Representations of matrix groups

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1. Introduction

Let G be a group and k a field. Recall that (finite-dimensional) k-linear representations of G can be understood in the following ways:

- Actions $G \times V \to V$ by linear transformations on a (finite-dimensional) k-vector space V.
- Homomorphisms $G \to \mathrm{GL}(V)$.
- Upon choosing a basis, homomorphisms $G \to \mathrm{GL}_n(k)$.

In this lecture, we will be interested in representations of (subgroups of) the group $\mathrm{GL}_n(k)$ itself.

Question 1.1. — *What are the representations of a subgroup of* $\mathrm{GL}_n(k)$*? I.e. what homomorphisms* $\rho\colon \mathrm{GL}_n(k) \to \mathrm{GL}_m(\ell)$ *are there?*

Remark 1.2. — There are quite a few different cases studied in the literature:

(1) $k = \mathbf{R}$ or $k = \mathbf{C}$ and $\ell = \mathbf{C}$, restricting to continuous representations: studied in relation to Lie groups, and also in the Langlands programme. In both cases, it is important to understand infinite-dimensional representations as well.
(2) $k = \mathbf{Q}_p$ and $\ell = \mathbf{C}$, again with a continuity hypothesis (don't ask). This is what the local Langlands programme is about.
(3) $k = \mathbf{F}_q$ and $\ell = \mathbf{C}$: linear algebraic groups over finite fields give an important class of finite simple groups. All their representations can be constructed purely geometrically (Deligne–Lusztig theory).
(4) $k = \mathbf{Z}$ (ok, not a field, but still important) and $\ell = \mathbf{C}$: these come up in relation to modular forms, but are also very interesting in their own right (for instance Margulis superrigidity).
(5) $k = \ell$ and $\rho\colon \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ given by polynomials: *algebraic representations* of *linear algebraic groups*.

Today's lecture is about (5), focusing in particular on relating properties of a subgroup $G \subseteq \mathrm{GL}_n$ to properties of its (algebraic) representations.

Exercise 1.3. — Verify that

$$\begin{aligned} \mathrm{GL}_2(k) &\to \mathrm{GL}_3(k) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \end{aligned}$$

is a group homomorphism.

Ok, so we need to be more systematic.

2. Definitions and examples

Recall from Monday that *affine n-space* $\mathbf{A}^n$ is the algebraic variety with $\mathbf{A}^n(k) = k^n$.

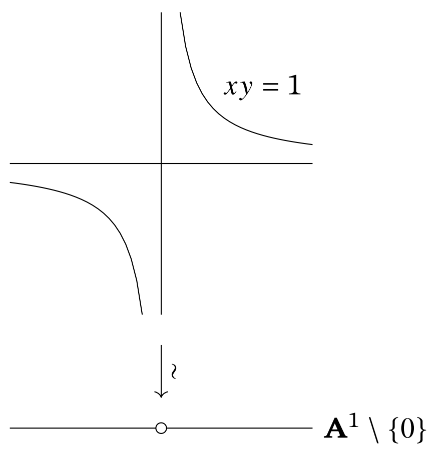
Definition 2.1. — An *affine k-variety* X is given by the vanishing locus

$$X = V(f_1, \ldots, f_r) = \{(x_1, \ldots, x_n) \in \mathbf{A}^n \mid f_1(x_1, \ldots, x_n) = \ldots = f_r(x_1, \ldots, x_n) = 0\}$$

for polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$.

Example 2.2. — The following are affine varieties:

- $\mathbf{A}^n$,
- the affine part of a conic, e.g. $z^2 = x^2 + y^2$,
- $\mathbf{A}^1 \setminus \{0\}$: it is isomorphic to the hyperbola $xy = 1$ in $\mathbf{A}^2$:



  (Algebraic geometers: draw pictures over $\mathbf{R}$. Also algebraic geometers: pretend that every field is algebraically closed.)
- In general, the nonvanishing locus U of a *single* polynomial g inside an affine variety X is affine: if $X = V(f_1, \ldots, f_r) \subseteq \mathbf{A}^n$, then

$$U = X \setminus V(g) \cong V(f_1, \ldots, f_r, gz - 1) \subseteq \mathbf{A}^{n+1},$$

since $g(x_1,\dots,x_n)\neq 0$ if and only if there exists z such that $g(x_1,\dots,x_n)z-1=0$, and such z is unique.

Warning 2.3. — The nonvanishing of more than one polynomial is usually not affine: e.g. $\mathbf{A}^n\setminus\{0\}=\mathbf{A}^n\setminus V(x_1,\dots,x_n)$ is not affine if $n\geq 2$.

Definition 2.4. — An *algebraic group* is a variety G with a group structure $m\colon G\times G\to G$ given by polynomials.

We are mostly interested in affine algebraic groups.

Example 2.5. — The following are algebraic groups:

(0) The zero group.

(1) $\mathbf{A}^n$ with addition:

$$\begin{aligned}\mathbf{A}^n\times\mathbf{A}^n&\to\mathbf{A}^n\\ \big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)&\mapsto\big(x_1+y_1,\dots,x_n+y_n\big).\end{aligned}$$

It is often denoted $\mathbf{G}_a^n$, where $\mathbf{G}_a$ is the *additive group*. More generally, any finite-dimensional vector space V gives an algebraic group $\mathbf{A}(V)$.

(2) The *multiplicative group* $\mathbf{G}_m$ is $\mathbf{A}^1\setminus\{0\}$ with multiplication

$$\begin{aligned}\mathbf{G}_m\times\mathbf{G}_m&\to\mathbf{G}_m\\ (x,y)&\mapsto xy.\end{aligned}$$

(3) An elliptic curve $(E,0)$ is an algebraic group (but not affine).

(4) $\mathrm{GL}_n=\mathbf{A}^{n^2}\setminus V(\det)$ is algebraic, where det is the 'universal determinant' of the matrix $(x_{ij})_{i,j=1}^n$ in the polynomial variables x_{ij}. The multiplication is matrix multiplication:

$$\begin{aligned}\mathrm{GL}_n\times\mathrm{GL}_n&\to\mathrm{GL}_n\\ \Big(\big(x_{ij}\big)_{i,j=1}^n,\big(y_{ij}\big)_{i,j=1}^n\Big)&\mapsto\left(\sum_{k=1}^n x_{ik}y_{kj}\right)_{i,j=1}^n.\end{aligned}$$

We saw above that it is affine (as the nonvanishing locus of det in $\mathbf{A}^{n^2}$).

(5) $\mathrm{SL}_n=V(\det-1)=\left\{(x_{ij})_{i,j=1}^n\in\mathrm{GL}_n\;\middle|\;\det(x_{ij})=1\right\}$ is a subgroup of GL_n, called the *special linear group*.

(6) The subgroup $\mathbf{D}_n\subseteq\mathrm{GL}_n$ of *diagonal matrices*, i.e. $x_{ij}=0$ if $i\neq j$:

$$\begin{pmatrix}* & 0 & \cdots & 0\\ 0 & * & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & *\end{pmatrix}.$$

(7) The upper triangular matrices $\mathbf{T}_n \subseteq \mathrm{GL}_n$ given by $x_{ij} = 0$ if $i > j$:

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

(8) The unipotent upper triangular matrices $\mathbf{U}_n \subseteq \mathbf{T}_n$ given by $x_{ij} = 0$ if $i > j$ and $x_{ii} = 1$ for all i:

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(9) The orthogonal group $O(n) = \{A \in \mathrm{GL}_n \mid A^\top A = I_n\}$.

~~(10)~~ The unitary group $U(n) = \{A \in \mathrm{GL}_n \mid \bar{A}^\top A = I_n\}$. Non-example: this is not an algebraic variety, as complex conjugation is not given by polynomials.

(11) Parabolic subgroups of GL_n: block upper triangular matrices

$$P = \left(\begin{array}{c|c|c|c} * & * & \cdots & * \\ \hline 0 & * & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & * \end{array}\right).$$

For instance,

$$\left(\begin{array}{cc|ccc} * & * & * & * & * \\ * & * & * & * & * \\ \hline 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array}\right).$$

Exercise 2.6. — Check that (5)–(9) and (11) are indeed subgroups of GL_n. Can you define $\mathbf{D}(V)$, $\mathbf{T}(V)$, and $\mathbf{U}(V)$ for a finite-dimensional vector space V?

Exercise 2.7. — Show that $\mathbf{U}_2 \cong \mathbf{G}_a$. Thus, $\mathbf{G}_a$ is a subgroup of GL_2. Can you embed $\mathbf{G}_a^n$ in GL_m for some m?

Exercise 2.8. — Construct a surjective group homomorphism $\mathbf{T}_n \to \mathbf{D}_n$ with kernel $\mathbf{U}_n$.

Thus, all affine algebraic groups we have seen so far are subgroups of GL_n. This is no coincidence:

Theorem 2.9. — *Every affine algebraic group G admits a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ for some finite-dimensional vector space V.*

For this reason, affine algebraic groups are also called *linear algebraic groups.*

The proof of the theorem is not very deep, but we omit it because it plays no role in studying the representations of any of the groups above.

3. Jordan decomposition

Recall the *Jordan normal form*:

Lemma 3.1 (Jordan normal form). — *If $A \in \mathrm{GL}_n(k)$ is a matrix with all eigenvalues in k, there exists $S \in \mathrm{GL}_n(k)$ such that SAS^{-1} is a block diagonal matrix*

$$\left(\begin{array}{c|c|c} A_1 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & A_s \end{array}\right)$$

where each A_i is of the form

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}. \tag{3.1}$$

First proof. Let $P_A(t) = \det(tI - A)$ be the characteristic polynomial of A. Since all eigenvalues are in k, the characteristic polynomial factors as $\prod_{i=1}^r (t-\lambda_i)^{n_i}$ where the λ_i are the eigenvalues of A. The *generalised eigenspace* E_{λ_i} with eigenvalue λ_i is the kernel of $(A-\lambda_i)^{n_i}$, and we have a decomposition $k^n \cong \bigoplus_i E_{\lambda_i}$. We may argue for one E_{λ_i} at a time, so we reduce to the case where A has only one eigenvalue λ. Replacing A by $A - \lambda I$, we may also assume $\lambda = 0$, so $A^n = 0$. Note that A has a Jordan normal form with all eigenvalues 0 if and only if k^n has a basis of the form

$$A^{m_1-1}v_1, \ldots, Av_1, v_1, A^{m_2-1}v_2, \ldots, Av_2, v_2, \ldots, A^{m_s-1}v_s, \ldots, Av_s, v_s, \tag{3.2}$$

where s is the number of Jordan blocks and m_i is the size of each block. We induct on n, the case $n \leq 1$ being obvious. Since $\ker A \neq 0$, we have $\dim \operatorname{im} A < n$ by the rank-nullity theorem. Thus, the induction hypothesis applies to $A\colon \operatorname{im} A \to \operatorname{im} A$, so we can choose a basis (3.2) for $\operatorname{im} A$. We may choose w_i with $Aw_i = v_i$ for all i, and extend the basis $A^{m_1-1}v_1, \ldots, A^{m_s-1}v_s$ of $\operatorname{im} A \cap \ker A$ to a basis $A^{m_1-1}v_1, \ldots, A^{m_s-1}v_s, u_1, \ldots, u_r$ for $\ker A$. We then claim that

$$A^{m_1}w_1, \ldots, Aw_1, w_1, A^{m_2}w_2, \ldots, Aw_2, w_2, \ldots, A^{m_s}w_s, \ldots, Aw_s, w_s, u_1, \ldots, u_r \tag{3.3}$$

are linearly independent. Indeed, given a linear relation between them, applying A kills the $A^{m_i}w_i$ and u_i and gives a linear relation between the others, which is zero since they form a basis of $\operatorname{im} A$. Thus we are left with a linear relation between $A^{m_i}w_i = A^{m_i-1}v_i$ and u_i, which is again zero because they form a basis for $\ker A$. Counting shows that (3.3) is a basis for k^n, as it consists of

$$\dim \operatorname{im} A + s + r = \dim \operatorname{im} A + s + \dim \ker A - s = n$$

elements. □

Second proof. We view $V = k^n$ as a $k[t]$-module where t acts by $A\colon V \to V$. Note that V is a torsion $k[t]$-module since $P_A(t)$ acts by 0 on V by Cayley–Hamilton. By the structure theorem of finitely generated modules over a principal ideal domain, there is a decomposition $V \cong V_1 \oplus \ldots \oplus V_s$ with $V_i \cong k[t]/f_i^{m_i}$ for some monic irreducible polynomial f_i and some $m_i > 0$. The characteristic polynomial for t acting on V_i is $f_i^{m_i}$, so we conclude that $P_A(t) = \prod_{i=1}^s f_i^{m_i}$. Since all eigenvalues of A are in k, we see that $f_i = t - \lambda_i$ for some $\lambda_i \in k$. The m_i-dimensional vector space $k[t]/(t-\lambda_i)^{m_i}$ has a basis $(t-\lambda_i)^{m_i-1}, \ldots, (t-\lambda_i), 1$, on which multiplication by t acts by

$$t \cdot (t-\lambda_i)^j = (t-\lambda_i)^{j+1} + \lambda_i (t-\lambda_i)^j,$$

so the matrix for A with respect to this basis is exactly the Jordan block (3.1). □

Definition 3.2. — Let V be a finite-dimensional vector space over a field k and let $A \in \mathrm{GL}(V)$. Then A is *semisimple* if it becomes diagonalisable over $\bar{k}$. It is *nilpotent* if $A^n = 0$ for some $n > 0$, and *unipotent* if $I - A$ is nilpotent. Equivalently, A is nilpotent (resp. unipotent) if its characteristic polynomial is t^n (resp. $(t-1)^n$).

Exercise 3.3. — Show that every element in $\mathbf{U}_n$ is unipotent.

A basis-independent reformulation of the Jordan normal form:

Lemma 3.4 (Jordan decomposition). — *Let k be a perfect field, let V be a finite-dimensional k-vector space, and let $A \in \mathrm{GL}(V)$.*

(1) *There exists a unique decomposition $A = A_s + A_n$ where A_s is semisimple and A_n is nilpotent and $A_s A_n = A_n A_s$.*
(2) *Suppose A is invertible. There exists a unique decomposition $A = A_s \cdot A_u$ where A_s is semisimple and A_u is unipotent and $A_s A_u = A_u A_s$.*

Proof. First assume that all eigenvalues of A are in k. Then there exists a basis of V for which A has a Jordan normal form. Let A_s be the diagonal matrix with the same diagonal entries as A, and let $A_n = A - A_n$ be the upper triangular part. Clearly A_s is semisimple and A_n is nilpotent. To check that they commute, we may work one block at a time, where the result is clear since A_s is a scalar matrix.

For uniqueness, assume $A = B + C$ with B semisimple and C nilpotent and $BC = CB$. Then B commutes with A, hence with $(A - \lambda_i I)^j$ for each i, j, so B preserves the generalised eigenspaces E_{λ_i} of A. Since $(A-B)|_{E_{\lambda_i}}$ is nilpotent, the eigenvalues of A and B agree on E_{λ_i}. Since $A|_{E_{\lambda_i}}$ has only eigenvalue λ_i and B is semisimple, we must have $B|_{E_{\lambda_i}} = \lambda_i I$.

This shows (i). For (ii), note that A_s is invertible if and only if A is, as its eigenvalues are nonzero. Take $A_u = I + A_n \cdot A_s^{-1} = I + A_s^{-1} \cdot A_n$, which is unipotent and clearly satisfies $A = A_s A_u = A_u A_s$. Conversely, from $A = A_s A_u = A_u A_s$, taking $A_n = A_s(A_u - I)$ gives the additive Jordan decomposition, so uniqueness in (ii) follows from uniqueness in (i).

Finally, if k is perfect, then the eigenvalues are defined over some finite Galois extension $k \to \ell$. For all $\sigma \in \mathrm{Gal}(\ell/k)$, the additive and multiplicative Jordan decompositions of $A = \sigma(A)$ are given by $\sigma(A_s) + \sigma(A_n)$ and $\sigma(A_s)\sigma(A_u)$, which by uniqueness means that

A_s, A_n, and A_u are fixed by $\mathrm{Gal}(\ell/k)$, hence defined over k. □

Example 3.5. — Let $A = \left(\begin{smallmatrix} 5 & 2 \\ -2 & 1 \end{smallmatrix}\right)$. Find the additive and multiplicative Jordan decomposition of A.

Solution. We have $P_A(t) = \det(tI - A) = \det\left(\begin{smallmatrix} t-5 & -2 \\ 2 & t-1 \end{smallmatrix}\right) = (t-5)(t-1)+4 = t^2-6t+9 = (t-3)^2$. Thus the only eigenvalue is $\lambda = 3$. Since $A \neq 3I$, we get $SAS^{-1} = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ (no need to compute S!), which splits as $3I + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Thus, A splits as $3S^{-1}IS + S^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)S$. The first part is just $3I$, so $A_s = 3I$, hence $A_n = A - 3I = \left(\begin{smallmatrix} 2 & 2 \\ -2 & -2 \end{smallmatrix}\right)$ (which is indeed nilpotent). Finally, $A_u = I + A_n/3 = \frac{1}{3}\left(\begin{smallmatrix} 5 & 2 \\ -2 & 1 \end{smallmatrix}\right) = A/A_s$. □

Theorem 3.6 (Jordan decomposition for algebraic groups). — *Let G be a linear algebraic group over a perfect field k, and let $g \in G(k)$. Then there exist unique elements $g_s, g_u \in G(k)$ such that $g = g_s g_u = g_u g_s$ and for every finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$, the multiplicative Jordan decomposition of $\rho(g)$ is $\rho(g_s) \cdot \rho(g_u)$. Moreover, if $\phi\colon G \to H$ is a homomorphism of linear algebraic groups and $g \in G(k)$, then $\phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$.*

This is proven using *Tannaka duality*:

Theorem 3.7 (Tannaka duality). — *Let G be a linear algebraic group over a field k. Let $U\colon \mathbf{Rep}_k(G) \to \mathbf{Vec}_k$ be the forgetful functor, and let $\mathrm{Aut}_k^{\otimes}(U)$ be the group of tensor-product preserving k-linear natural isomorphisms $U \to U$. Then $G(k) \xrightarrow{\sim} \mathrm{Aut}_k^{\otimes}(U)$, where $g \in G(k)$ maps to the natural transformation $U \to U$ given on $\rho\colon G \to \mathrm{GL}(V)$ by $\rho(g)\colon V \to V$.*

This is applied to the semisimple and unipotent parts $\rho(g)_s, \rho(g)_u$ for all $\rho\colon G \to \mathrm{GL}(V)$. (It is not hard to check that these are k-linear tensor automorphisms of U.)

Corollary 3.8. — *If $G \subseteq \mathrm{GL}_n(k)$ is a linear algebraic group and $g \in G(k)$ is semisimple (resp. unipotent) when viewed as element of $\mathrm{GL}_n(k)$, then $\rho(g)$ is semisimple (resp. unipotent) for every representation $\rho\colon G \to \mathrm{GL}(V)$.*

This can also be proven directly (without Tannaka duality).

Warning 3.9. — We have really used that the representations are algebraic: the above corollary is very much false for continuous representations $G(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ (and likewise for $G(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$), even when G is an algebraic group.

For instance, the group $\mathbf{R} = \mathbf{G}_a(\mathbf{R}) \cong \mathbf{U}_2(\mathbf{R})$ has a 1-dimensional representation

$$\begin{aligned} \mathbf{R} &\to \mathrm{GL}_1(\mathbf{R}) = \mathbf{R}^\times \\ t &\mapsto e^t. \end{aligned}$$

Although every element in $\mathbf{R}$ is unipotent, the image in $\mathrm{GL}_1(\mathbf{R})$ is semisimple.

Likewise, the 1-parameter subgroup $\rho\colon \mathbf{R} \hookrightarrow \mathrm{GL}_2(\mathbf{R})$ given by $t \mapsto \left(\begin{smallmatrix} 2^t & 2^{t-1}t \\ 0 & 2^t \end{smallmatrix}\right)$ does not contain the semisimple and unipotent parts of $\rho(1) = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, contrary to the situation of Theorem 3.6. Of course, this homomorphism is not given by polynomials.

The exercise below is an approximation of the situation described in this last example.

Exercise 3.10. — Suppose char $k = 0$.

(1) Prove that the map

$$\rho\colon \mathbf{G}_a \longrightarrow \mathrm{GL}_n(k)$$
$$x \longmapsto \begin{pmatrix} 1 & x & \binom{x}{2} & \cdots & \binom{x}{n-1} \\ 0 & 1 & x & \cdots & \binom{x}{n-2} \\ 0 & 0 & 1 & \cdots & \binom{x}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

is a group homomorphism. (It might help notationally to write $J = \rho(1) - I$ and express everything as polynomial in J.)

(2) Suppose k is moreover algebraically closed, and let $A \in \mathrm{GL}_n(k)$. Show that there is an injective homomorphism $\mathbf{G}_m^r \times \mathbf{G}_a \hookrightarrow \mathrm{GL}_n(k)$ whose image contains A. (Hint: write $A = A_s \cdot A_u$. Use $\mathbf{G}_m^r$ to cover A_s and $\mathbf{G}_a$ to cover A_u.)

(3) If none of the eigenvalues of A is a root of unity (meaning $\lambda^n = 1$ for some n), show that the subgroup constructed in (2) is the smallest algebraic subgroup of GL_n containing A. What happens if one of the eigenvalues is a root of unity?

4. Representations versus elements

From now on, k will be algebraically closed.

We saw that every representation $\rho\colon G \to \mathrm{GL}(V)$ preserves the (multiplicative) Jordan decomposition.

Corollary 4.1. — *If ρ is a representation of $\mathbf{D}_n$ (resp. $\mathbf{U}_n$), then $\rho(g)$ is semisimple (resp. unipotent) for all $g \in G(k)$.*

Proof. Every element of $\mathbf{D}_n$ (resp. $\mathbf{U}_n$) is semisimple (resp. unipotent). □

We will see that every representation of $\mathbf{D}_n$ (resp. $\mathbf{U}_n$ or $\mathbf{T}_n$) in GL_m lands in $\mathbf{D}_m$ (resp. $\mathbf{U}_m$ or $\mathbf{T}_m$) (for a suitable choice of basis!). We will then translate these properties to categorical properties of $\mathbf{Rep}_k(G)$.

4.1 Diagonal groups

Lemma 4.2. — *Let $T \subseteq \mathrm{GL}_n(k)$ be a set of commuting diagonalisable matrices. Then there exist $S \in \mathrm{GL}_n(k)$ such that SAS^{-1} is diagonal for* all $A \in T$.

Proof. We first prove the result when $T = \{A_1, \ldots, A_r\}$ is finite by induction on the number of elements. If $r \leq 1$, there is nothing to prove. Assume the result is proven for $r - 1$ commuting matrices, and consider the eigenspace decomposition $V = \bigoplus_i E_{\lambda_i}$ of A_r.

Then each E_{λ_i} is preserved by each A_j: if $v \in E_{\lambda_i}$, then $A_r v = \lambda_i v$, so

$$A_r(A_j v) = A_j(A_r v) = A_j(\lambda_i v) = \lambda_i A_j v,$$

so $A_j v$ is again in E_{λ_i}. Thus, we may apply the induction hypothesis to $A_1, \ldots, A_{r-1}$ acting on E_{λ_i} to get a further decomposition into subspaces on which $A_1, \ldots, A_{r-1}$ act via scalars. But A_r also acts as a scalar on E_{λ_i}. Putting together these decompositions for all i gives $V = \bigoplus_j V_j$ where each V_j is preserved by $A_1, \ldots, A_r$ and each A_i acts by a scalar on V_j.

This proves the result when T is finite. For infinite T, use that this process terminates as decompositions cannot infinitely shrink. Alternatively, note that the span of T in $\operatorname{End}(V)$ is finite-dimensional, so we may pick a finite subset $T' \subseteq T$ such that all other matrices are linear combinations of T'. If each $A \in T'$ acts by a scalar on V_j, then so do linear combinations. □

Theorem 4.3. — *Let G be a linear algebraic group. Then the following are equivalent:*

1. $G \subseteq \mathbf{D}_n$ *for some* n.
2. *Every representation* $\rho\colon G \to \mathrm{GL}(V)$ *lands in* $\mathbf{D}_n$ *for a suitable choice of basis of* V.
3. *Every representation* V *of* G *is a direct sum of* 1*-dimensional representations.*

A group satisfying the equivalent properties of the theorem is called *diagonalisable.*

Proof. If $G \subseteq \mathbf{D}_n$ and $\rho\colon G \to \mathrm{GL}(V)$, we saw that each $\rho(g)$ is diagonalisable. But $\mathbf{D}_n$ is commutative, so $\rho(G)$ consists of commuting diagonalisable matrices, hence they are simultaneously diagonalisable. This proves (1) ⇒ (2), and the converse follows by choosing a faithful representation $G \hookrightarrow \mathrm{GL}(V)$.

If $\rho\colon G \to \mathbf{D}_n \subseteq \mathrm{GL}_n$ is a representation, then $k^n = \bigoplus_{i=1}^n \operatorname{span}(e_i)$ is a direct sum decomposition into 1-dimensional subrepresentations. Conversely, if $V = \bigoplus_{i=1}^n V_i$ where all V_i are 1-dimensional, then choosing a basis $(v_1, \ldots, v_n)$ with $v_i \in V_i$ makes all matrices diagonal since it is a direct sum of representations. □

Exercise 4.4. — Show that a direct sum $V = \bigoplus_i V_i$ of representations $\rho_i\colon G \to \mathrm{GL}(V_i)$ is given by

$$\begin{aligned} G &\to \mathrm{GL}(V) \\ g &\mapsto \left(\begin{array}{c|c|c|c} \rho_1(g) & 0 & \cdots & 0 \\ \hline 0 & \rho_2(g) & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \rho_r(g) \end{array}\right). \end{aligned}$$

Likewise, if $W \subseteq V$ is a subspace and $(w_1, \ldots, w_r, v_{r+1}, \ldots, v_n)$ is a basis of V obtained by extending a basis $(w_1, \ldots, w_r)$ of W, then W is a subrepresentation if and only if ρ is given by

$$\rho(g) = \left(\begin{array}{c|c} \rho_W(g) & * \\ \hline 0 & \rho_{V/W}(g) \end{array}\right).$$

If V and W are G-representations, can you describe the matrix for $V \otimes W$?

4.2 Unipotent groups

Recall Schur's lemma:

Lemma 4.5. — *Assume k is algebraically closed, and let $\rho\colon G \to \mathrm{GL}(V)$ be a simple representation of G. Then $\mathrm{End}_G(V) = kI$.*

Here, $\mathrm{End}_G(V)$ means all matrices $A \in \mathrm{End}(V)$ such that $\rho(g)A = A\rho(g)$ for all $g \in G$, and V is simple if it has no nontrivial subrepresentations. The proof is the same as for finite groups.

Exercise 4.6. — Let $V_1, \ldots, V_r$ be G-representations, and set $V = \bigoplus_i V_i$. Show that

$$\mathrm{End}_G(V) = \left(\begin{array}{c|c|c|c} \mathrm{Hom}_G(V_1,V_1) & \mathrm{Hom}_G(V_2,V_1) & \cdots & \mathrm{Hom}_G(V_r,V_1) \\ \hline \mathrm{Hom}_G(V_1,V_2) & \mathrm{Hom}_G(V_2,V_2) & \cdots & \mathrm{Hom}_G(V_r,V_2) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{Hom}_G(V_1,V_r) & \mathrm{Hom}_G(V_2,V_r) & \cdots & \mathrm{Hom}_G(V_r,V_r) \end{array}\right).$$

In other words, a block matrix

$$\left(\begin{array}{c|c|c} A_{11} & \cdots & A_{1r} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{r1} & \cdots & A_{rr} \end{array}\right)$$

commutes with the G-action on V if and only if $A_{ij}\rho_i(g) = \rho_j(g)A_{ij}$ for all $i, j \in \{1, \ldots, r\}$ and all $g \in G(k)$. In particular, if V is simple, then $\mathrm{End}_G(V^n) \cong M_n(k)$, the algebra of $n \times n$ matrices, acting by blockwise scalar matrices.

Lemma 4.7. — *Let $V_1, \ldots, V_n$ be simple representations of a group G, let $V = \bigoplus_i V_i$, and let $U \subseteq V$ be a subrepresentation. Then there exists a complementary subrepresentation $W \subseteq V$, i.e. $V = U \oplus W$.*

Proof. Let $I = \{1, \ldots, n\}$, and for each $J \subseteq I$ write $V_J = \bigoplus_{j \in J} V_j$. Choose $J \subseteq I$ inclusionwise maximal such that $V_J \cap U = 0$. Then we claim that $V = U \oplus V_J$. By definition, we have $V_J \cap U = 0$, so we only need to prove $V_J + U = V$. If not, there exists $k \in I$ with $V_k \not\subseteq V_J + U$. Note that $k \notin J$ as $V_j \subseteq V_J$ for all $j \in J$. Since V_k is simple, we must have $V_k \cap (V_J + U) = 0$. Then $V_{J \cup \{k\}} \cap U = (V_J + V_k) \cap U = 0$ as well, contradicting maximality of J. □

Proposition 4.8 (Density theorem). — *If $G \subseteq \mathrm{GL}(V)$ is a subgroup acting irreducibly on V, then G spans the matrix algebra $\mathrm{End}(V)$.*

Proof. Let $(v_1, \ldots, v_n)$ be a basis of V, and let $A \in \mathrm{End}(V)$ be any matrix. If $W = V^n$, we can identify the matrix A with the vector $(Av_1, \ldots, Av_n) \in W$. Let $w = (v_1, \ldots, v_n) \in W$, and consider the linear map $B\colon W \to W$ given by

$$\left(\begin{array}{c|c|c} A & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & A \end{array}\right),$$

so that $(Av_1, \ldots, Av_n) = Bw$. Let $U \subseteq W$ be the submodule generated by w (i.e. the subspace generated by $\rho_W(g)w$ for all $g \in G$). By Lemma 4.7, there exists a complement $X \subseteq W$ with $W \cong U \oplus X$ as G-representations. Let $C\colon W \to W$ be the projection to U, which by Exercise 4.6 is a blockwise scalar matrix (for the decomposition $W = V^n$). In particular, C commutes with B, so $Bw = BCw = CBw \in U$ since $\operatorname{im} C \subseteq U$. □

Definition 4.9. — If $\rho\colon G \to \mathrm{GL}(V)$ is a representation, the *G-invariant subspace* is $V^G = \{v \in V \mid \rho(g)v = v \text{ for all } g \in G(k)\}$. It is the maximal subrepresentation of V on which G acts trivially.

Theorem 4.10. — *Let G be a linear algebraic group. Then the following are equivalent:*

(1) *G is a subgroup of $\mathbf{U}_n$ for some n.*
(2) *All elements of $G(k)$ are unipotent.*
(3) *If V is a simple G-representation, then $V = k$ with the trivial action.*
(4) *If V is a nonzero G-representation, then $V^G \neq 0$.*
(5) *If $\rho\colon G \to \mathrm{GL}(V)$ is a representation, then ρ lands in $\mathbf{U}_n$ for a suitable choice of basis.*

A group satisfying the equivalent properties of the theorem is called *unipotent*.

Exercise 4.11. — Let $A \in M_n(k)$ be a nonzero matrix. Then

$$\operatorname{span}\left\{BAC \mid B, C \in M_n(k)\right\} = M_n(k).$$

Exercise 4.12. — If all elements of a linear algebraic group G are conjugate to an element in $\mathbf{T}_n$, is it true that G is a subgroup of $\mathbf{T}_n$ for a suitable choice of basis?

Proof of Theorem. All elements of $\mathbf{U}_n$ are unipotent, proving (1) $\Rightarrow$ (2).

If all elements $g \in G(k)$ are unipotent, then $\rho(g)$ is unipotent for any representation ρ by Theorem 3.6. Thus, we may replace G by its image in $\mathrm{GL}(V)$. Assume V is simple, so $G(k)$ spans $\mathrm{End}(V)$ by Proposition 4.8. Suppose there exists $g \neq 1$ in $G(k)$, and set $x = g_n = I - g$ (since g is unipotent). Since x is nonzero, Exercise 4.11 shows that $\{BxC \mid B, C \in \mathrm{End}(V)\}$ spans $\mathrm{End}(V)$. Since $\mathrm{End}(V)$ is spanned by $G(k)$, it in fact suffices to take $\{hxh' \mid h, h' \in G(k)\}$. But $hxh' = h(I-g)h' = hh' - hgh'$ is a difference of unipotent matrices, so has trace 0. We can never get all of $M_n(k)$ this way, so we conclude that G is the trivial group, proving (2) $\Rightarrow$ (3).

Suppose every simple representation is k with the trivial action, and let V be a nonzero representation. Take a simple subrepresentation $W \subseteq V$, which is trival by assumption, hence lives inside V^G. This proves (3) $\Rightarrow$ (4).

If $V^G \neq 0$ for any nonzero representation V, we prove by induction on $n = \dim V$ that ρ

lands in $\mathbf{U}_n$ for a suitable choice of basis. The result is clear for $n \leq 1$ as then V is the trivial representation. In general, pick a nonzero element $v_1 \in V^G$, so $W = \operatorname{span}(v_1)$ is a subrepresentation. By Exercise 4.4, the elements in G look like

$$g = \left(\begin{array}{c|c} \rho_W(g) & * \\ \hline 0 & \rho_{V/W}(g) \end{array}\right).$$

By induction, there exists a basis of V/W for which $\rho_{V/W}(g)$ is upper triangular unipotent for all $g \in G(k)$, proving the statement for V as well. This proves (4) $\Rightarrow$ (5). The final implication (5) $\Rightarrow$ (1) follows by taking a faithful representation $\rho \colon G \hookrightarrow \mathrm{GL}_n$. ◻

4.3 Triangular groups

Combining the results for $\mathbf{D}_n$ and $\mathbf{U}_n$ gives a statement for $\mathbf{T}_n$. Recall from Exercise 2.8 that there is a surjection $\mathbf{T}_n \to \mathbf{D}_n$ with kernel $\mathbf{U}_n$.

Lemma 4.13. — *Let $N \subseteq G$ be a normal algebraic subgroup, and let V be a G-representation. Then V^N is a sub-G-representation.*

Proof. Suppose $\rho(n)v = v$ for all $n \in N(k)$, and let $g \in G(k)$. Since N is normal, the element $n' = g^{-1}ng$ is in N, so

$$\rho(n)\rho(g)v = \rho(g)\rho(n')v = \rho(g)v$$

since $v \in V^N$ and $n' \in N$. Thus, $\rho(g)v \in V^N$ as well. ◻

Theorem 4.14. — *Let G be a linear algebraic group. Then the following are equivalent:*

(1) *G is a subgroup of $\mathbf{T}_n$ for some n.*
(2) *There exists a normal unipotent subgroup $U \subseteq G$ such that $D = G/U$ is diagonalisable.*
(3) *Every nonzero G-representation contains a 1-dimensional subrepresentation.*
(4) *If $\rho \colon G \to \mathrm{GL}(V)$ is a representation, then ρ lands in $\mathbf{T}_n$ for a suitable choice of basis.*

A group satisfying the equivalent properties of the theorem is called *trigonisable*.

Proof. If $G \subseteq \mathbf{T}_n$, then $U = G \cap \mathbf{U}_n$ is unipotent and $G/U \hookrightarrow \mathbf{T}_n/\mathbf{U}_n \cong \mathbf{D}_n$ is diagonalisable, proving (1) $\Rightarrow$ (2).

Suppose $U \subseteq G$ is normal and G/U is diagonalisable, and let V be a nonzero G-representation. By Theorem 4.10, we have $V^U \neq 0$, and this is a subrepresentation by Lemma 4.13. The action of G on V^U is trivial on U, hence factors via $D = G/U$. By Theorem 4.3, every representation of D is a direct sum of 1-dimensional representations, so it has a 1-dimensional sub-D-representation $W \subseteq V^U$. Then W is also a sub-G-representation, proving (2) $\Rightarrow$ (3).

Suppose every nonzero representation has a 1-dimensional subrepresentation, and let $\rho \colon G \to \mathrm{GL}(V)$ be a representation. We will prove by induction on $n = \dim V$ that ρ lands in $\mathbf{T}_n$ for a suitable choice of basis. The result is clear if $n \leq 1$ as $\mathbf{T}_n = \mathrm{GL}_n$ in that case. For arbitrary V, there exists a 1-dimensional subrepresentation $W \subseteq V$. By Exercise 4.4, the action of G is given by

$$\rho(g) = \left(\begin{array}{c|c} \rho_W(g) & * \\ \hline 0 & \rho_{V/W}(g) \end{array}\right).$$

By induction, there exists a basis of V/W for which $\rho_{V/W}(g)$ is upper triangular for all $g \in G(k)$, proving the statement for V as well. This proves (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1) follows by taking a faithful representation $\rho \colon G \hookrightarrow \mathrm{GL}(V)$. □

5. Reductive groups

To summarise the situation so far:

- Diagonalisable groups: all representations are direct sums of 1-dimensional representations.
- Unipotent groups: the only simple representation is the trivial representation.
- Trigonisable groups: all simple representations are 1-dimensional.

What about representations of GL_n, SL_n, or parabolic subgroups?

Example 5.1. — The defining representation of $\mathrm{GL}(V)$ on V is clearly irreducible. The same does not hold for parabolic subgroups $P \subseteq \mathrm{GL}_n$: the block upper triangular form means certain subspaces are P-invariant.

Definition 5.2. — A linear algebraic group G is *connected* if it does not have algebraic subgroups $H \subsetneq G$ of the same dimension.

Example 5.3. — Everything we've seen is connected. But finite groups are also allowed.

Example 5.4. — The group $\mathbf{G}_m$ is connected, but it doesn't look connected over $\mathbf{R}$ since $\mathbf{G}_m(\mathbf{R}) = \mathbf{R}^\times \cong \{\pm 1\} \times \mathbf{R}_{>0}$. However, $\mathbf{R}_{>0} \subseteq \mathbf{R}^\times$ is not algebraic. On the other hand, a linear algebraic group G over $\mathbf{C}$ is connected if and only if $G(\mathbf{C})$ is connected in the complex topology.

Proposition 5.5. — *Let G be a linear algebraic group. There is a unique maximal connected normal unipotent subgroup $U \subseteq G$.*

The group U is called the *unipotent radical* of G, and denoted $R_u(G)$.

Proof (idea). Take a connected normal unipotent subgroup $U \subseteq G$ of maximal dimension. If H is any other connected normal unipotent subgroup, then UH is again normal. Show that it is also connected and unipotent. By maximality, this forces $UH = U$, so $H \subseteq U$. □

Example 5.6. — The unipotent radical of a subgroup $G \subseteq \mathbf{T}_n$ is the group $G \cap \mathbf{U}_n$ that came up in the proof of Theorem 4.14.

Example 5.7. — The unipotent radical of the parabolic subgroup

$$P = \left(\begin{array}{c|c|c|c} * & * & \cdots & * \\ \hline 0 & * & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & * \end{array}\right).$$

is the group

$$R_u(P) = \left(\begin{array}{c|c|c|c} I & * & \cdots & * \\ \hline 0 & I & \cdots & * \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & I \end{array}\right).$$

If the blocks have size $m_1, \ldots, m_s$, then $P/R_u(P) \cong \mathrm{GL}_{m_1} \times \ldots \times \mathrm{GL}_{m_s}$, as block diagonal matrices

$$P/R_u(P) \cong \left(\begin{array}{c|c|c|c} * & 0 & \cdots & 0 \\ \hline 0 & * & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & * \end{array}\right).$$

Definition 5.8. — A linear algebraic group G is *reductive* if G is connected and $R_u(G) = 1$.

Example 5.9. — Connected diagonalisable groups are reductive. They are all isomorphic to $\mathbf{G}_m^n$.

Lemma 5.10. — *If G is connected and G has a faithful and semisimple G-representation V, then G is reductive.*

Proof. Write $U = R_u(G)$, and write $V = \bigoplus_i V_i$ as a sum of simple G-representations. Then each V_i^U is a subrepresentation of V_i since U is normal (Lemma 4.13). Since U is unipotent, we have $V_i^U \neq 0$, which by simplicity means $V_i^U = V_i$. Thus, U acts trivially on V, hence $U = 1$ since $\rho\colon G \to \mathrm{GL}(V)$ is injective. □

Example 5.11. — Since $\mathrm{GL}(V)$ acts irreducibly on V, we see that $\mathrm{GL}(V)$ is reductive.

Fact 5.12. — If G is connected, then $G/R_u(G)$ is the maximal reductive quotient of G.

Example 5.13. — For a parabolic subgroup $P \subseteq \mathrm{GL}_n$, the quotient $P/R_u(P) \cong \prod_i \mathrm{GL}_{m_i}$ is reductive by Lemma 5.10 and Example 5.11

Theorem 5.14. — *If* char $k = 0$*, then a connected linear algebraic group G is reductive if and only if every G-representation splits as a direct sum of simple representations.*

The proof (the one I know, anyway) is quite involved, going pretty far into Lie algebras. We did see one implication in Lemma 5.10, but the other implication is more useful.

There is also a quite good understanding of what reductive groups look like:

- Classify all *simple* algebraic groups: they come in four families plus six exceptional types. (This is a very beautiful theory.)

- Then take finite products of these, and quotient out by a finite subgroup to get all *semisimple* algebraic groups.
- Finally, G is reductive if and only if it has a diagonalisable normal subgroup $D \subseteq G$ such that G/D is semisimple. For instance, $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbf{G}_m$ (quotient by the diagonal matrices) is simple.

The representation theory of GL_n and SL_n is very well understood (especially in characteristic 0), and can be expressed purely combinatorially. There is also a beautiful connection between representations of SL_n and representations of the symmetric groups S_m.