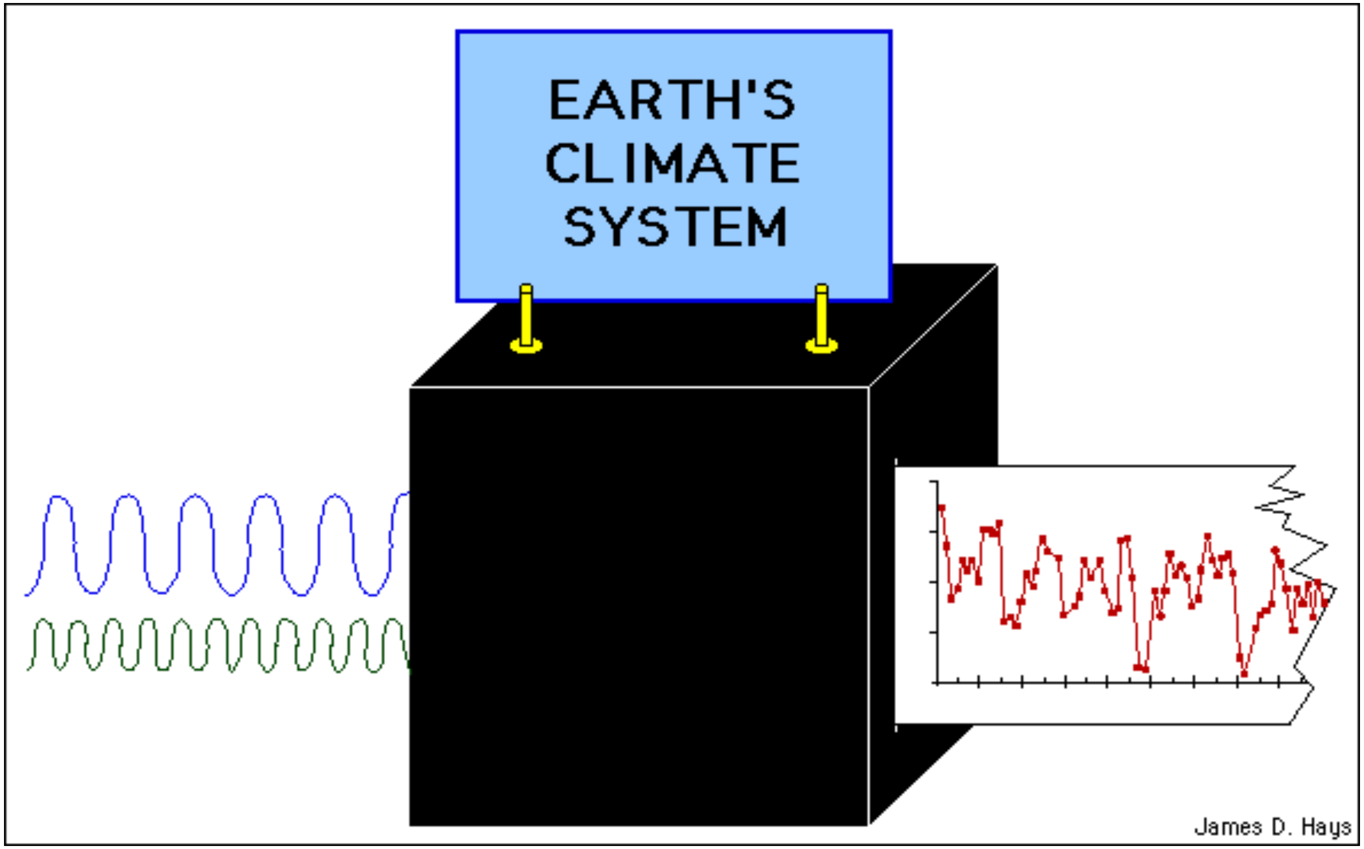
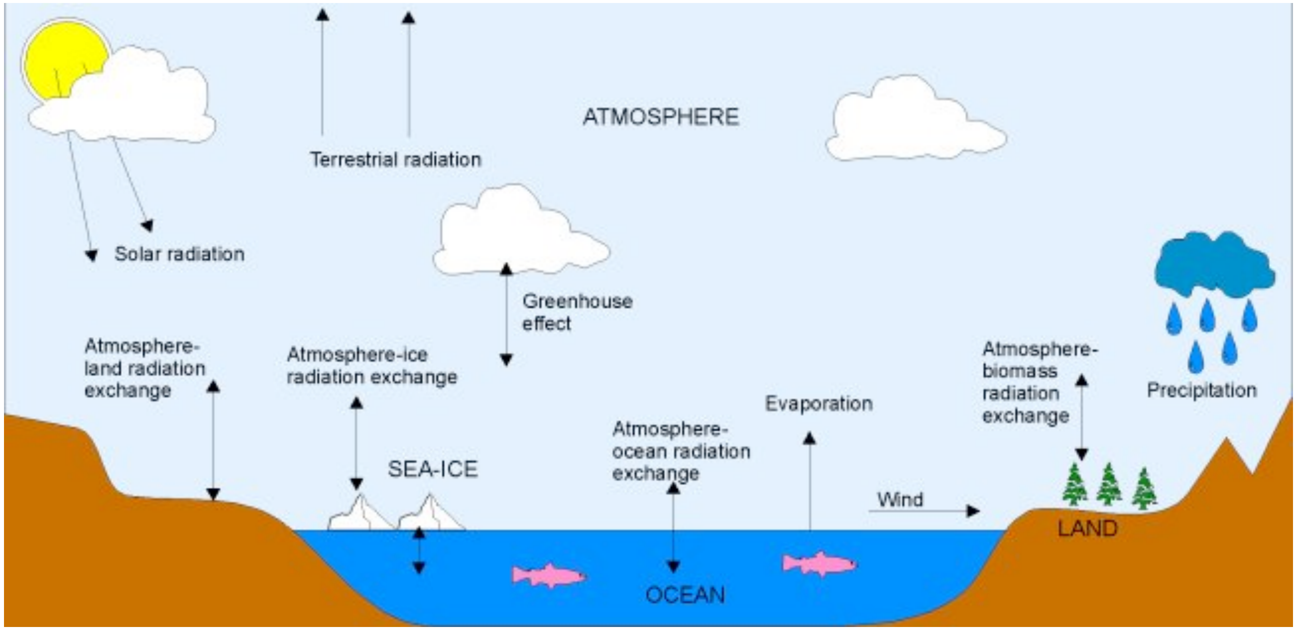
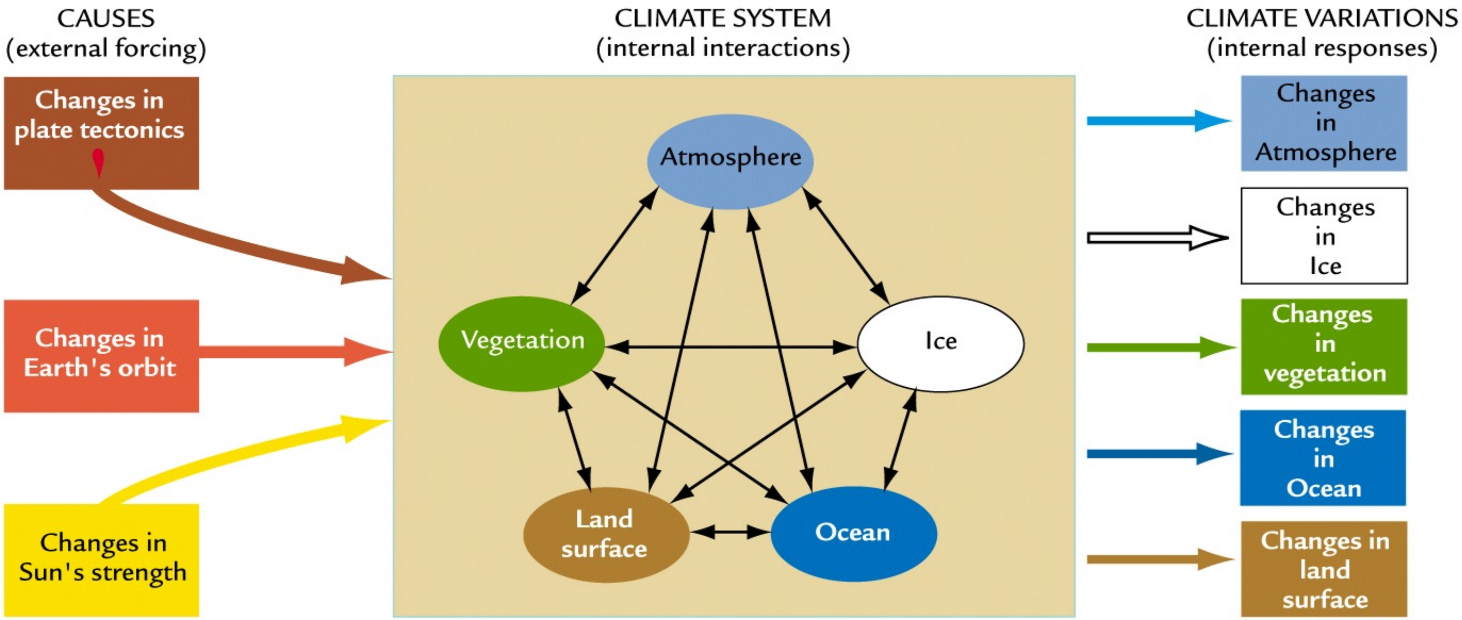
Stochastic Climate Dynamics



Henk Dijkstra, IMAU & CCSS
Physics Department, Utrecht University, Utrecht, Netherlands

Climate System

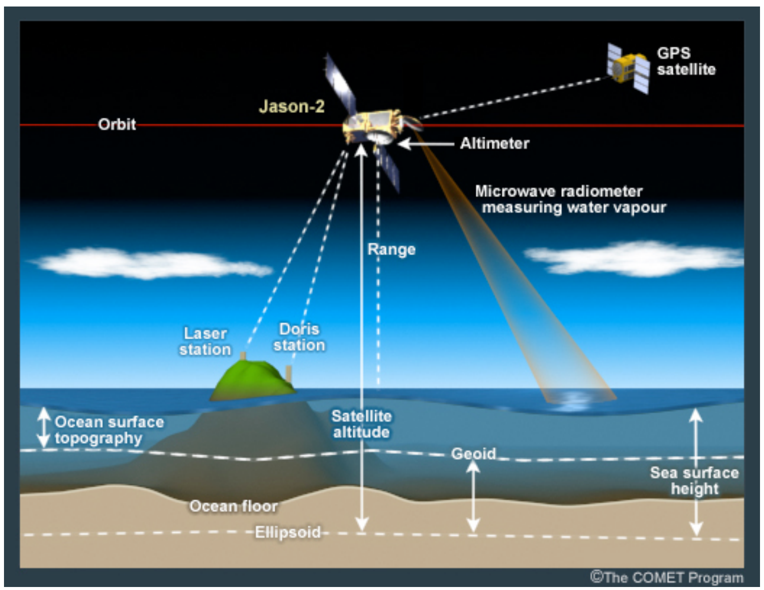




Observations

Instrumental data
(~ 1880 -)

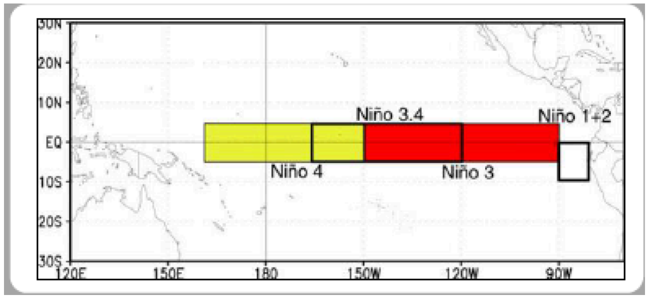
(1992 -)



Proxy data
(geological past)

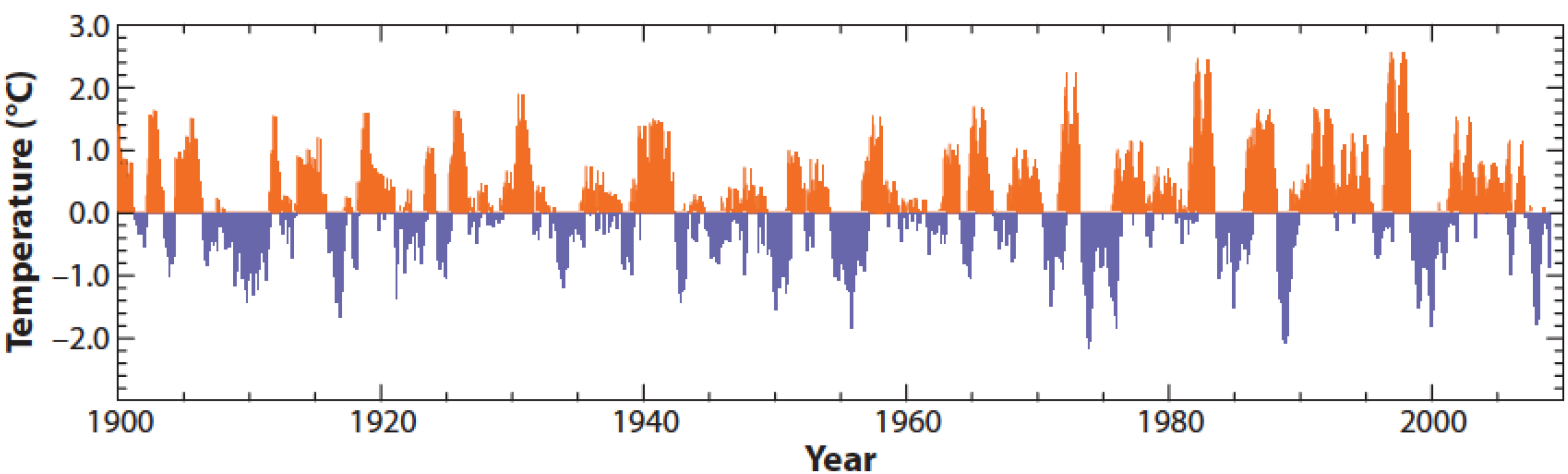
Very limited for hypotheses/theory falsification

Example instrumental data: El Nino/Southern Oscillation (ENSO)



NINO3.4 index

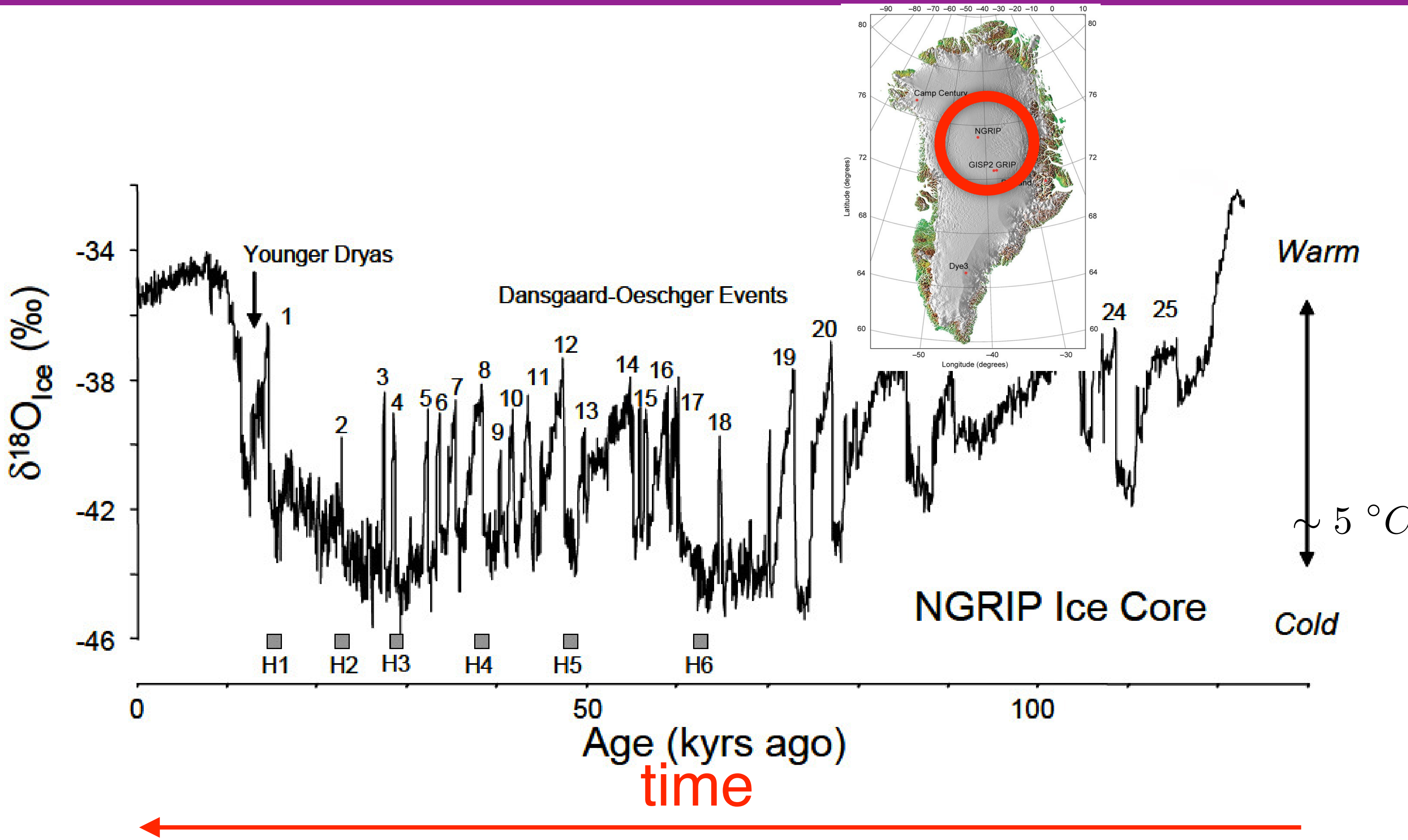
sea surface temperature anomaly equatorial Pacific



http://climexp.knmi.nl

Deser et al., Ann. Rev. Mar. Sc. (2010)

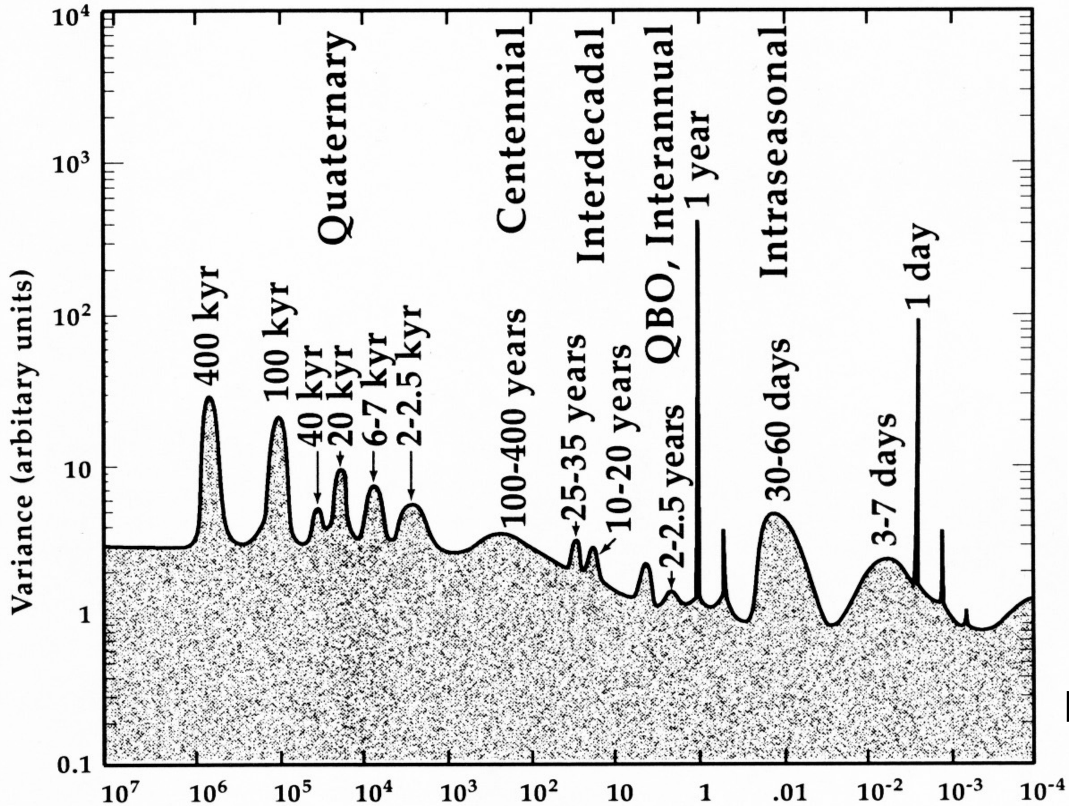
Example proxy data:
Ice core oxygen isotope record



https://www.ncdc.noaa.gov/data-access/paleoclimatology-data/datasets

Traditional (artist's) view of Variability in the Climate System

'Energy'

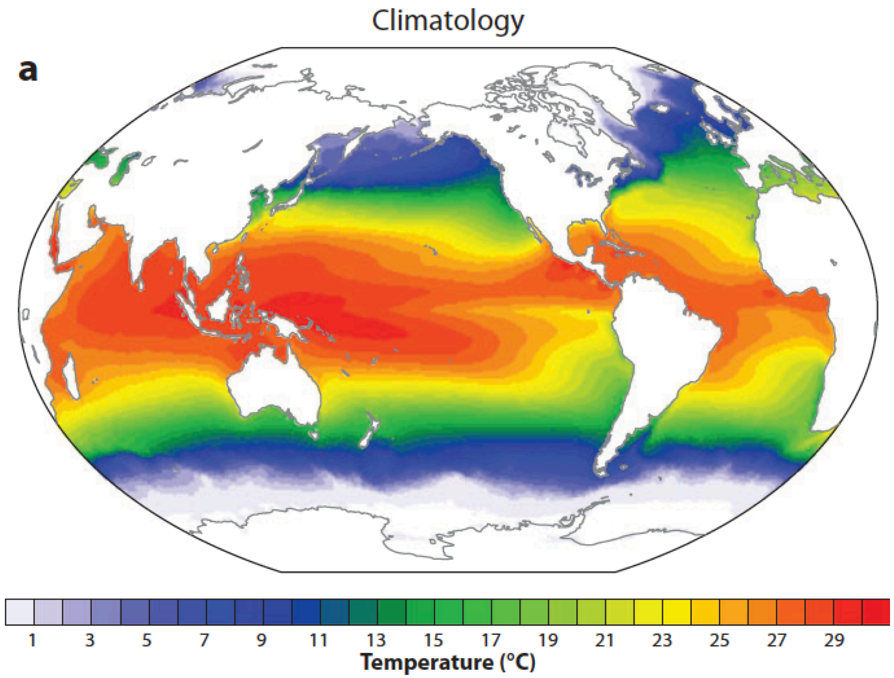


Mitchell, 1978

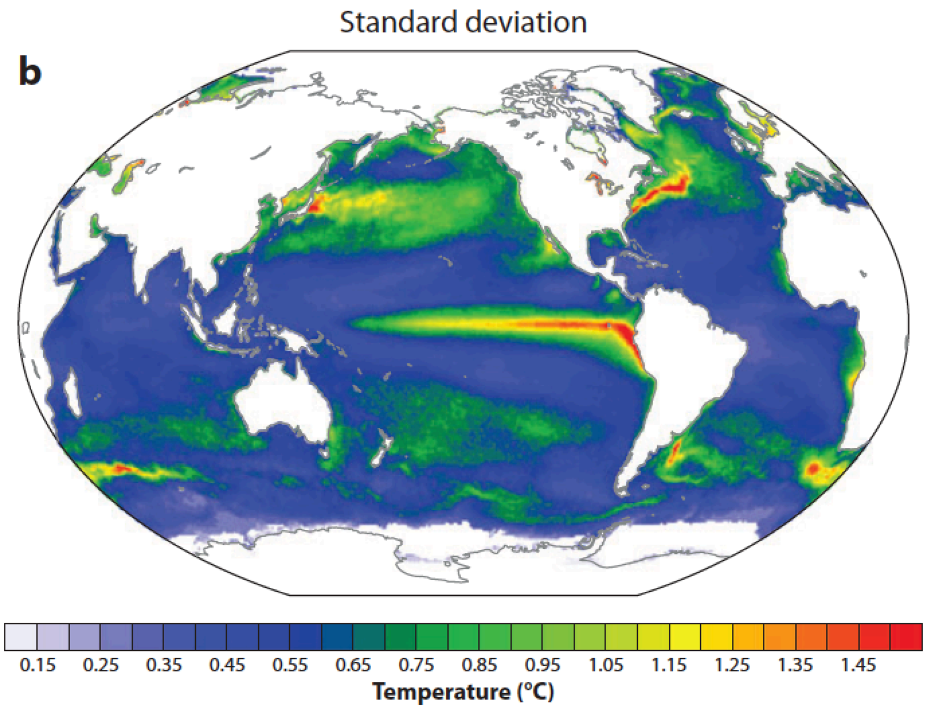
Period (years)

1. Sharp peaks
2. Continuous background
3. Elevated energy in frequency bands

Sea surface temperature (SST) variability

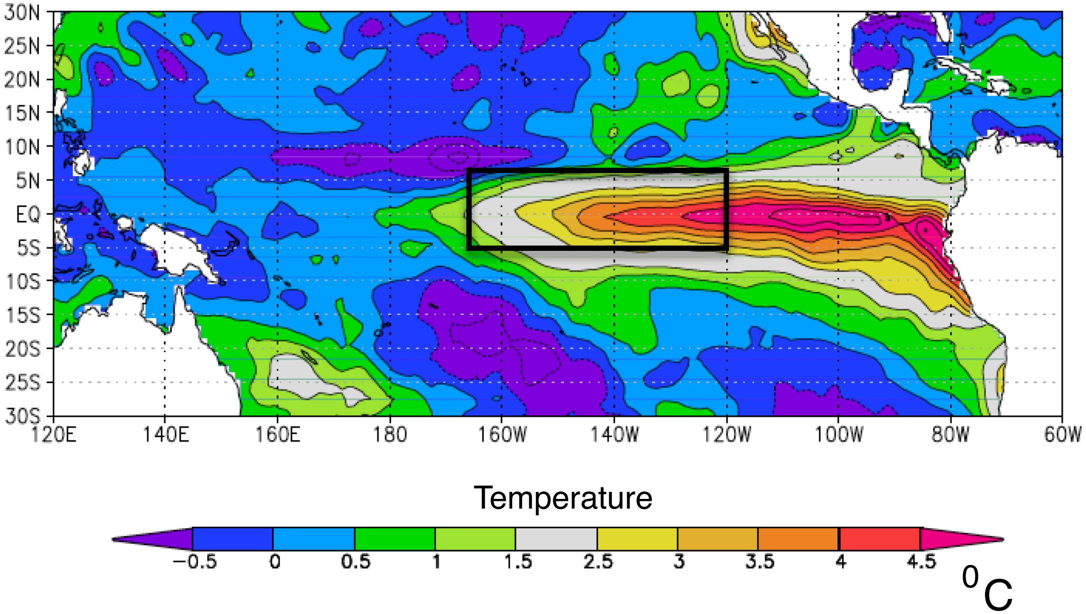


1982-2008

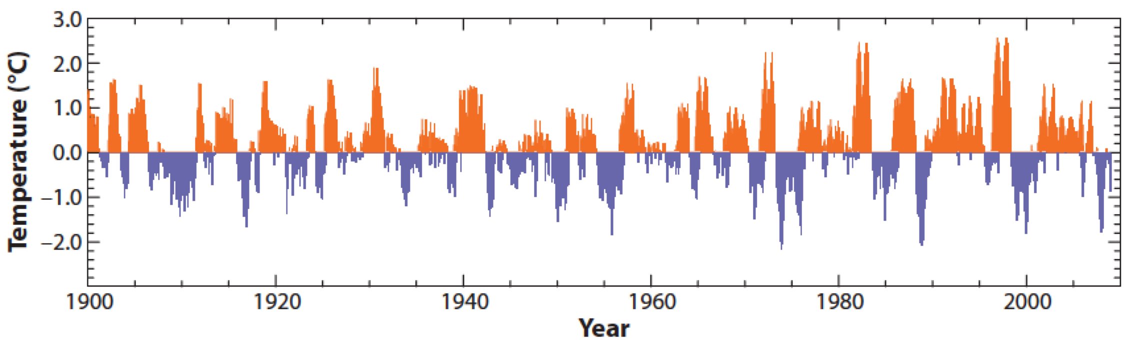


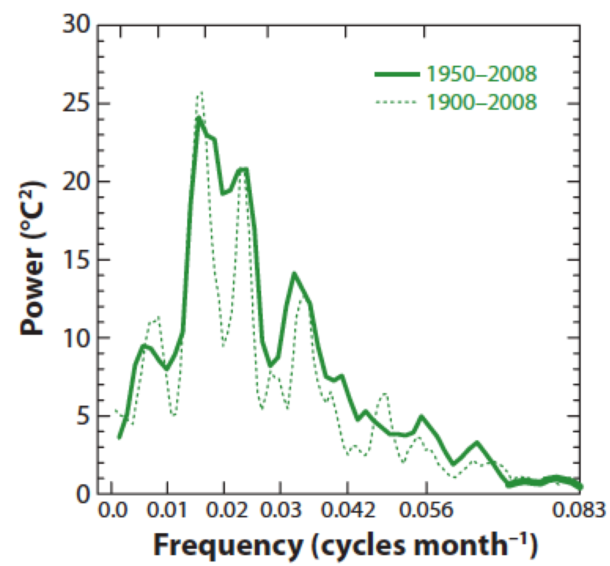
Deser et al., Ann. Rev. Mar. Sc. (2010)

Patterns of variability:
El Nino/Southern Oscillation (ENSO)



SST anomaly (1982-2010 mean) of December 1997

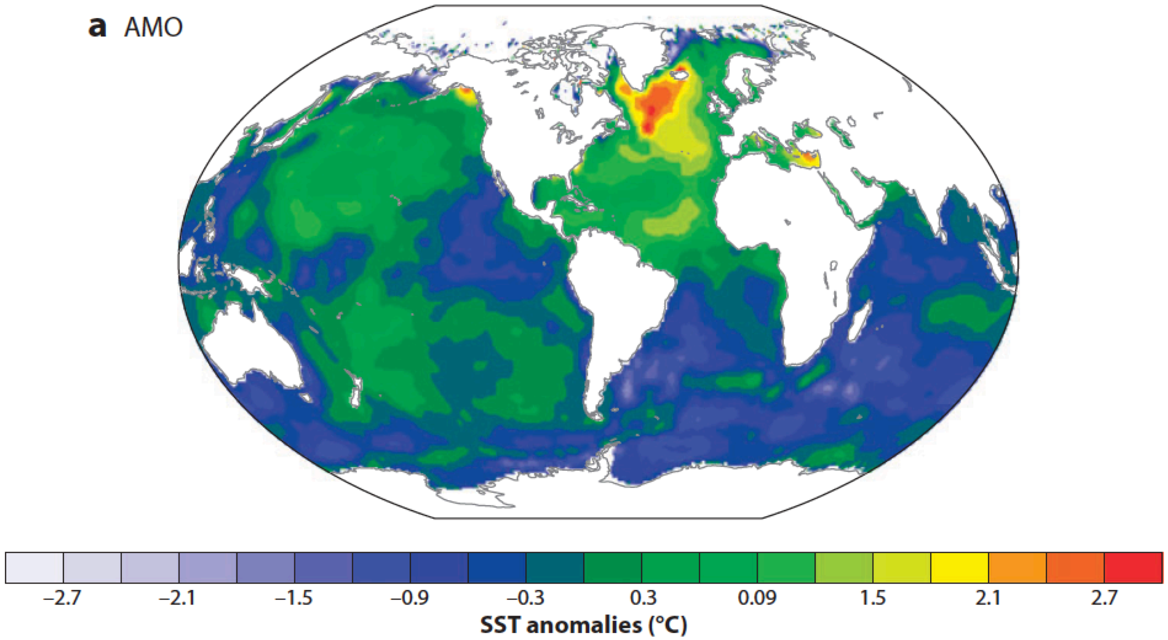




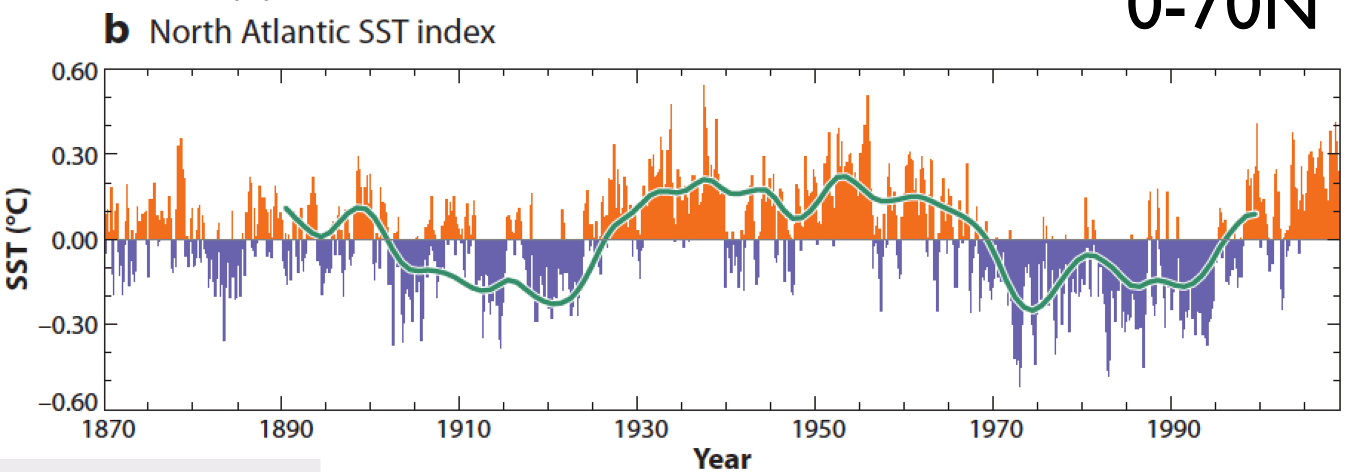
NINO3.4 index

Deser et al., Ann. Rev. Mar. Sc. (2010)

Patterns of variability:
Atlantic Multidecadal Variability (AMV)



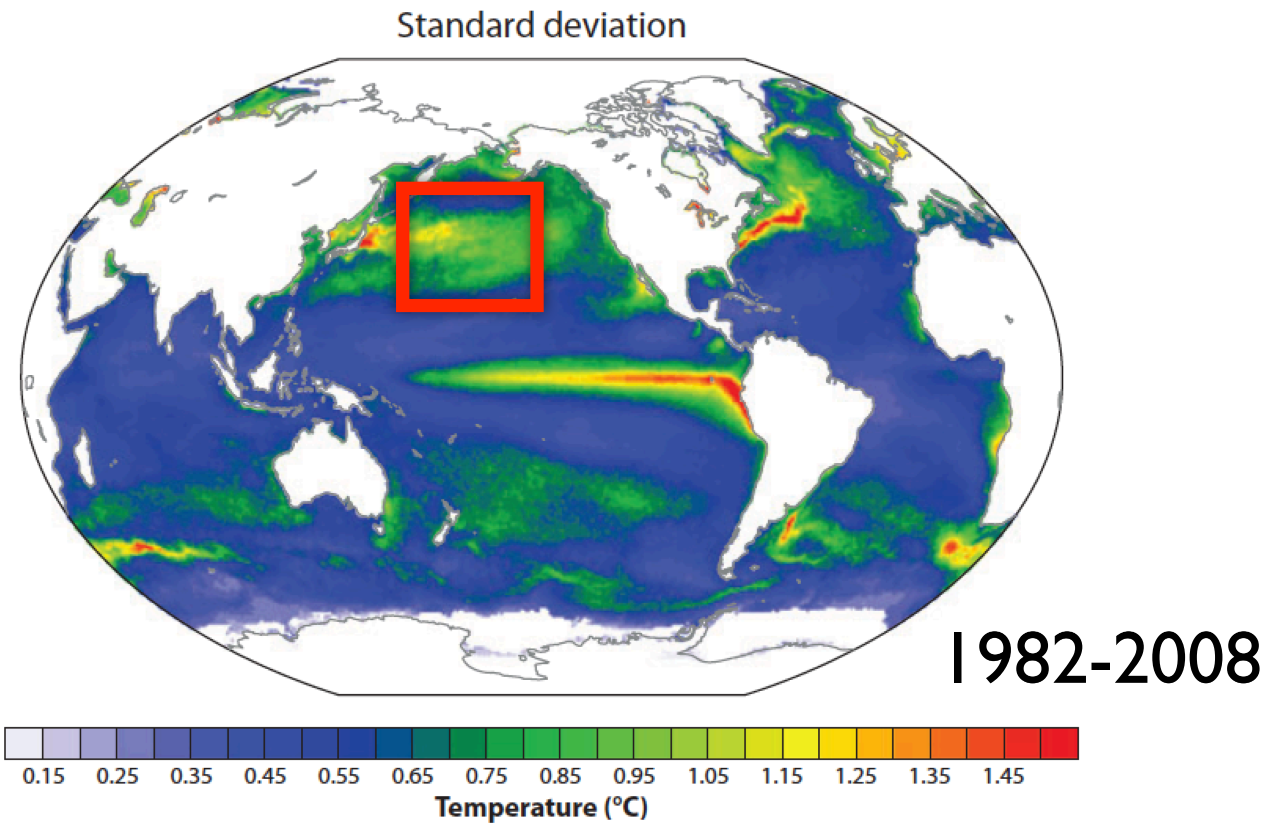
0-70N



Deser et al., Ann. Rev. Mar. Sc. (2010)

Question time

Sea surface temperature variability

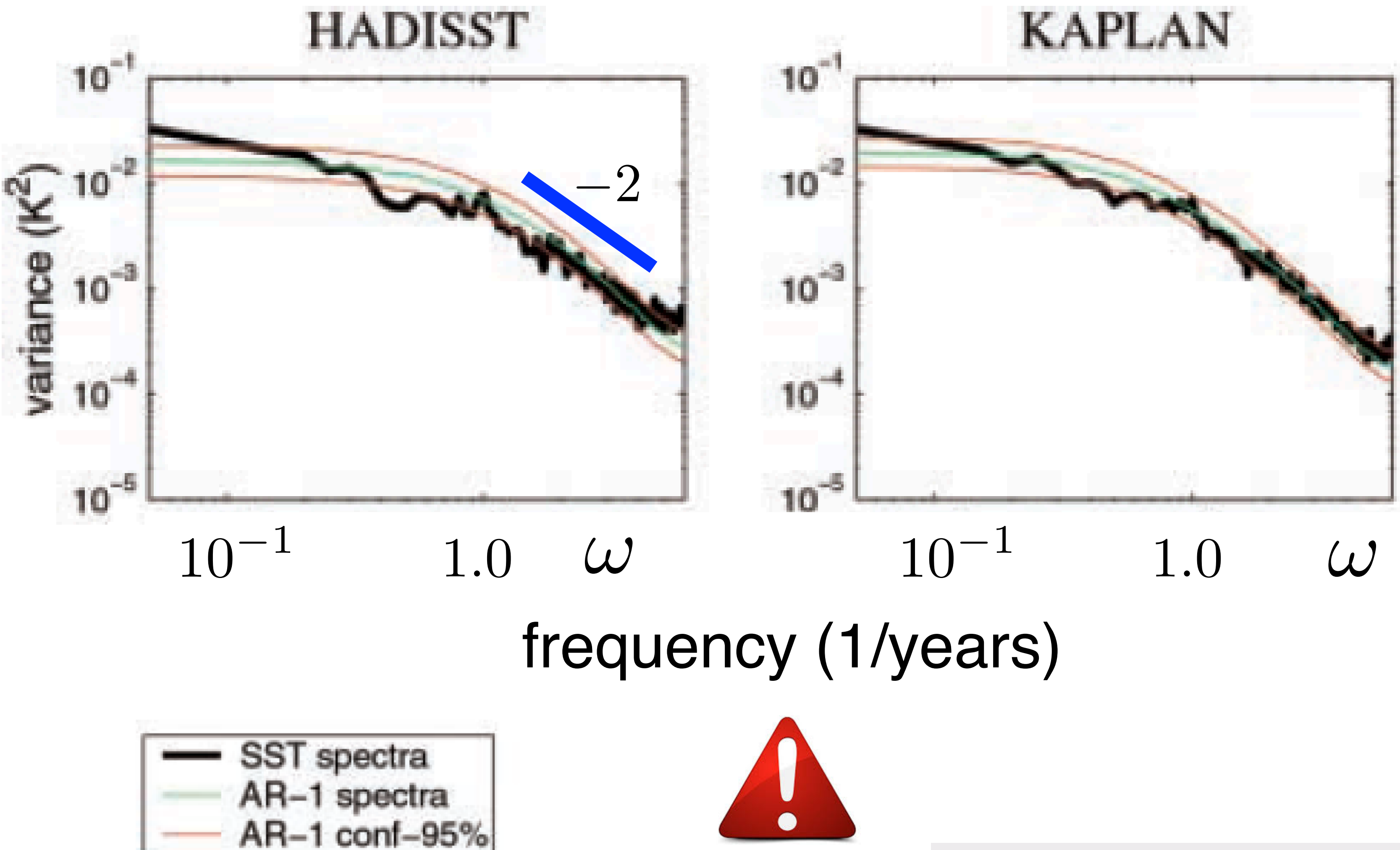


What is the null-hypothesis of sea surface temperature variability?

Deser et al., Ann. Rev. Mar. Sc. (2010)

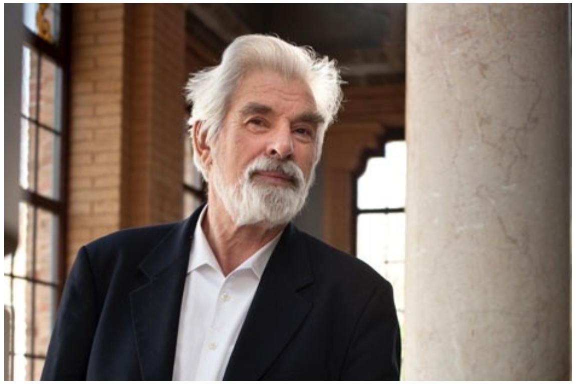
Midlatitude SST Spectra

25N – 50N Pacific (1903-1994)



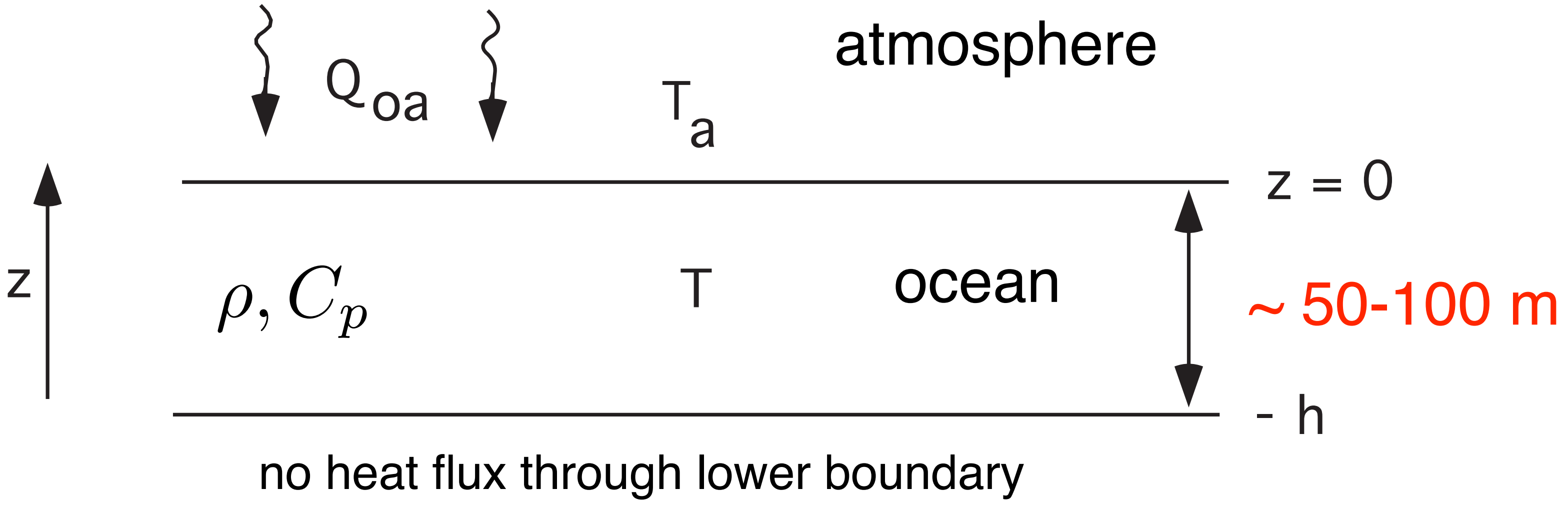
Dommenget & Latif, Clim. Dyn. (2002)

Hasselmann approach (1976)



Klaus Hasselmann (1931-)

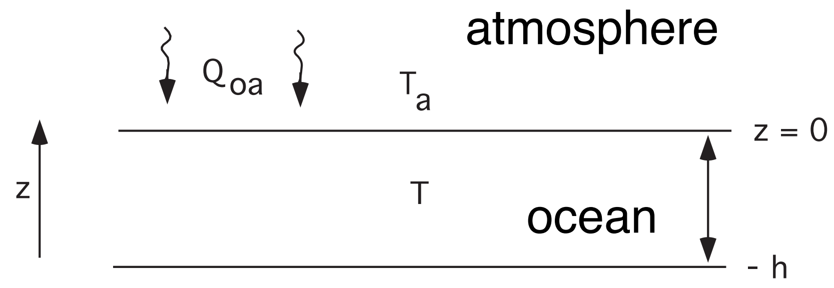
Hasselmann K. (1976), "Stochastic climate models, Part 1: Theory", *Tellus*, 28: 473-485.



Ocean Mixed-layer Model

Equation:

$$\rho C_p \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial z^2}$$



Boundary conditions:

$$z = 0 : K \frac{\partial T}{\partial z} = Q_{oa}$$

$$z = -h : \frac{\partial T}{\partial z} = 0$$

Solution

Use: $$\bar{T} = \frac{1}{h}\int_{-h}^{0} T\, dz$$ and $$Q_{oa} = \alpha(T_a - \bar{T})$$

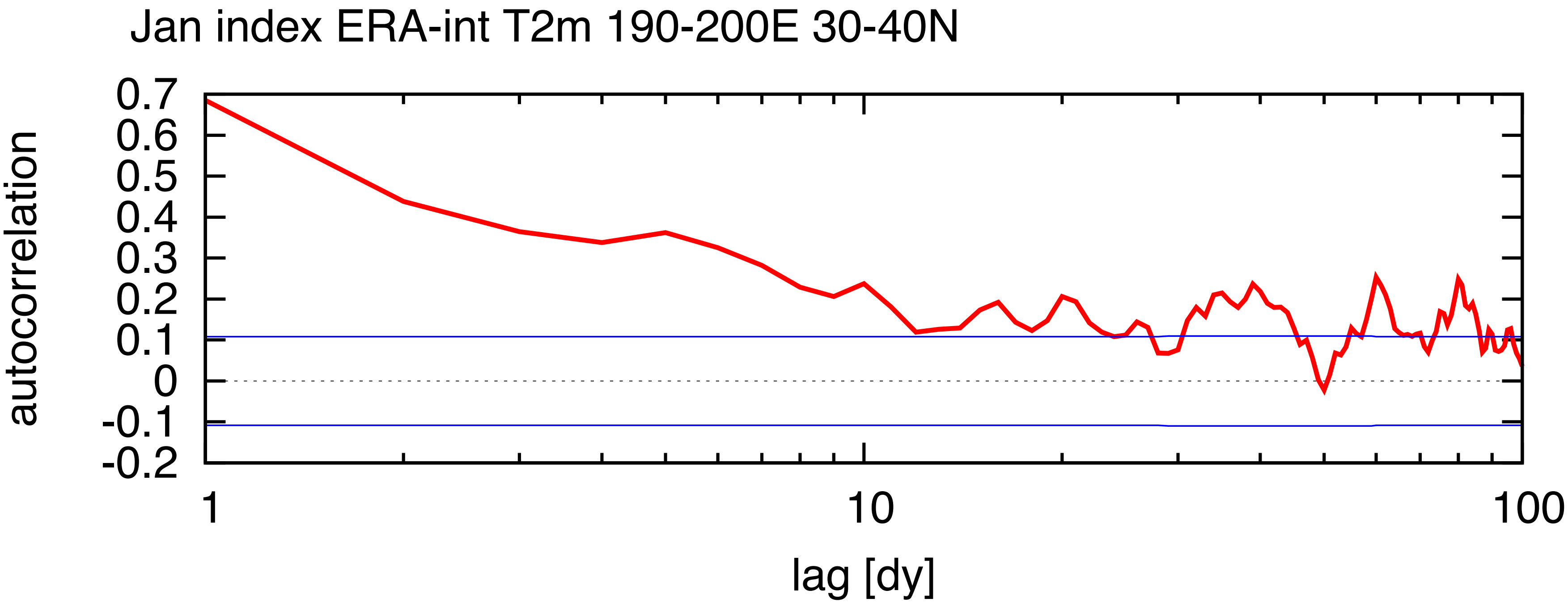
$$\bar{T} =< \bar{T} > +\tilde{T}$$

$$T_a =< T_a > +\tilde{T}_a$$

Result: $$\frac{d\tilde{T}}{dt} = \frac{\alpha}{\rho C_p h}(\tilde{T}_a - \tilde{T})$$

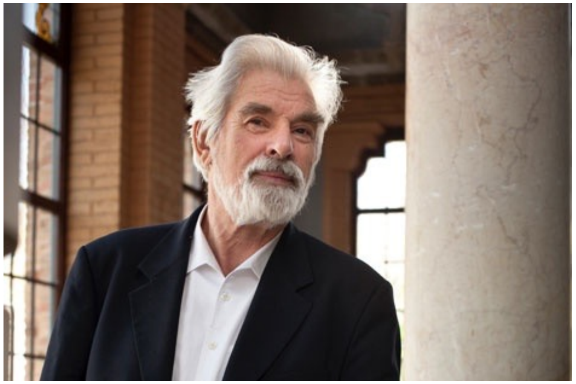
$$\gamma = \frac{\alpha}{\rho C_p h}$$ ~ 1/(100 days)

Example: Autocorrelation
Pacific atmospheric surface temperatures



Decorrelation time scale atmospheric forcing << ocean damping time scale

The Hasselmann (1976) stochastic climate model



Hasselmann K. (1976), "Stochastic climate models, Part 1: Theory", *Tellus*, 28: 473-485.

$$\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi \qquad \gamma = \frac{\alpha}{\rho C_p h}$$

white noise

$$E[\xi(t)] = 0$$

$$E[\xi(t)\xi(s)] = \delta(t-s)$$

“The choice between a deterministic and a stochastic formulation of the equations …. [is] dictated by convenience” Lorenz (1987)

Question time

Stochastic Differential Equations (SDEs)

from

$$\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi$$

Stochastic process: $X_t = \tilde{T}$

Wiener process: W_t $\quad N(0, t)$ distributed

$$E[(dW_t)^2] = dt$$

to

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

or

$$X_t = X_0 + \int_0^t (-\gamma X_s) ds + \int_0^t \sigma dW_s$$

Stochastic integrals

Kiyoshi *Itô* (1915-2008)

$$\int_0^T h(t)\, dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

`left end point'

Ruslan Stratonovich (1930-1997)

`mid point'

$$\int_0^T h(t) \circ dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j))$$

both in the mean-square sense

$$\lim_{N\to\infty} E[(I - I_N)^2] \to 0$$

Stochastic integrals: evoluation

Problem: Evaluate $\int_0^T W_t dW_t$

$It\hat{o}$

$$f(W_T) - f(W_0) = \int_0^T f'(W_t)dW_t + \int_0^T \frac{1}{2} f''(W_t)dt$$

Take: $f(t) = t^2$

Answer: $$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

Solution of SDEs

$$f(t+dt, X_t+dX_t) - f(t, X_t) = f_1 dt + f_2 dX_t + \frac{1}{2}(f_{11}(dt)^2 + 2f_{12}dtdX_t + f_{22}(dX_t)^2) + \ldots$$

$It\hat{o}$

$$dX_t = A_t dt + B_t dW_t$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Example: Hasselmann

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

$$A_t = -\gamma X_t \; ; \; B_t = \sigma$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Take: $f(t, x) = x e^{\gamma t}$

Result: $X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$

Ornstein-Uhlenbeck process

Numerical solution of SDEs

$$X(t) = X(0) + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s)$$

$$\tau_j = j\Delta t, j = 0, \ldots, n \text{ on } [0, T]$$

$$\Delta t = T/n$$

Euler-Maruyama scheme:

Gisiro Maruyama (1916-1986)

$$X_j - X_{j-1} = f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1}))$$

Ornstein-Uhlenbeck process

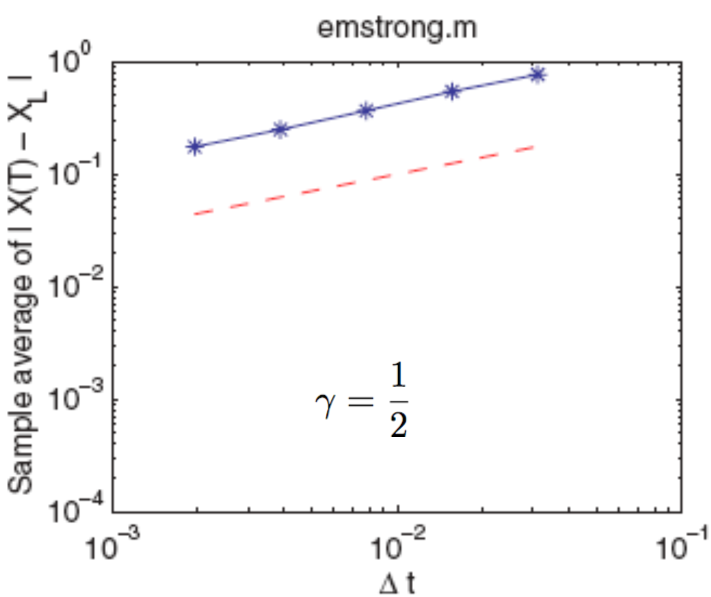
Convergence of EM-method

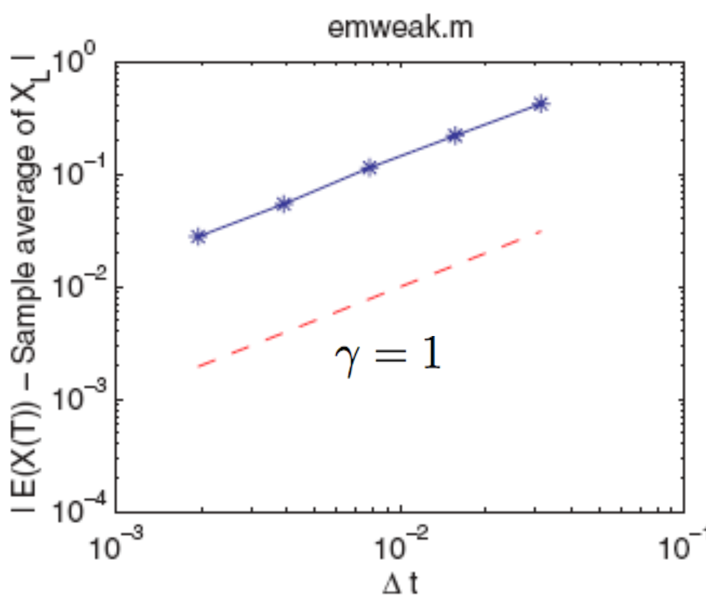
Strong convergence:

$$E[|X_k - X(\tau_k)|] \leq (\Delta t)^\eta$$

Weak convergence:

$$|E[X_k] - E[X(\tau_k)]| \leq (\Delta t)^\eta$$





Question time

Probability density

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

$$E[f(X_t)] = \int f(x)p(x,t)dx$$

Probability Density Function (PDF)

Fokker-Planck Equation

$$\frac{\partial p}{\partial t} = -\frac{\partial (ap)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 p)}{\partial x^2}$$

+ Bc's + IC

Example

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

$$a = -\gamma x \ ; \ b = \sigma$$

$$\frac{\partial p}{\partial t} = -\frac{\partial (ap)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 p)}{\partial x^2} = 0$$

equilibrium

$$p_e(x) = Ce^{-\frac{\gamma x^2}{\sigma^2}}$$

Result:

$$\int_{-\infty}^{\infty} p_e(x) = 1 \rightarrow C = \frac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}$$

Statistics Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Solution: $$X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$

Wanted: $$E[X_t X_{t+s}], s > 0$$

Result: $$E[X_t X_{t+s}] = e^{-\gamma(2t+s)}(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma})$$

$$E[X_t X_{t+s}] \rightarrow \frac{\sigma^2}{2\gamma} e^{-\gamma s}, \quad t \rightarrow \infty$$

Spectrum: $$S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}(e^{-\gamma s}) = \frac{\sigma^2}{\omega^2 + \gamma^2}$$

Summary: Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Leonard Ornstein (1880-1941)

George Uhlenbeck (1900-1988)

Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \frac{\partial(\gamma x p)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$$

$$x \to \pm\infty : p \to 0$$

Stationary density:

$$p_{stat} = \sqrt{\frac{\gamma}{\pi}}\frac{1}{\sigma}e^{\frac{-\gamma x^2}{\sigma^2}}$$

Stationary autocorrelation:

$$E[X_t X_s] = \frac{\sigma^2}{2\gamma}e^{-\gamma|\tau|}, \tau = t - s$$

Stationary spectrum:

$$S(\omega) = \frac{\sigma^2}{\gamma^2 + \omega^2}$$

The Red Noise (AR(1)) process

$$X_t = e^{-\gamma t}\left(X_0 + \sigma \int_0^t e^{\gamma s} dW_s\right)$$

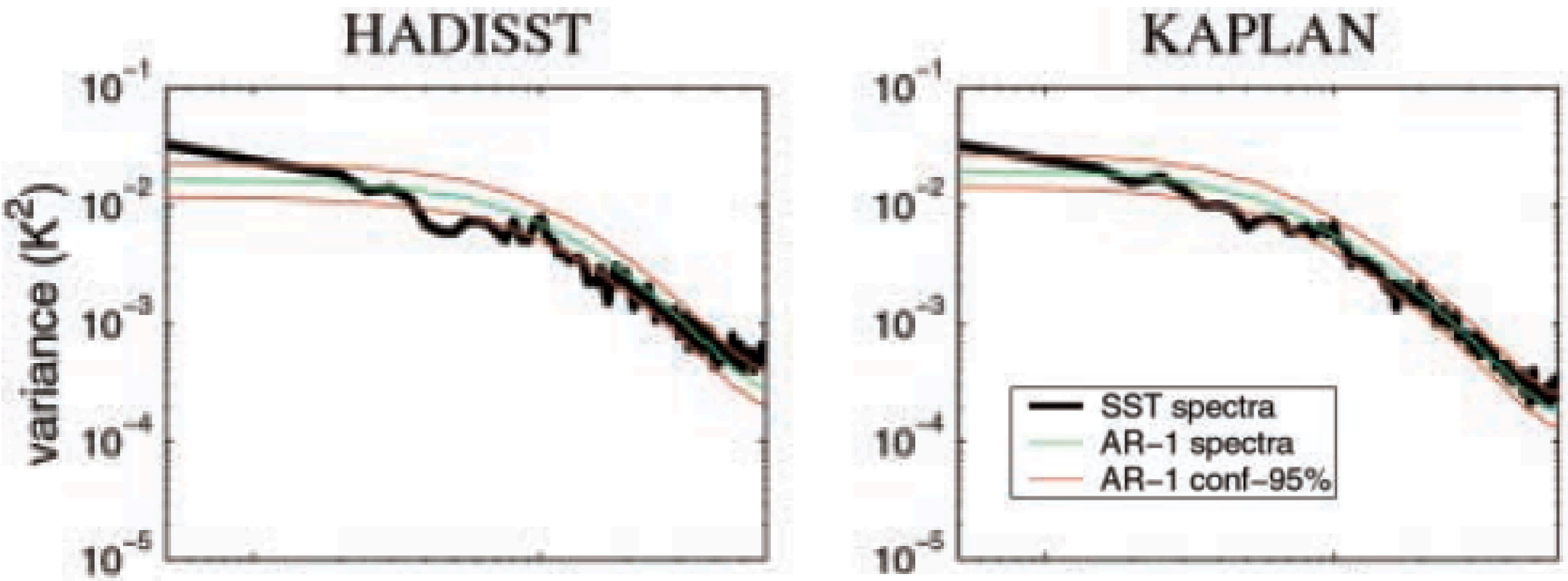
Discrete time: $$t_n = n\Delta t \; ; \; X_n = X_{t_n}$$

$$X_{n+1} = \alpha X_n + Z_{n+1}$$

$$\alpha = e^{-\gamma \Delta t}, Z_n = \sigma dW_n$$

α can be estimated from the autocorrelation of the time series

Red noise representation



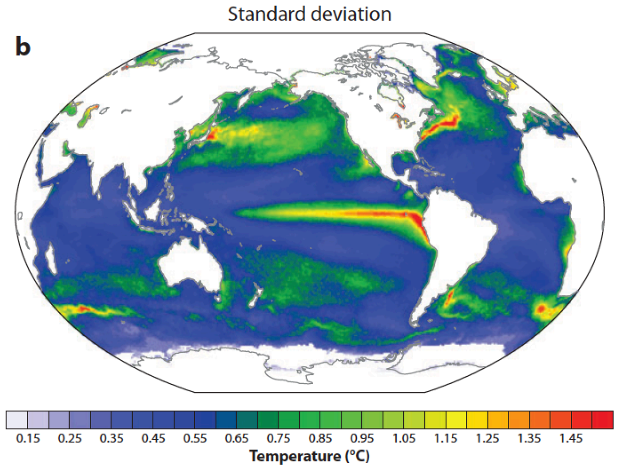
The red noise spectrum serves as a null-hypothesis for climate variability!

Question time

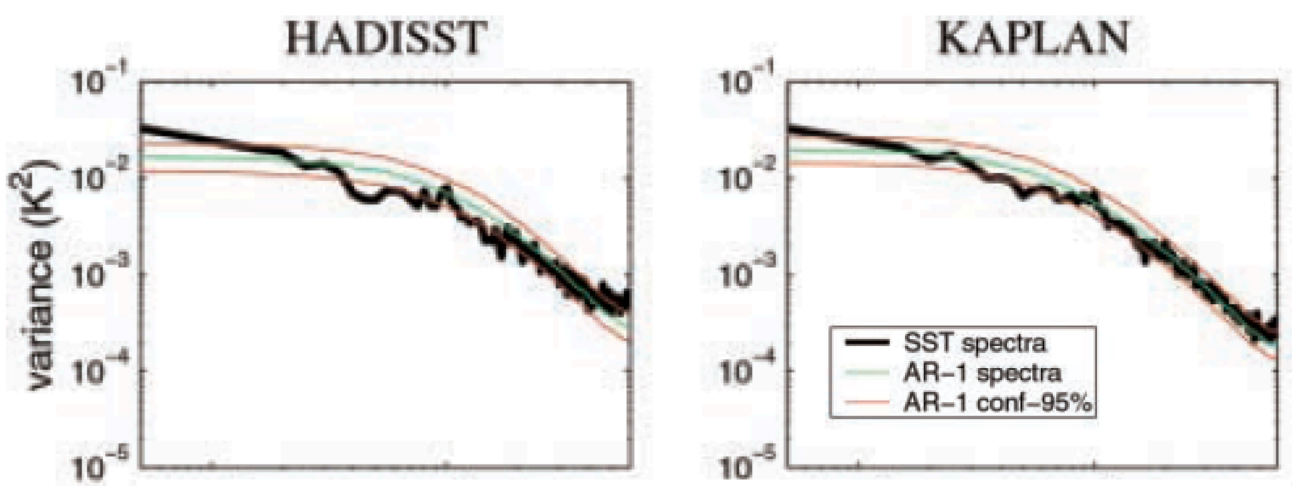
Summary: SST variability

Ocean mixed layer temperature: $$dX_t = -\gamma X_t dt + \sigma dW_t$$

Wiener process



Continuous: Ornstein-Uhlenbeck process
Discrete: red noise or AR(1) process



The red noise spectrum serves as a null-hypothesis for climate variability!

End of Lecture 1