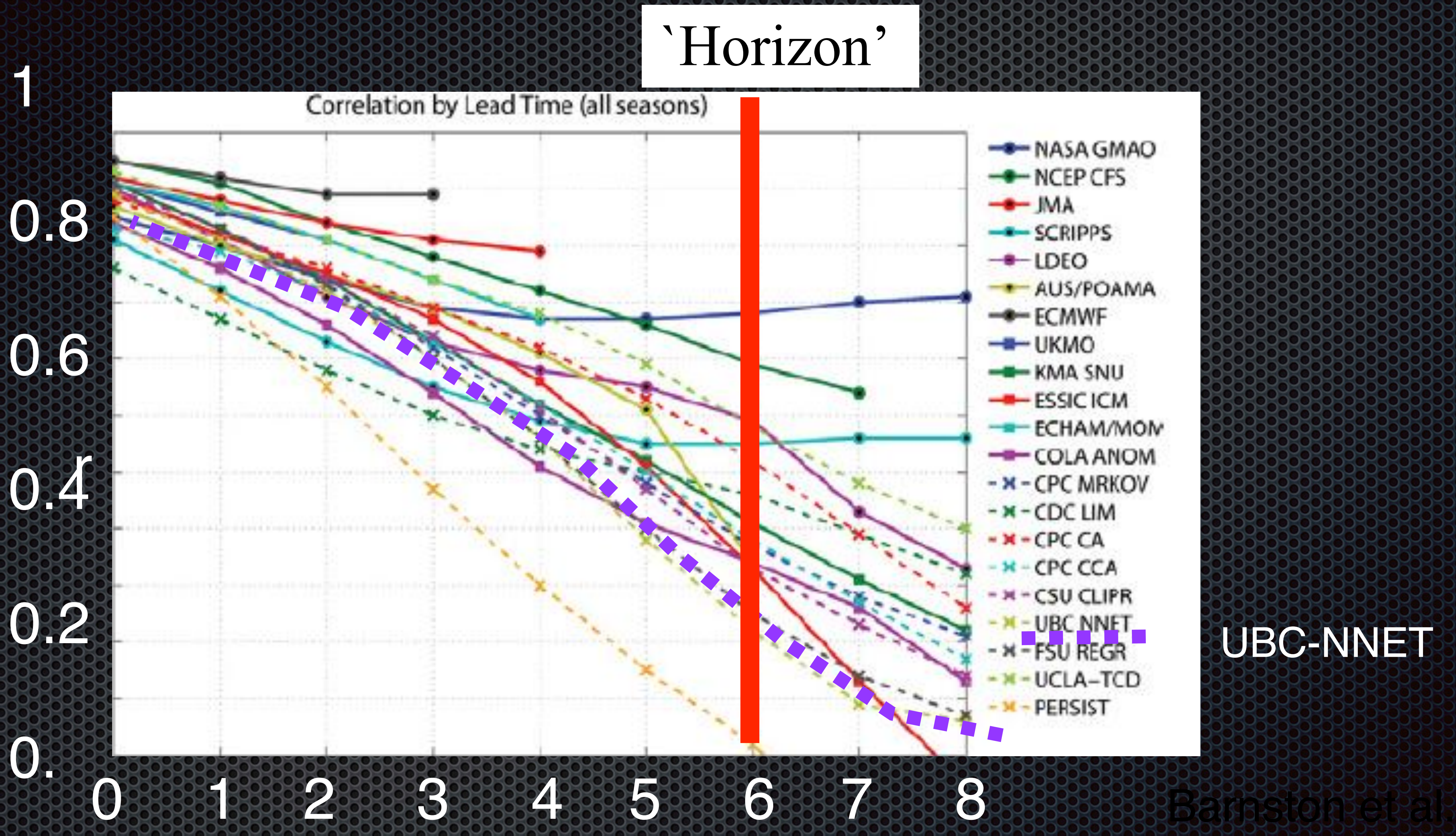
different models

Spring Predictability Barrier, models

Nov-Jan forecast



ENSO forecast skill 2002-2011

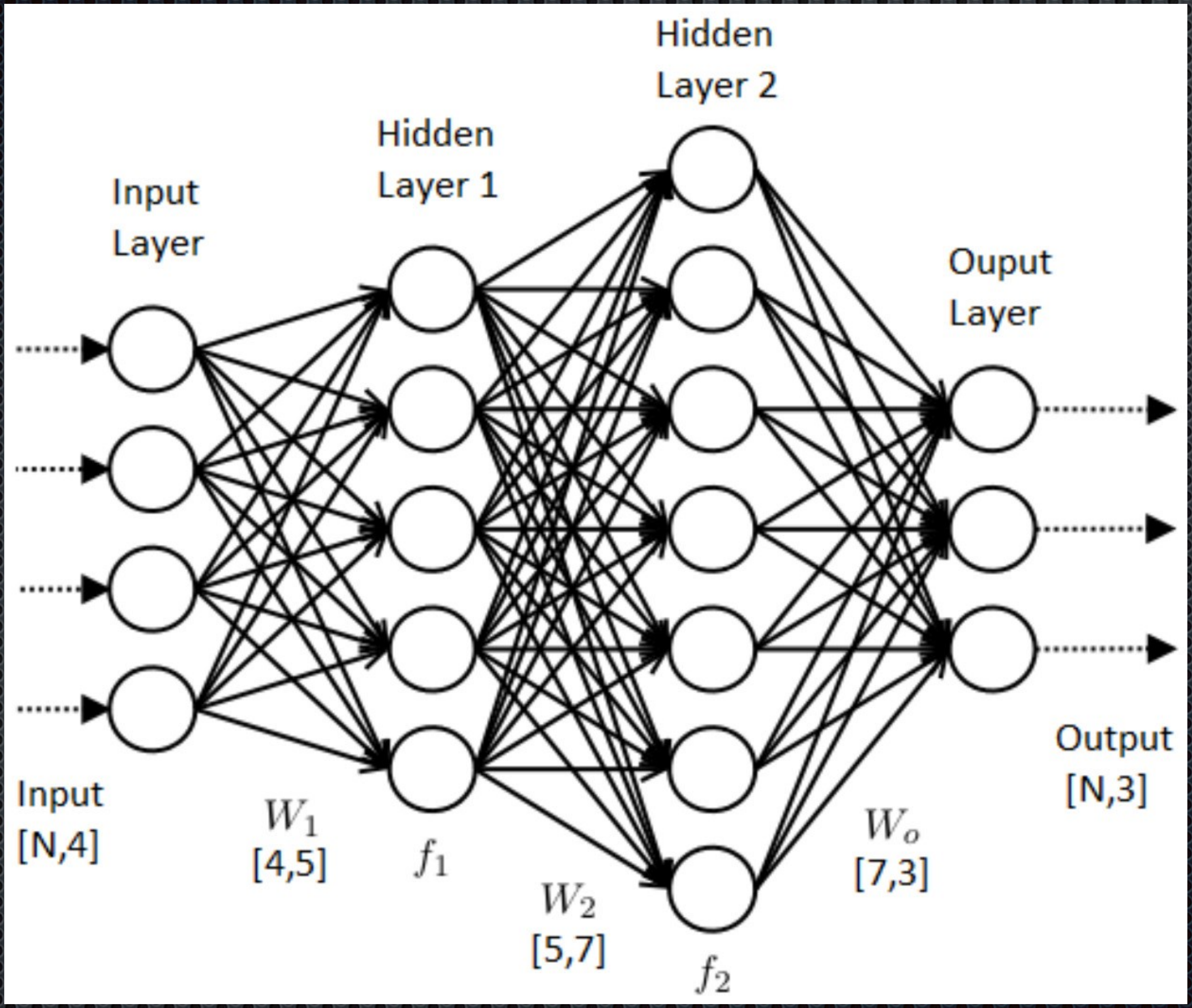


Barnston et al. (2012)

Lead Time [Month]

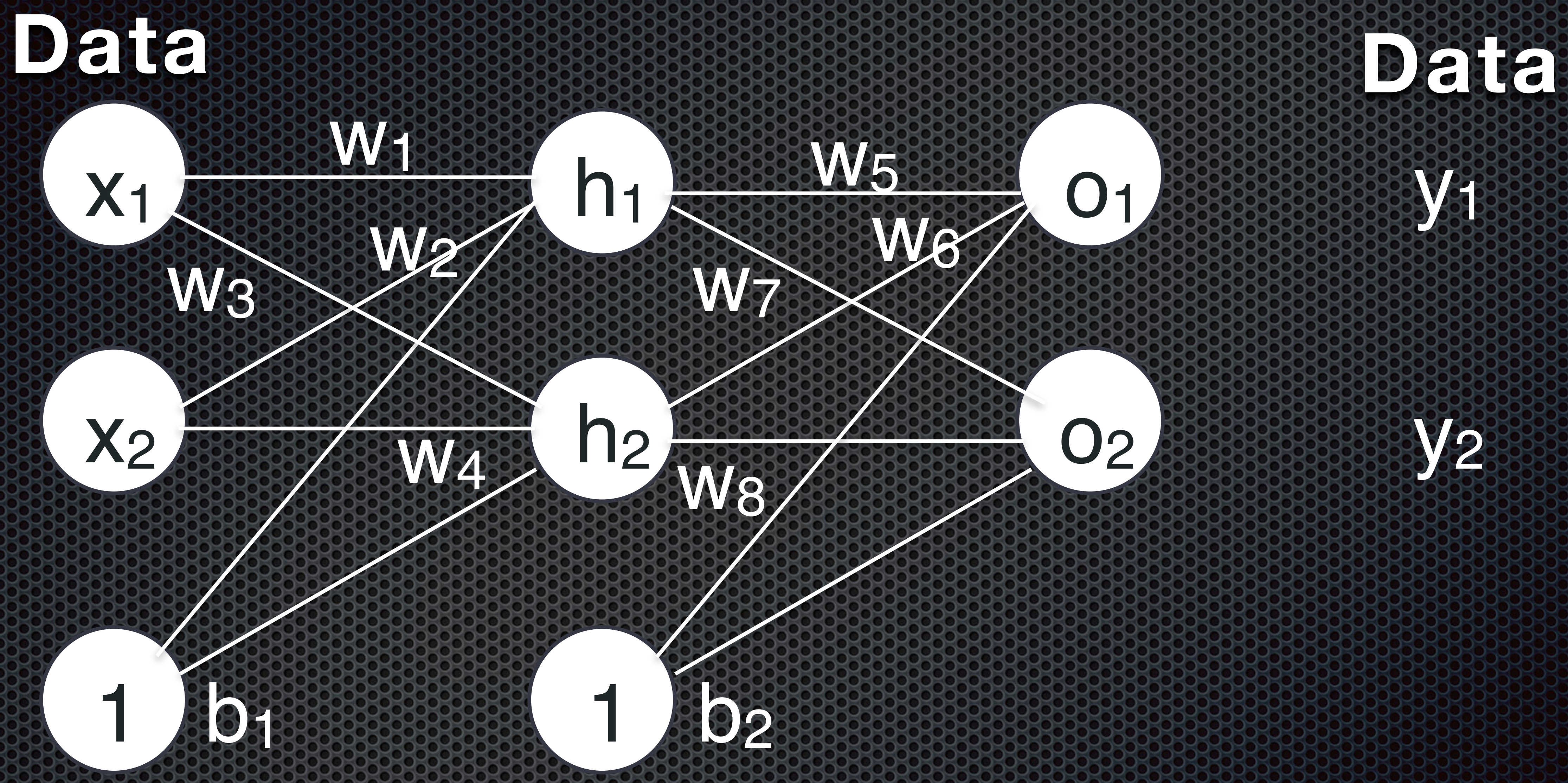
Barnston et al. (2011)

Machine Learning: Artificial Neural Networks (ANNs)



Input: Attributes/Features

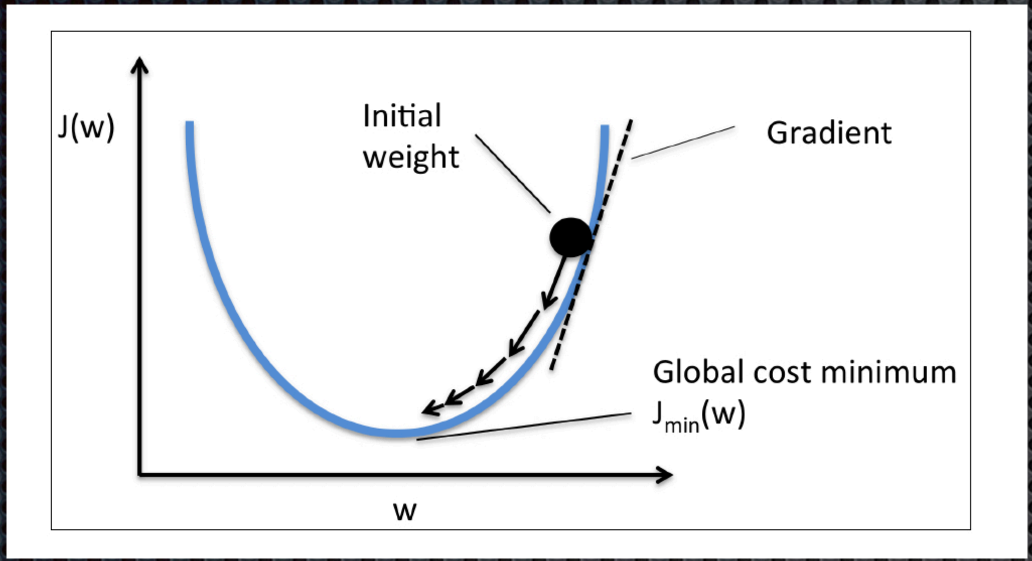
Example ANN with simple cost function



Cost function: $J(\mathbf{w}) = \frac{1}{2}((y_1 - o_1)^2 + (y_2 - o_2)^2)$

Training problem

Solve: $min_{\mathbf{w}}\ J(\mathbf{w})$



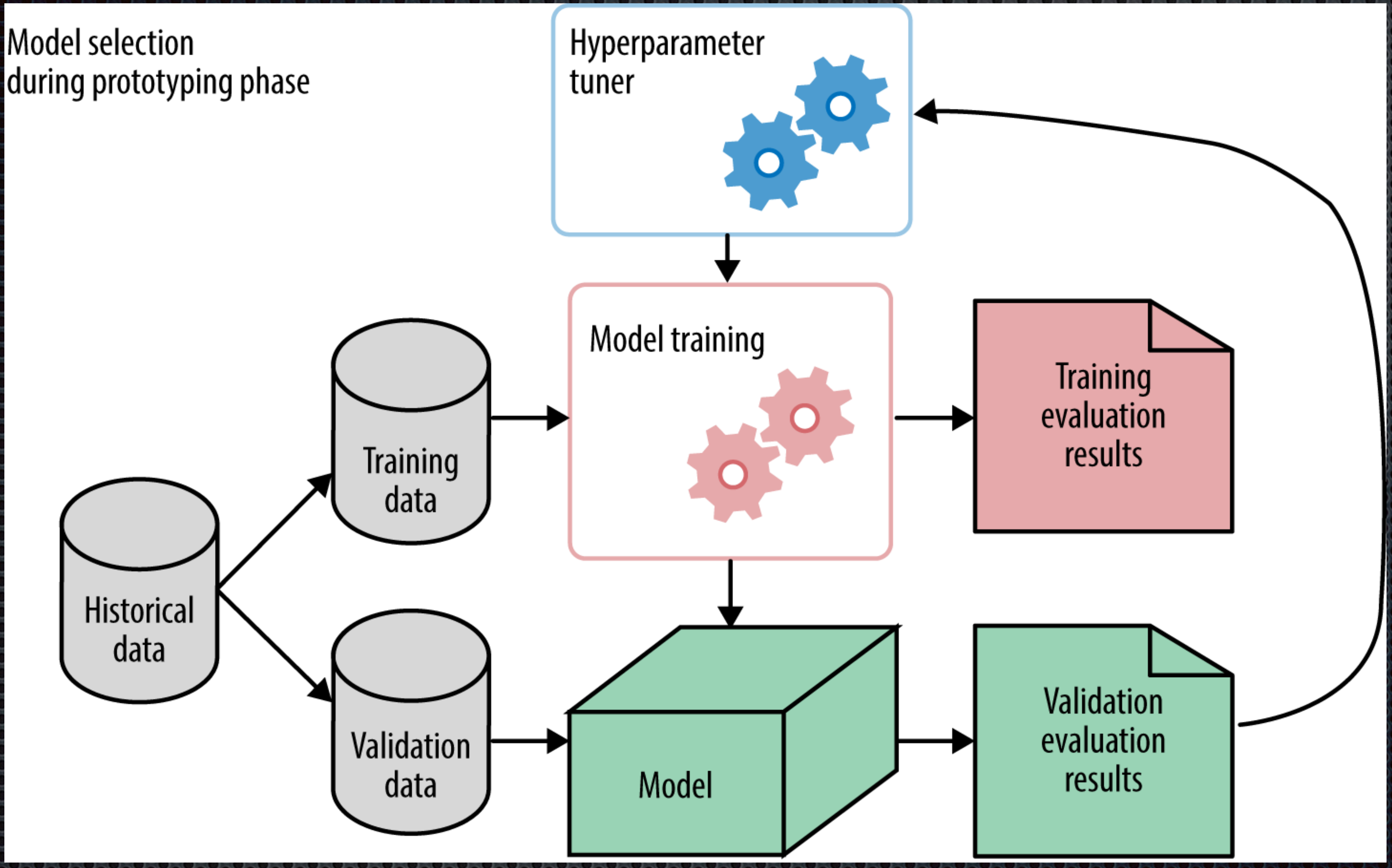
Method: Gradient Descent

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \Delta\mathbf{w}_{k+1}$$

$$\Delta\mathbf{w}_{k+1} = -\eta\nabla J(\mathbf{w}_k)$$

Solution: back propagation

Approach: model selection



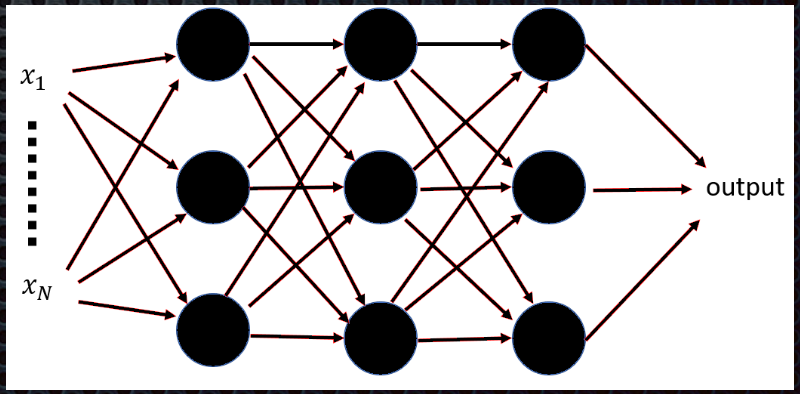
Hybrid Prediction Model

- ARIMA(p,d,q) (standard taken p = 12, d = 1, q = 1)

- Artificial Neural Network (ANN) applied to residual (2 x 1 x 1: three layers with 2, 1 and 1 neuron)

Error measure:

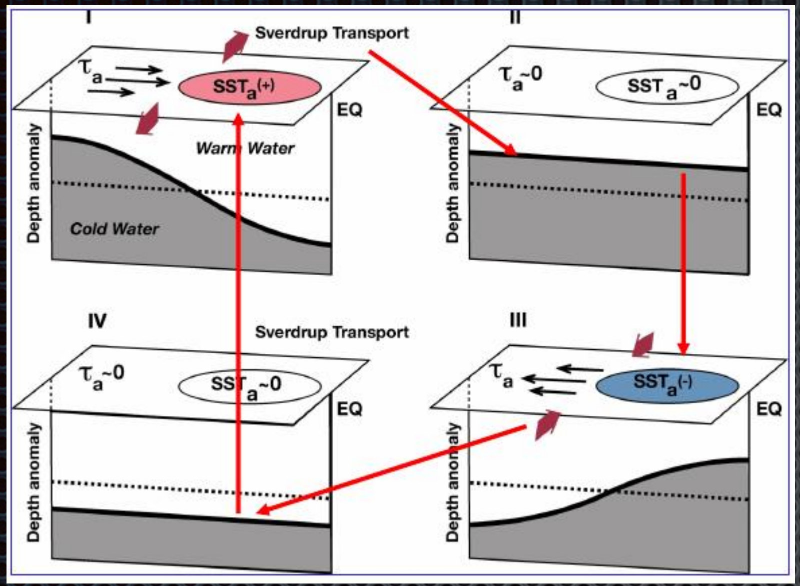
$$\mathrm{NRMSE}(y^A, y^B) = \frac{1}{\max\left(y^A, y^B\right) - \min\left(y^A, y^B\right)} \times \sqrt{\frac{\sum_{t_1^{\text{test}} \le t_k \le t_n^{\text{test}}} \left(y_k^A - y_k^B\right)^2}{n}}.$$



Nooteboom et al. ESD (2018)

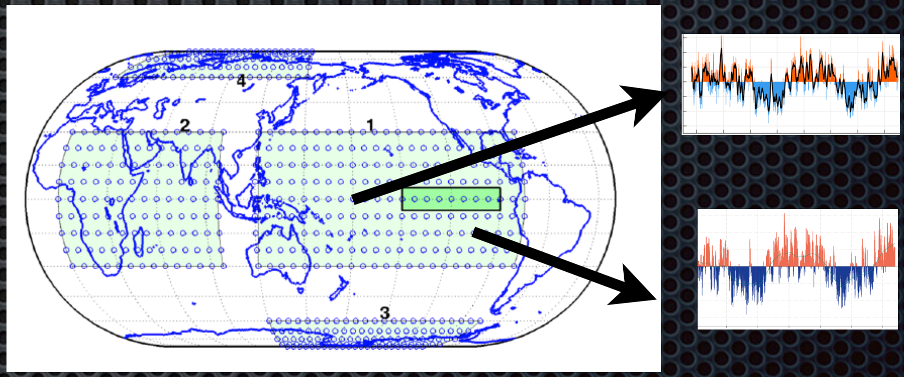
Attributes

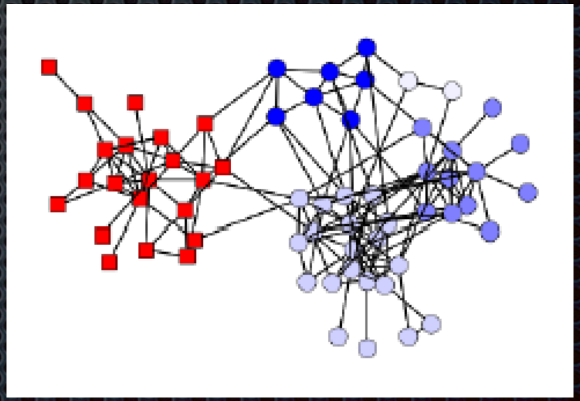
Warm Water Volume (WWV)



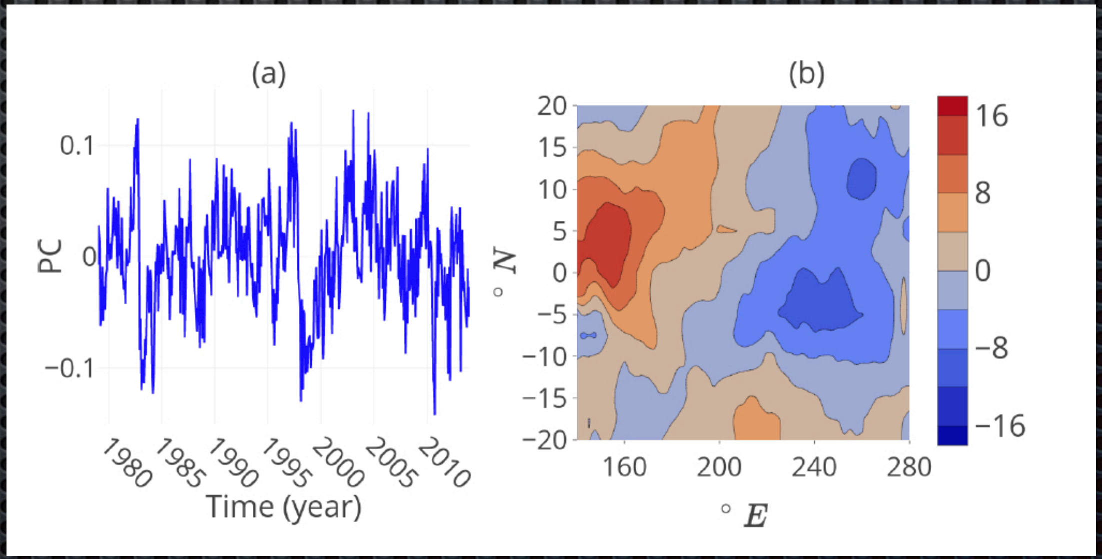
Evolving Complex Network measures (SSH, c_2)

$$c_s = \frac{sn_s}{N}$$

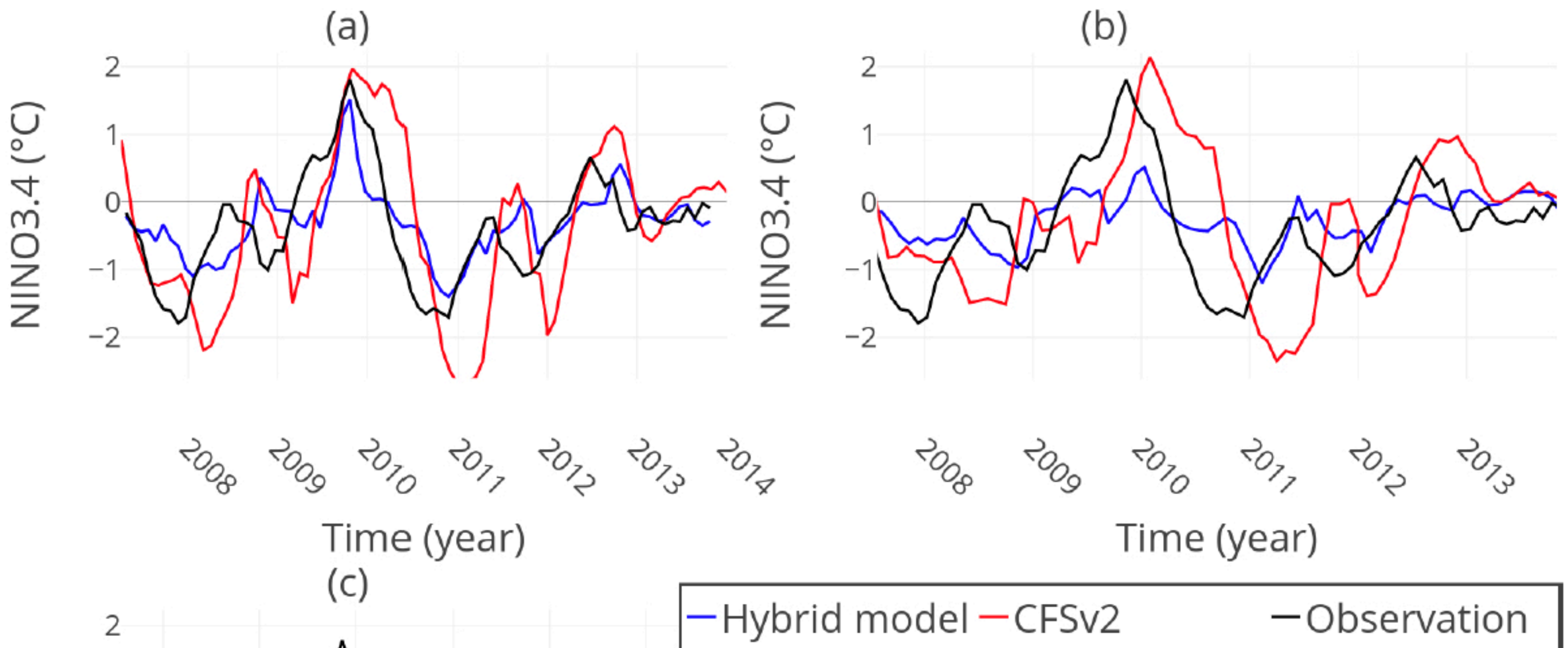


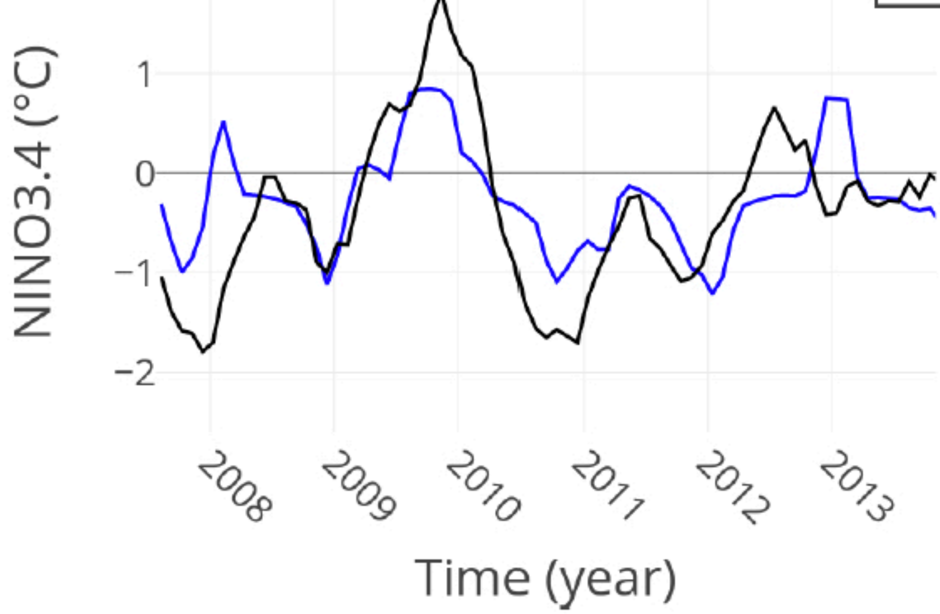


Wind-stress noise (PC_2)



Results

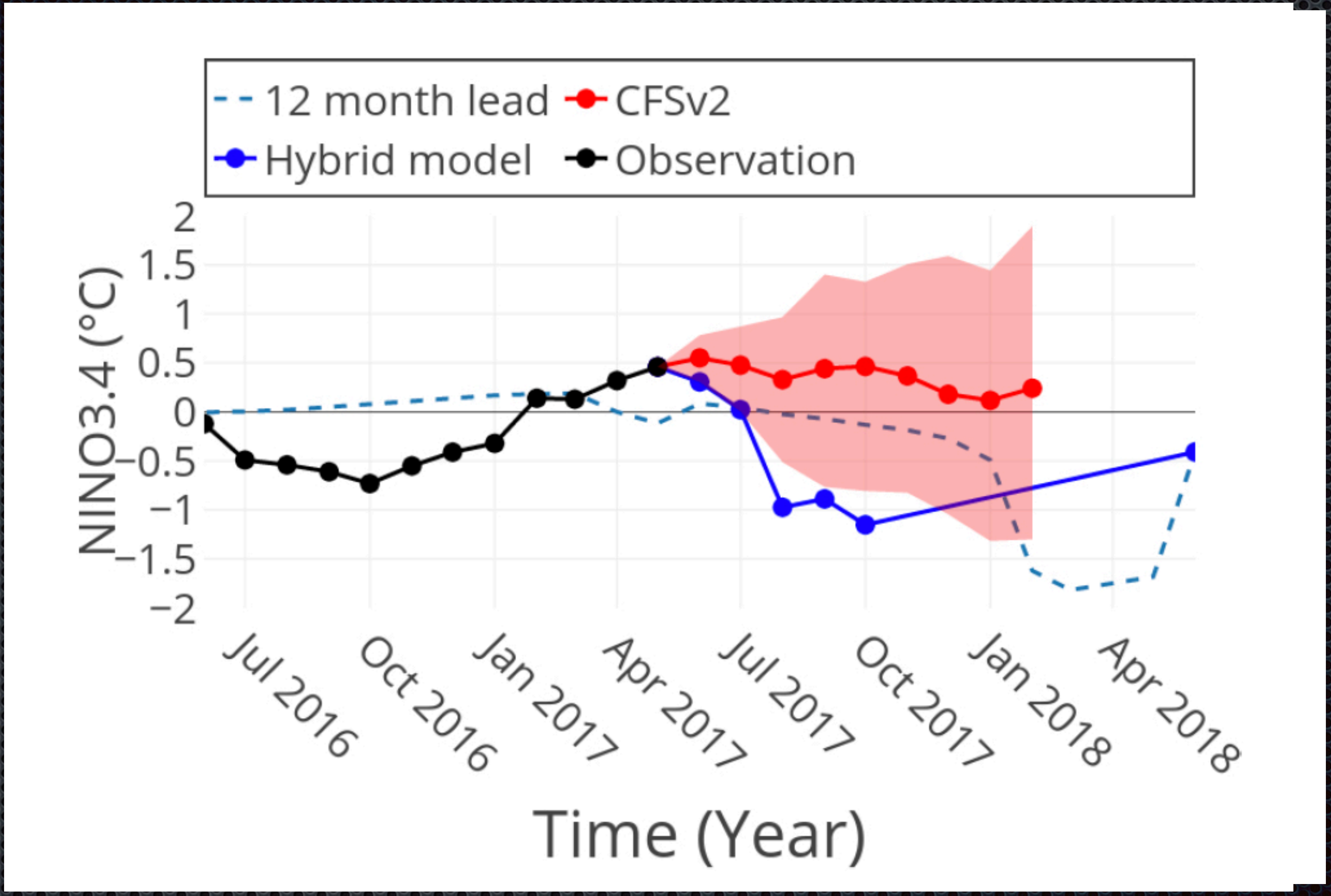




(d)

Lead time	3/4 months	6 months	12 months
Structure	$4 \times 3 \times 1$	$4 \times 2 \times 2$	$3 \times 1 \times 4$
NRMSE	$0.17/0.16$	$0.21/0.18$	$-/0.17$
Attributes	PC_2, WWV, SC	PC_2, WWV, SC	PC_2, c_2, SC

Prediction hybrid model



Question time

Summary: ENSO Prediction

The predictability of El Nino is limited by a Spring Predictability Barrier where growth of model errors is largest. Skill at 6 months lead time is only about 0.5.

Deep Learning (Artificial Neural Networks) is a powerful method for skillful ENSO forecasting (with many pitfalls) beyond the Spring Predictability Barrier

Main problem: insufficient data