Project 3: Delayed-oscillator ENSO model

A 'toy' model of the ENSO wave oscillator is the delay equation for the spatially averaged eastern Pacific temperature

$$dT_t = (aT_t - bT_{t-d} - cT_t^3)dt + \sigma dW_t \quad (1)$$

Here a represents the growth rate of the temperature disturbance T in the eastern Pacific. The quantity d is the delay time due to the propagation of equatorial waves, and b measures its influence with respect to the local feedbacks. The nonlinear term with coefficient c is needed for equilibration of the temperature to finite amplitude. Again, additive noise is assumed with variance σ.

(i) Consider first the deterministic case. Apply a scaling in temperature and time to derive the equation

$$\frac{dT(t)}{dt} = T(t) - \alpha T(t-\delta) - T^3(t) \quad (2)$$

with $\alpha = b/a$ and $\delta = ad$.

(ii) Determine the fixed points (steady states) of this model.

(iii) Consider the stability of the fixed points in both regimes $\alpha < 1$ and $\alpha > 1$. Which regime is more realistic for ENSO dynamics (and why)?

(iv) Next consider the stochastic case. Write an Euler-Maruyama scheme to integrate equation (1) for different σ. Make an appropriate choice for the initial conditions (over the interval $[0, \delta]$ or $[-\delta, 0]$). Provide a check of the code using the analytical results of (ii) and (iii).

(v) Fix $\alpha = 0.9$. Determine the probability density function of the temperature variability, both in the (deterministically) subcritical (e.g. $\delta = 1$) and supercritical case (e.g. $\delta = 2$). For different values of σ. Does a stochastic Hopf bifurcation occur at the transition from steady to oscillatory behavior?

(vi) What is the underlying dynamical mechanism (in a stochastic Hopf bifurcation) that noise excites an oscillatory mode under (deterministic) subcritical conditions?