Locally compact abelian groups

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1. Locally compact abelian groups

Definition. A topological space X is *locally compact* if X is Hausdorff and every point $x \in X$ has a compact neighbourhood, i.e. there exist subsets $x \in U \subseteq K$ with U open and K compact.

This implies that X is compactly generated, but not vice versa. For instance, $\mathbf{Q} \subseteq \mathbf{R}$ with the induced topology is not locally compact by the Heine–Borel theorem, yet it is compactly generated (hint: use compact subspaces of the form $\{x_i \mid i \in \mathbf{N}\} \cup \{x\}$ for convergent sequences $x_i \to x$ in $\mathbf{Q}$).

Definition. Write **HAb** for the category of Hausdorff abelian groups and **LCA** for the category of locally compact abelian groups.

Example. The following are examples of locally compact abelian groups:

- Any discrete abelian group, e.g. $\mathbf{Z}$, $\mathbf{Z}/n\mathbf{Z}$, or $\mathbf{Q}$. This gives a fully faithful embedding $\mathbf{Ab} \subseteq \mathbf{LCA}$.
- $\mathbf{R}^n$ for any $n \in \mathbf{N}$.
- The circle group $\mathbf{T} := \mathbf{R}/\mathbf{Z}$.
- Any profinite (compact Hausdorff totally disconnected) abelian group, e.g. $\mathbf{Z}_p$ or $\hat{\mathbf{Z}}$.
- Any locally profinite (locally compact Hausdorff totally disconnected) abelian group, e.g. $\mathbf{Q}_p$ or the *finite adèles* $\mathbf{A}_{\text{fin}} = \hat{\mathbf{Z}} \otimes \mathbf{Q} = \varinjlim_n \frac{1}{n}\hat{\mathbf{Z}}$ (topologised as a colimit of open immersions).

Two non-examples:

- An infinite-dimensional topological vector space over $\mathbf{R}$ or $\mathbf{Q}_p$ is *never* locally compact. For instance, $\mathbf{C}_p$ is not locally compact.
- $\mathbf{Q} \subseteq \mathbf{R}$ with the subspace topology is not locally compact, as we saw before. We will only ever consider $\mathbf{Q}$ with the discrete topology.

Warning. Some authors say that $A \in \mathbf{LCA}$ is *compactly generated* if there exists a compact subset $Z \subseteq A$ such that Z generates A as abelian group. We will mostly avoid this terminology as it clashes with the topological notion. Every locally compact topological space is compactly generated, but not every locally compact abelian group is compactly generated in this new sense. For instance, for a discrete group A this is equivalent to A being finitely generated.

Remark. The categories $\mathbf{HAb}$ and $\mathbf{LCA}$ are not abelian: the inclusion $\mathbf{R}^{\text{disc}} \to \mathbf{R}$ is both monic and epic, but not an isomorphism. However, they are *quasi-abelian categories*:

- they are additive;
- kernels and cokernels exist (for the latter, take $\operatorname{coker} f = B/\overline{f(A)}$);
- kernels are stable under pushout and cokernels are stable under pullback.

Definition. A morphism $f\colon A \to B$ in a quasi-abelian category is *strict* if the natural map

$$\begin{array}{ccc} \operatorname{coim} f & \longrightarrow & \operatorname{im} f \\ \| & & \| \\ \operatorname{coker}(\ker f) & & \ker(\operatorname{coker} f) \end{array}$$

is an isomorphism. A complex $C^\bullet$ is *strictly exact* if all maps $d^i\colon C^i \to C^{i+1}$ are strict and the complex is exact.

In $\mathbf{HAb}$ or $\mathbf{LCA}$, a morphism $f\colon A \to B$ is strict if and only if $A/\ker f \hookrightarrow B$ is a closed embedding. A short exact sequence

$$0 \to A \to B \to C \to 0$$

is strictly exact if and only if $A \hookrightarrow B$ is a closed embedding and $B \to C$ is a topological quotient (equivalently, $B \to C$ is open).

Example. Some examples of strict short exact sequences:

- $0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{T} \to 0$;
- $0 \to \mathbf{Z}_p \to \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \to 0$.

Note that $\mathbf{Z}$ and $\mathbf{Q}_p/\mathbf{Z}_p$ are discrete, and $\mathbf{T}$ and $\mathbf{Z}_p$ are compact. We will see later that any $A \in \mathbf{LCA}$ can be obtained as a successive extension of discrete and compact abelian groups.

2. Pontryagin duality

Definition. For $A, B \in \mathbf{LCA}$, endow $\mathrm{Hom}(A, B) \subseteq \mathrm{Cont}(A, B)$ with the *compact-open topology*: a subbase is given by the sets

$$V(K, U) := \{f : A \to B \mid f(K) \subseteq U\}$$

for $K \subseteq A$ compact and $U \subseteq B$ open (subsets, not subgroups). We denote this internal Hom by $\mathbf{Hom}(A, B)$.

Warning. In general, $\mathbf{Hom}(A, B)$ is only in $\mathbf{HAb}$, not in $\mathbf{LCA}$. For instance, $\mathbf{Hom}(\mathbf{Z}^{\oplus I}, \mathbf{R}) = \mathbf{R}^I$ is locally compact if and only if I is finite.

Definition. The *Pontryagin dual* of $A \in \mathbf{LCA}$ is $A^\vee := \mathbf{Hom}(A, \mathbf{T})$.

Note that if $0 \to A \to B \to C \to 0$ is strictly exact, then

$$0 \to C^\vee \to B^\vee \to A^\vee$$

is strictly exact, so $(-)^\vee$ is "left exact" in a suitable sense.

Theorem (Pontryagin, van Kampen)**.**

(i) If $A \in \mathbf{LCA}$, then $A^\vee \in \mathbf{LCA}$. If A is compact, then $A^\vee$ is discrete. If A is discrete, then $A^\vee$ is compact.
(ii) The natural map $A \to A^{\vee\vee}$ is an isomorphism for all $A \in \mathbf{LCA}$.
(iii) The functor $(-)^\vee$ takes strict exact sequences to strict exact sequences.

We will sketch a proof below; details can for instance be found in [Pon39], [HR79], and [Mor77]. The one deep ingredient in the proof is:

Theorem (Peter–Weyl, abelian case)**.** *If A is compact, then the natural map $A \to A^{\vee\vee}$ is injective, i.e. for $a \in A \setminus \{0\}$ there exists $\psi : A \to \mathbf{T}$ such that $\psi(a) \neq 0$.*

This is usually proven using Haar measures, but there is also an elementary (albeit somewhat long and slightly strange) proof due to Prodanov; see for instance [DS11] and the references therein.

Proof of duality (outline). Statement (i) is an easy verification. Statement (iii) for C discrete follows since $\mathbf{T}$ is divisible and any homomorphism $B \to \mathbf{T}$ extending a continuous homomorphism $A \to \mathbf{T}$ is automatically continuous. Statement (ii) for $A \in \{\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}, \mathbf{T}, \mathbf{R}\}$ is verified by computation. This gives (ii) when A is a finitely generated discrete abelian group.

To prove (ii) for any discrete abelian group A, note that $A^{\vee\vee}$ is discrete by (i). An element $a \in A$ gives an injection $\mathbf{Z}/n\mathbf{Z} \hookrightarrow A$ for some $n \in \mathbf{N}$ (possibly 0 or 1), which by (iii) for discrete groups and (ii) for finitely generated discrete groups gives a commutative diagram

$$\begin{tikzcd} \mathbf{Z}/n\mathbf{Z} \arrow[r, "\sim"] \arrow[d, hook] & (\mathbf{Z}/n\mathbf{Z})^{\vee\vee} \arrow[d, hook] \\ A \arrow[r] & A^{\vee\vee}. \end{tikzcd}$$

This shows that $A \to A^{\vee\vee}$ is injective.

For any subgroup $H \subseteq A$, write $\mathrm{Ann}(H) := \{f\colon A \to \mathbf{T} \mid f(H) = 0\} \subseteq A^\vee$, which is canonically isomorphic to $(A/H)^\vee$ by left exactness of $(-)^\vee$. Note that

$$\bigcap_{\substack{H \subseteq A \\ \text{f.g.}}} \mathrm{Ann}(H) = 0. \tag{2.1}$$

If $\psi\colon A^\vee \to \mathbf{T}$ is an element of $A^{\vee\vee}$, then $W = \psi^{-1}\big((-\varepsilon, \varepsilon)\big) \subseteq A^\vee$ is an open subset containing 0, and its complement Z in $A^\vee$ is compact by (i). Then (2.1) and compactness of Z show that there exists a finitely generated subgroup $H \subseteq A$ such that $\mathrm{Ann}(H) \subseteq W$, i.e. $\psi(\mathrm{Ann}(H)) \subseteq (-\varepsilon, \varepsilon)$. Since ψ is a group homomorphism and $\mathrm{Ann}(H) \subseteq A^\vee$ a subgroup, this forces $\mathrm{Ann}(H) \subseteq \ker \psi$. In the commutative diagram with strictly exact rows

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] \arrow[d, "\wr"] & A \arrow[r] \arrow[d] & A/H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{\vee\vee} \arrow[r] & A^{\vee\vee} \arrow[r] & (A/H)^{\vee\vee} \end{tikzcd}$$

we see that $\psi \in A^{\vee\vee}$ comes from $H^{\vee\vee}$, proving surjectivity of $A \to A^{\vee\vee}$. This proves (ii) for A discrete, and (ii) for A compact follows formally since $A^\vee$ is discrete and $A \to A^{\vee\vee}$ is injective by Peter–Weyl (exercise).

Now one needs to bootstrap from compact to locally compact abelian groups. This usually involves proving difficult structure theorems about locally compact abelian groups, e.g. that every $A \in \mathbf{LCA}$ sits in a strictly exact sequence $0 \to \mathbf{R}^n \times C \to A \to D$ with C compact and D discrete. Then (ii) for A follows from (ii) for $\mathbf{R}^n$, C, and D and (iii) since D is discrete. Finally, the general case of (iii) is obtained as a formal consequence of (ii). $\square$

Remark. Note that if we already knew (iii), then the bootstrap from the compact to the locally compact case of (ii) would be substantially easier. For instance, any compact neighbourhood K of 0 in A generates an open subgroup $B \subseteq A$ that is compactly generated in the sense of topological abelian groups. Then one shows that there exists a closed lattice $\mathbf{Z}^n \subseteq B$ whose quotient $C = B/\mathbf{Z}^n$ is compact. This gives strict short exact sequences

$$0 \longrightarrow B \longrightarrow A \longrightarrow D \longrightarrow 0$$
$$0 \longrightarrow \mathbf{Z}^n \longrightarrow B \longrightarrow C \longrightarrow 0$$

with D discrete and C compact, and (ii) for A follows from the same statement for C, D, and $\mathbf{Z}^n$ if we knew (iii). However, I am not aware of a proof of (iii) that does not rely on (ii).

Example. Some examples of Pontryagin duals:

A	$A^\vee$
$\mathbf{Z}/n\mathbf{Z}$	$\mathbf{Z}/n\mathbf{Z}$
$\mathbf{Z}$	$\mathbf{T}$
$\mathbf{R}$	$\mathbf{R}$
$\mathbf{Z}_p$	$\mathbf{Q}_p/\mathbf{Z}_p$
$\mathbf{Q}_p$	$\mathbf{Q}_p$
discrete	compact
$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{T} \to 0$	$0 \leftarrow \mathbf{T} \leftarrow \mathbf{R} \leftarrow \mathbf{Z} \leftarrow 0$
$0 \to \mathbf{Z}^{\oplus J} \to \mathbf{Z}^{\oplus I} \to D \to 0$	$0 \leftarrow \mathbf{T}^J \leftarrow \mathbf{T}^I \leftarrow C \leftarrow 0$
(for any discrete abelian group D)	(for any compact abelian group C)

Lemma. *The "weak Serre subcategory" $\mathscr{C} \subseteq$ **LCA** generated by $\mathbf{Z}^{\oplus I}$ and $\mathbf{T}^I$ for all sets I is* **LCA**.

Here, "weak Serre subcategory" should be taken to mean the smallest (full) subcategory $\mathscr{C} \subseteq$ **LCA** that contains the given objects and has the property that for any strict short exact sequence $0 \to A \to B \to C \to 0$, if two out of three are in $\mathscr{C}$ then so is the third.

Proof. Any discrete abelian group D sits in a strict short exact sequence $0 \to \mathbf{Z}^{\oplus J} \to \mathbf{Z}^{\oplus I} \to D \to 0$, and dually every compact abelian group C sits in a strict short exact sequence $0 \to C \to \mathbf{T}^I \to \mathbf{T}^J \to 0$. So $\mathscr{C}$ contains all discrete and compact abelian groups. By the remark on the previous page, every $A \in$ **LCA** is a successive extension of discrete and compact abelian groups. □

3. The derived category of locally compact abelian groups

For any quasi-abelian category $\mathscr{C}$, Schneiders [Sch99] has constructed a derived category $D^b(\mathscr{C})$. It is constructed as $K^b(\mathscr{C})/\mathscr{N}(\mathscr{C})$, where $\mathscr{N}(\mathscr{C})$ is the saturated triangulated subcategory on the strictly exact complexes in $\mathscr{C}$.

In the special case $\mathscr{C} =$ **LCA**, this was studied in great detail by Hoffman and Spitzweck [HS07]. In particular, they construct an internal Hom

$$R\mathbf{Hom}\colon D^b(\mathbf{LCA})^{\mathrm{op}} \times D^b(\mathbf{LCA}) \to D^b(\mathbf{HAb})$$

extending $\mathbf{Hom}\colon \mathbf{LCA}^{\mathrm{op}} \times \mathbf{LCA} \to \mathbf{HAb}$.

Theorem (Hoffman–Spitzweck)**.**

(i) $R\mathbf{Hom}(\mathbf{Z}, B) \cong B[0]$ *for any* $B \in \mathbf{LCA}$*;*
(ii) $R\mathbf{Hom}(\mathbf{T}^I, M) \cong M^{\oplus I}[-1]$ *for* M *discrete;*
(iii) $R\mathbf{Hom}(\mathbf{T}^I, \mathbf{R}) = 0$.

We do not attempt to sketch the proof, as even the definition of $R\mathbf{Hom}$ is involved and a little counterintuitive.

4. Comparison with condensed abelian groups

Lemma. *The functor* $\mathbf{LCA} \to \mathbf{Cond}(\mathbf{Ab})$ *given by* $A \mapsto \underline{A}$ *takes strictly exact complexes to acyclic complexes.*

Thus it induces a functor $D^b(\mathbf{LCA}) \to D^b(\mathbf{Cond}(\mathbf{Ab}))$.

Proof. A strictly exact complex splits up into strict short exact sequences $0 \to A \to B \to C \to 0$, so it suffices to treat that case. Then clearly $0 \to \underline{A} \to \underline{B} \to \underline{C}$ is exact. Given $E \in \mathbf{ED}$ and a continuous map $f \colon E \to C$, we want to lift to a dashed arrow

$$\begin{tikzcd} & B \arrow[d, two heads, "\pi"] \\ E \arrow[ur, dashed] \arrow[r, "f"'] & C . \end{tikzcd}$$

The image $f(E)$ is compact and E is projective in $\mathbf{CHaus}$, so it suffices to find $Z \subseteq B$ compact with $f(E) \subseteq \pi(Z)$. Pick $0 \in U \subseteq K \subseteq B$ with U open and K compact. Since π is open, the open sets $\pi(U+b) \subseteq C$ for $b \in B$ cover $f(E)$, so finitely many $\pi(U+b_1), \ldots, \pi(U+b_n)$ suffice. Then the compact set

$$Z := \bigcup_{i=1}^{n} (K + b_i)$$

satisfies $\pi(Z) \supseteq f(E)$. $\square$

To separate notation, we will write $\mathscr{H}om$ for the internal Hom in the category of sheaves on a site; for instance for condensed abelian groups.

Proposition. *Let* $A, B \in \mathbf{LCA}$*. Then* $\mathscr{H}om(\underline{A}, \underline{B}) \cong \underline{\mathbf{Hom}(A, B)}$*.*

Proof. Recall the resolution $\mathbf{Z}[\underline{A} \times \underline{A}] \to \mathbf{Z}[\underline{A}] \to \underline{A} \to 0$, whose maps are given by $[(a_1, a_2)] \mapsto [a_1 + a_2] - [a_1] - [a_2]$ and $[a] \mapsto a$ respectively. Tensoring with the projective module $\mathbf{Z}[S]$ for any $S \in \mathbf{ED}$ gives an exact sequence

$$\mathbf{Z}\big[\underline{A} \times \underline{A} \times S\big] \xrightarrow{\phi} \mathbf{Z}\big[\underline{A} \times S\big] \to A \otimes \mathbf{Z}[S] \to 0.$$

Thus we get

$$\begin{aligned}
\mathscr{H}om(\underline{A},\underline{B})(S) &= \mathrm{Hom}(\underline{A}\otimes \mathbf{Z}[S],\underline{B}) \\
&= \big\{ f\in \mathrm{Hom}(\mathbf{Z}[\underline{A}\times S],\underline{B}) \bigm| f\circ\phi = 0\big\} \\
&= \big\{ f\in \mathrm{Hom}_{\mathbf{Cond(Set)}}(\underline{A}\times S,\underline{B}) \bigm| f\circ\phi = 0\big\} \\
&= \left\{ f\in \mathrm{Cont}(A\times S, B) \,\middle|\, \begin{array}{c} f(a_1+a_2,s) = f(a_1,s)+f(a_2,s) \\ \text{for all } (a_1,a_2,s)\in A\times A\times S \end{array}\right\} \\
&= \big\{ f\in \mathrm{Cont}(A\times S,B) \bigm| f(s)\in \mathrm{Hom}(A,B) \text{ for all } s\in S\big\} \\
&= \mathrm{Cont}(S,\mathbf{Hom}(A,B)) = \underline{\mathbf{Hom}(A,B)}(S). \qquad \square
\end{aligned}$$

Theorem. *The functor $D^b(\mathbf{LCA}) \to D^b(\mathbf{Cond}(\mathbf{Ab}))$ is fully faithful. If $A,B\in\mathbf{LCA}$, then $\mathscr{E}xt^i(\underline{A},\underline{B}) = 0$ if $i\geq 2$.*

Proof. For the first statement, it suffices to show that the natural map

$$\underline{R\mathbf{Hom}(A,B)} \to R\mathscr{H}om(\underline{A},\underline{B}) \tag{4.1}$$

is an isomorphism for all $A,B\in\mathbf{LCA}$. By the results above, this reduces to the case $A,B\in\{\mathbf{T}^I,\mathbf{Z}^{\oplus I}\}$ as these generate $\mathbf{LCA}$. For the second statement we reduce to the case where A and B are either compact or discrete, as any locally compact abelian group is a successive extension of such.

We will use the Hoffman–Spitzweck theorem from §3 above and the results from the previous talk (amounting to [CS19, Thm. 4.3]):

(i) If $A = \mathbf{Z}^{\oplus I}$, then both sides of (4.1) are $\underline{B}^I[0]$. (For the LCA case, use part (i) of Hoffman–Spitzweck.) This also shows that if A is any discrete abelian group, then $\mathscr{E}xt^i(\underline{A},\underline{B})$ for $i\geq 2$ and $B\in\mathbf{LCA}$.

(ii) If $A = \mathbf{T}^I$ and B is discrete, then both sides of (4.1) are $\underline{B}^{\oplus I}[-1]$ by [CS19, Thm. 4.3(i)] from last talk together with Hoffman-Spitzweck part (ii). This also shows that $\mathscr{E}xt^i(\underline{A},\underline{B}) = 0$ for $i\geq 2$ in this case.

(iii) If $A = \mathbf{T}^I$ and $B = \mathbf{R}$, then [CS19, Thm. 4.3(ii)] from last talk and Hoffman-Spitzweck part (iii) give $\underline{R\mathbf{Hom}(A,B)} = R\mathscr{H}om(\underline{A},\underline{B}) = 0$.

(iv) If $A = \mathbf{T}^I$ and $B = \mathbf{T}$, then (ii) and (iii) and the strictly exact sequence $0\to\mathbf{Z}\to\mathbf{R}\to\mathbf{T}\to 0$ give $\underline{R\mathbf{Hom}(A,B)} = R\mathscr{H}om(\underline{A},\underline{B}) = \underline{\mathbf{Z}}^{\oplus I}[0]$.

(v) If $A = \mathbf{T}^I$ and $B = \mathbf{T}^J$, then (iv) shows that (4.1) is an isomorphism and $\mathscr{E}xt^i(\underline{A},\underline{B}) = 0$ for $i\geq 1$. Thus if B is any compact abelian group, we conclude that $\mathscr{E}xt^i(\underline{A},\underline{B}) = 0$ for $i\geq 2$, since there exists a strictly short exact sequence $0\to B\to\mathbf{T}^J\to\mathbf{T}^K\to 0$.

(vi) If A is compact and B is either discrete or compact, then choosing a strictly short exact sequence $0\to A\to\mathbf{T}^I\to\mathbf{T}^J\to 0$ shows that $\mathscr{E}xt^i(\underline{A},\underline{B}) = 0$ for $i\geq 2$ by (ii) and (v).

The first statement of the theorem now follows from (i), (ii), and (v), and the second from (i) and (vi). $\square$

Remark. We finally note that the computations in (i) and (iv) also imply that $R\mathscr{H}om(\underline{A}, \underline{\mathbf{T}}) = \underline{A}^\vee[0]$ for any $A \in \mathbf{LCA}$. As far as I understand, this is not a formal consequence of exactness of $(-)^\vee$, as $R\mathbf{Hom}$ is not constructed as a universal δ-functor, and it is not at all clear whether $\underline{\mathbf{T}}$ remains injective in the (much larger) category $\mathbf{Cond}(\mathbf{Ab})$.

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