ICS on Condensed Mathematics—Notes for talk 4

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These notes are based on Section 3 of [3], but I give somewhat different proofs for the main results. The main theorems in these notes are entirely due to Clausen and Scholze. All mistakes are due to me.

1 Cohomology in a topos

1.1 *Notation.* If C is a site, we write

$$\begin{aligned} \mathrm{PSh}(\mathrm{C}) &= \text{category of presheaves of sets (also called } \hat{\mathrm{C}}) \\ \mathrm{Sh}(\mathrm{C}) &= \text{category of sheaves of sets (also called } \tilde{\mathrm{C}}) \\ \mathrm{PAb}(\mathrm{C}) &= \text{category of presheaves of abelian groups} \\ \mathrm{Ab}(\mathrm{C}) &= \text{category of sheaves of abelian groups} \end{aligned}$$

1.2 The left adjoint of the inclusion $\mathrm{Sh}(\mathrm{C}) \to \mathrm{PSh}(\mathrm{C})$ is the sheafification functor, which we denote by $\mathscr{F} \mapsto \mathscr{F}^\sharp$. If $\mathscr{F}$ is a presheaf of abelian groups then $\mathscr{F}^\sharp$ (meaning: the sheafification of the underlying presheaf of sets) is a sheaf of abelian groups, and the functor $\mathrm{PAb}(\mathrm{C}) \to \mathrm{Ab}(\mathrm{C})$ given by $\mathscr{F} \mapsto \mathscr{F}^\sharp$ is again the left adjoint of the inclusion functor.

The inclusion $\mathrm{PAb}(\mathrm{C}) \to \mathrm{PSh}(\mathrm{C})$ has a left adjoint, which we denote by $\mathscr{F} \mapsto \mathbb{Z}_{\mathscr{F}}$. Concretely, $\mathbb{Z}_{\mathscr{F}}$ sends an object T to the free abelian group on the set $\mathscr{F}(T)$. The left adjoint of $\mathrm{Ab}(\mathrm{C}) \to \mathrm{PSh}(\mathrm{C})$ is then $\mathscr{F} \mapsto \mathbb{Z}_{\mathscr{F}}^\sharp$, where the latter denotes the sheafification of $\mathbb{Z}_{\mathscr{F}}$. Note that even $\mathscr{F}$ is a sheaf, $\mathbb{Z}_{\mathscr{F}}$ is usually not a sheaf. For instance, if U and V are disjoint open subsets of a topological space X then $\mathscr{F}(U \sqcup V) = \mathscr{F}(U) \times \mathscr{F}(V)$, and therefore $\mathbb{Z}_{\mathscr{F}}(U \sqcup V) = \mathbb{Z}_{\mathscr{F}}(U) \otimes \mathbb{Z}_{\mathscr{F}}(V)$; but we want to have $\mathbb{Z}_{\mathscr{F}}^\sharp(U \sqcup V) = \mathbb{Z}_{\mathscr{F}}^\sharp(U) \times \mathbb{Z}_{\mathscr{F}}^\sharp(V)$.

Scholze uses the notation $\mathbb{Z}[\mathscr{F}]$ instead of $\mathbb{Z}_{\mathscr{F}}^\sharp$. We shall follow this.

1.3 If $X \in \tilde{\mathrm{C}} = \mathrm{Sh}(\mathrm{C})$ and $\mathscr{F} \in \mathrm{Ab}(\mathrm{C})$, the cohomology groups of X with coefficients in $\mathscr{F}$ are defined by

$$H^i(X, \mathscr{F}) := \mathrm{Ext}^i\big(\mathbb{Z}[X], \mathscr{F}\big),$$

where the Ext-groups are calculated in the abelian category $\mathrm{Ab}(\mathrm{C})$ (which has enough injectives).

If $X = h_X$ for some object X of C then

$$\mathrm{Mor}_{\mathrm{Ab}(\mathrm{C})}(\mathbb{Z}[X], \mathscr{F}) = \mathrm{Mor}_{\tilde{\mathrm{C}}}(X, \mathscr{F}) = \mathrm{Mor}_{\tilde{\mathrm{C}}}(h_X, \mathscr{F}) = \mathscr{F}(X)$$

by Yoneda. Hence the $H^i(X, -)$ are the right derived functors of $\mathscr{F} \mapsto \mathscr{F}(X)$.

The Ext-groups can be calculated in either variable, i.e., using an injective resolution of $\mathscr{F}$ or a projective resolution of $\mathbb{Z}[X]$.

2 Hypercoverings

Reference: Deligne [1] or the Stacks Project [4], especially Chapters 0162 and 01FX.

2.1 Let Δ be the category whose objects are the sets $[n] = \{0, \ldots, n\}$ for $n \in \mathbb{N}$, and in which the morphisms $[m] \to [n]$ are the nondecreasing maps.

For $n \geq 1$, we have the morphisms $\delta_j^n \colon [n-1] \to [n]$ $(0 \leq j \leq n)$ that are characterised by the fact that δ_j^n is injective and $j \notin \mathrm{Im}(\delta_j^n)$. For $n \geq 0$ and $0 \leq j \leq n$ we also have the maps $\sigma_j^n \colon [n+1] \to [n]$ that are surjective and have $(\sigma_j^n)^{-1}\{j\} = \{j, j+1\}$. Every morphism in Δ can be obtained as a composition of maps of the form δ_j^n and σ_j^n, for varying indices n and j. The maps δ_j^n and σ_j^n satisfy a number of commutation relations that we will not spell out here.

If there is no risk of confusion, we often omit the indices n in the notation δ_j^n and σ_j^n.

2.2 If C is a category, a simplicial object in C is a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{C}$. One writes X_n for $X_\bullet\big([n]\big)$. Further, we write $d_j^n \colon X_n \to X_{n-1}$ for $\delta_n^{j,*}$ and $s_j^n \colon X_n \to X_{n+1}$ for $\sigma_j^{n,*}$. Concretely, a simplicial object is then given by a diagram

$$X_\bullet : \qquad \cdots\ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} X_3\ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} X_2\ \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} X_1\ \substack{\rightarrow\\ \leftarrow\\ \rightarrow} X_0$$

in which the various arrows d_j^n and s_j^n (not labeled in the diagram) satisfy the required commutation relations.

We write Simp(C) for the category of simplicial objects in C.

2.3 An object S of C can be viewed as a constant simplicial object $S_\bullet$: take $S_n = S$ for all n, and let all arrows d_i and s_i be the identity.

An augmentation $a \colon X_\bullet \to S$ of a simplicial object $X_\bullet$ to an object S in C is a morphism $a_0 \colon X_0 \to S$ such that $a_0 \circ d_0^1 = a_0 \circ d_1^1$. For a morphism $\varphi \colon [0] \to [n]$ in Δ, we obtain a morphism $a_0 \circ X_\bullet(\varphi) \colon X_n \to S$. This morphism is independent of the choice of φ, and we denote it by $a_n \colon X_n \to S$.

To give such an augmentation is the same as giving a morphism $X_\bullet \to S_\bullet$ in Simp(C).

2.4 For $n \geq 0$, let $\Delta_{\leq n}$ be the full subcategory of Δ whose objects are the $[m]$ with $m \leq n$. An n-truncated simplicial object in C is a functor $\Delta_{\leq n}^{\mathrm{op}} \to \mathrm{C}$. Write $\mathrm{Simp}_n(\mathrm{C})$ for the category of n-truncated simplicial object in C.

2.5 The inclusion $\Delta_{\leq n} \to \Delta$ induces functors

$$\mathrm{sk}_n \colon \mathrm{Simp}(\mathrm{C}) \to \mathrm{Simp}_n(\mathrm{C})\,,$$

called skeleton functors.

Assume that finite limits exist in C. Then the skeleton functors sk_n have right adjoints

$$\mathrm{cosk}_n \colon \mathrm{Simp}_n(\mathrm{C}) \to \mathrm{Simp}(\mathrm{C})\,,$$

called coskeleton functors. If $Y_\bullet$ is an n-truncated simplicial object, one may think of $\operatorname{cosk}_n(Y_\bullet)$ as the minimal way to extend $Y_\bullet$ to a simplicial object. The canonical morphism $\operatorname{sk}_n\operatorname{cosk}_n(Y_\bullet) \to Y_\bullet$ (the counit of the adjunction) is an isomorphism.

Given an n-truncated simplicial object $Y_\bullet$, the terms of $\operatorname{cosk}_n(Y_\bullet)$ are constructed as limits of certain (finite) diagrams. In general this seems of limited use because the combinatorics of these diagrams is somewhat involved. However, the term $\operatorname{sk}_n\operatorname{cosk}_n(Y_\bullet)_{n+1}$ can be made explicit, and this will be important for us. We return to this in Section 5.

2.6 Examples. (1) A 0-truncated simplicial object in C is simply an object Y_0 of C. We have

$$\operatorname{cosk}_0(Y_0)_n = Y_0 \times Y_0 \times \cdots \times Y_0 \qquad (n+1 \text{ factors}).$$

(2) A 1-truncated simplicial object in C is a diagram

$$Y_1 \underset{d_1}{\overset{d_0}{\underset{\xleftarrow{s_1}}{\rightrightarrows}}} Y_0$$

with $d_1 \circ s_1 = \mathrm{id}_{Y_0} = d_0 \circ s_1$. Then

$$\operatorname{cosk}_1(Y_{\le 1})_2 = \operatorname{Eq}\Big(Y_1 \times Y_1 \times Y_1 \rightrightarrows Y_0 \times Y_0 \times Y_0\Big)$$

where the two maps are given by

$$(x_0, x_1, x_2) \mapsto \big((d_0(x_0), d_1(x_0), d_1(x_1)\big), \quad \text{resp.} \quad (x_0, x_1, x_2) \mapsto \big((d_0(x_1), d_0(x_2), d_1(x_2)\big).$$

In other words, we have the "equations"

$$d_0(x_0) = d_0(x_1), \quad d_1(x_0) = d_0(x_2), \quad \text{and} \quad d_1(x_1) = d_1(x_2).$$

The three maps $d_i \colon \operatorname{cosk}_1(Y_{\le 1})_2 \to \operatorname{cosk}_1(Y_{\le 1})_1 = Y_1$ are given by

$$(x_0, x_1, x_2) \mapsto \begin{cases} d_0(x_0) = d_0(x_1) & \text{for } i = 0, \\ d_1(x_0) = d_0(x_2) & \text{for } i = 1, \\ d_1(x_1) = d_1(x_2) & \text{for } i = 2. \end{cases}$$

The two maps $s_i \colon Y_1 \to \operatorname{cosk}_1(Y_{\le 1})_2$ are given by sending $w \in Y_1$ to $(s_1 d_0(w), w, w)$, respectively $(w, w, s_1 d_1(w))$. (We use "pointwise" notation to describe this example, as otherwise the notation becomes too involved.) Already in this example, the terms $\operatorname{cosk}_1(Y_{\le 1})_n$ with $n \ge 3$ become somewhat complicated.

2.7 Let C be a site and let $X_\bullet$ be a simplicial object in $\tilde{\mathrm{C}} = \mathrm{Sh}(\mathrm{C})$. We obtain a chain complex $\mathbb{Z}[X_\bullet]$ in Ab(C) by

$$\mathbb{Z}[X_\bullet]: \qquad \cdots \xrightarrow{d} \mathbb{Z}[X_2] \xrightarrow{d} \mathbb{Z}[X_1] \xrightarrow{d} \mathbb{Z}[X_0]$$

where the differentials $d \colon \mathbb{Z}[X_n] \to \mathbb{Z}[X_{n-1}]$ are given by $d = \sum_{i=0}^n (-1)^i d_i$.

An augmentation $a \colon X_\bullet \to S$ induces a morphism of chain complexes $\mathbb{Z}[X_\bullet] \to \mathbb{Z}[S]$, where $\mathbb{Z}[S]$ is viewed as a chain complex whose only nonzero term is in degree 0.

2.8 Definition. Let C be a site that has finite limits. Let S be an object of C. A hypercovering of S is a simplicial object $X_\bullet$ in Simp(C) together with an augmentation $a\colon X_\bullet \to S$ such that:

(a) $a_0\colon X_0 \to S$ is a covering in C;
(b) the morphism $(d_0, d_1)\colon X_1 \to X_0 \times_S X_0$ is a covering in C;
(c) for every $n \geq 1$ the natural morphism $X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$ (the unit of the adjunction) is a covering in C.

2.9 Remarks. (1) The second condition is not the special case $n = 0$ of the third condition. A simplicial object with an augmentation to S may be considered as a simplicial object in the category $\mathrm{C}_{/S}$. For $n \geq 1$ the coskeleton functors cosk_n in $\mathrm{C}_{/S}$ can be calculated in C; see [4], Lemma 08NJ. For $n = 0$ this is not true: if $Y_0 \to S$ is a morphism in C, the terms of $\mathrm{cosk}_0(Y_0)$ are given by $Y_0 \times \cdots \times Y_0$ (see 2.5), whereas the terms of $\mathrm{cosk}_0(Y_0)$ when working in $\mathrm{C}_{/S}$ are given by the fibre products $Y_0 \times_S \cdots \times_S Y_0$.

(2) We can generalise this definition, working with the category $\mathrm{SR}(\mathrm{C}, S) = \mathrm{SR}(\mathrm{C}_{/S})$ of semirepresentable objects over S; see [4], Section 0DBB. We shall not need this.

2.10 Proposition. *Let* C *be a site in which finite limits exist. Let $a\colon X_\bullet \to S$ be a hypercovering in* C*. Then $a\colon \mathbb{Z}[X_\bullet] \to \mathbb{Z}[S]$ is a resolution of $\mathbb{Z}[S]$ in* Ab(C)*.*

For the proof, see [4], Lemma 01GF.

2.11 Corollary. *In the situation of the proposition, if $\mathscr{F}$ is a sheaf of abelian groups on* C*, we have a spectral sequence*

$$E_1^{pq} = H^q(X_p, \mathscr{F}) \quad \Rightarrow \quad H^{p+q}(S, \mathscr{F}). \tag{2.11.1}$$

3 Proper hypercoverings of topological spaces

Reference: The Stacks Project [4], Section 09XA.

3.1 A continuous map of topological spaces $f : X \to Y$ is said to be proper if f is separated (meaning that the diagonal in $X \times_Y X$ is closed) and universally closed (meaning that for every topological space Z the map $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is closed).

Every continuous map from a compact Hausdorff space to a Hausdorff space is proper.

3.2 Definition. Let S be a topological space. Then a proper hypercovering of S is an augmented simplicial topological space $a\colon X_\bullet \to S$ such that

(a) $a_0\colon X_0 \to S$ is proper surjective;
(b) the map $(d_0, d_1)\colon X_1 \to X_0 \times_S X_0$ is proper surjective;
(c) for every $n \geq 1$ the map $X_{n+1} \to \mathrm{cosk}_n(\mathrm{sk}_n X_\bullet)_{n+1}$ is proper surjective.

The main use of such proper hypercoverings is that sheaf cohomology can be computed on the simplicial covering:

3.3 Theorem. *Let $a\colon X_\bullet \to S$ be a proper hypercovering of a topological S. Then for every sheaf of abelian groups $\mathscr{F}$ on S we have a spectral sequence*

$$E_1^{pq} = H^q(X_p, a_p^{-1}\mathscr{F}) \quad\Rightarrow\quad H^{p+q}(S,\mathscr{F})\,. \tag{3.3.1}$$

3.4 Remark. Suppose all spaces S and X_n are compact Hausdorff. In this case the proper hypercovering $a\colon X_\bullet \to S$ is a hypercovering in CHaus. However, even in that case, (3.3.1) is not the same as (2.11.1). The E_1^{pq}-terms in (3.3.1) are sheaf cohomology groups on the spaces X_p, whereas the terms in (2.11.1) are cohomology groups in the topos Ab(C). A priori this is something different.

A proof of Theorem 3.3 is given by Deligne in [1], Section 5.3. He uses that a proper surjective map of topological spaces is a universal cohomological descent morphism; this is proven in SGA IV, Exp. $\mathrm{V}^{\mathrm{bis}}$ by Saint-Donat.

For locally compact spaces S (which is the only case we need), another proof of Theorem 3.3 can be found in the Stacks Project, [4], Section 09XA. The argument there is based on the study of a site called $\mathrm{LC}_{\mathrm{qc}}$, whose objects are the locally compact Hausdorff spaces. (The Grothendieck topology that is considered is a very nontrivial one.) There is a morphism of topoi $\alpha_S\colon \mathrm{Sh}(\mathrm{LC}_{/S}) \to \mathrm{Sh}(S)$; the functor α_S^{-1} sends a sheaf $\mathscr{F}$ on S to the sheaf on $\mathrm{LC}_{/S}$ whose value at an object $f\colon Y \to S$ is $\Gamma(Y, f^{-1}\mathscr{F})$. The key point ([4], Lemmas 0D91 and 09X4) is that the adjunction morphism $\mathrm{Id} \to R\alpha_{S,*}\alpha_S^{-1}$ on the derived category $D^+(S)$ is an isomorphism; this just expresses that the sheaf cohomology of a sheaf $\mathscr{F}$ on S is the same as the cohomology of $\alpha_S^{-1}\mathscr{F}$ in the topos $\mathrm{Sh}(\mathrm{LC}_{/S})$. This gives what we want because a proper hypercovering $a\colon X_\bullet \to S$ of S is a hypercovering in $\mathrm{LC}_{/S}$, so now Corollary 2.11 applies.

3.5 Lemma. *If X is a profinite space and $\mathscr{F}$ is a sheaf of abelian groups on X, we have $H^i(X,\mathscr{F}) = 0$ for all $i \geq 1$.*

For a proof, see [4], Lemma OA3F. The key point is that every open covering of a profinite space X has a refinement of the form $X = U_1 \sqcup \cdots \sqcup U_N$, where $U_1,\ldots,U_N$ are open and closed subsets of X.

3.6 Corollary. *Let $a\colon X_\bullet \to S$ be a proper hypercovering of a topological S such that all spaces X_n are profinite. Then for every sheaf of abelian groups $\mathscr{F}$ on S we have*

$$H^i(S,\mathscr{F}) \xrightarrow{\sim} \mathscr{H}^i\big[\Gamma(X_0, a_0^{-1}\mathscr{F}) \xrightarrow{d} \Gamma(X_1, a_1^{-1}\mathscr{F}) \xrightarrow{d} \cdots\big]$$

where the differentials $d\colon \Gamma(X_n, a_n^{-1}\mathscr{F}) \to \Gamma(X_{n+1}, a_{n+1}^{-1}\mathscr{F})$ are given by $d = \sum_{i=0}^{n+1} (-1)^i d_i^*$.

4 Condensed cohomology with discrete coefficients

4.1 *Notation.* Write

$$\mathrm{CHaus} = \text{category of compact Hausdorff spaces}$$
$$\mathrm{ProFin} = \text{category of profinite spaces } (= *_{\text{proét}})$$

As topology on either of these we take the one for which the coverings of a space X are the jointly surjective finite collections of maps $\{Y_i \to X\}_{i=1,\dots,n}$ (with all Y_i in the given category).

The inclusion functor ProFin $\to$ CHaus is continuous and defines a morphism of topoi

$$\mathrm{Sh}(\mathrm{CHaus}) \to \mathrm{Sh}(\mathrm{ProFin})\,. \tag{4.1.1}$$

4.2 Proposition. *The morphisms of topoi* (4.1.1) *is an equivalence.*

Reference: [3], Lecture II. (This result was discussed in Noah Olander's talk.)

4.3 *Notation.*

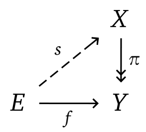
$$\begin{aligned} \mathrm{Cond}(\mathrm{Set}) &:= \mathrm{Sh}(\mathrm{ProFin}) && \text{the category of condensed sets;} \\ \mathrm{Cond}(\mathrm{Ab}) &:= \mathrm{Ab}(\mathrm{ProFin}) && \text{the category of condensed abelian groups.} \end{aligned}$$

By the proposition, we may (and will) instead work in Sh(CHaus) and Ab(CHaus).

If S is any topological space, the functor $X \mapsto \mathrm{Mor}_{\mathrm{Top}}(X,S)$ defines a condensed set $\underline{S}$.

4.4 If S is any compact Hausdorff space and $\mathscr{F}$ is a condensed abelian group, we define the condensed cohomology groups $H^i_{\mathrm{cond}}(S,\mathscr{F})$ as in Section 1.1, working in the category Ab(CHaus). For $i = 0$ we get $H^0_{\mathrm{cond}}(S,\mathscr{F}) = \mathscr{F}(S)$.

4.5 A topological space X is said to be extremally disconnected if the closure of every open subset $U \subset X$ is again open. If in addition X is compact Hausdorff then X is profinite. We write EDCH for the category of extremely disconnected compact Hausdorff spaces. The extremally disconnected compact Hausdorff spaces are the projective objects in CHaus. We shall repeatedly make use of the fact that whenever we have a solid diagram



in CHaus with E extremally disconnected, there exists a continuous $s\colon E \to X$ with $\pi \circ s = f$. In particular, every continuous surjection $X \twoheadrightarrow Y$ in CHaus with Y extremally disconnected has a continuous section.

The Stone–Čech compactification of any discrete space is an object of EDCH. It follows that for every compact Hausdorff space S there exists an extremally disconnected Hausdorff space T and a continuous surjection $T \to S$. (Consider $\mathrm{id}\colon S_{\mathrm{discr}} \to S$ and take $T = \beta(S_{\mathrm{discr}})$.)

4.6 Lemma. *Let S be an object in* EDCH *(i.e., an extremally disconnected compact Hausdorff space). Then $\mathbb{Z}[\,\underline{S}\,]$ is a projective object in* Cond(Ab).

Reference: [3], Lecture II. (This result was discussed in Francesca Leonardi's talk.)

4.7 Proposition. *Let S be a compact Hausdorff space. Let $a\colon X_\bullet \to S$ be a hypercovering in* CHaus *such that all X_n are extremally disconnected. Then for every condensed abelian group $\mathscr{F}$ we have*

$$H^i_{\mathrm{cond}}(S,\mathscr{F}) \cong \mathscr{H}^i\bigl[\mathscr{F}(X_0) \to \mathscr{F}(X_1) \to \mathscr{F}(X_2)\cdots\bigr].$$

Proof. Immediate from Corollary 2.11 and Lemma 4.6. □

4.8 Proposition. *Let S be a compact Hausdorff space. Then there exists a hypercovering $X_\bullet$ of S in* CHaus *such that all terms X_n are extremally disconnected.*

For the proof we refer to [4], Lemma 094A, in which we take $\mathscr{C} = \mathrm{CHaus}$ and $\mathscr{B} \subset \mathrm{Ob}(\mathscr{C})$ the extremally disconnected spaces. Replacing each $K_n = \{U_{n,i}\}_{i\in I_n}$ (with I_n finite) that is produced in the proof of that lemma by the disjoint union of the spaces $U_{n,i}$ gives us a hypercovering in which each X_n is a single extremally disconnected compact Hausdorff space.

4.9 Theorem. *If S is a compact Hausdorff space and A is a discrete abelian topological group,*

$$H^i(S,A_S) \cong H^i_{\mathrm{cond}}(S,\underline{A})$$

for all $i \geq 0$.

Here A_S denotes the constant sheaf on S given by A, and $\underline{A}$ is the condensed abelian group defined by A.

Proof. Choose a hypercovering $a\colon X_\bullet \to S$ as before. By Corollary 3.6 and Proposition 4.7 we have

$$H^i(S,A_S) \cong \mathscr{H}^i\bigl[\underline{A}(X_0) \to \underline{A}(X_1) \to \underline{A}(X_2)\cdots\bigr] \cong H^i_{\mathrm{cond}}(S,\underline{A})$$

for all i. □

4.10 Remark. On any paracompact Hausdorff space, sheaf cohomology agrees with Čech cohomology. See Godement's book [2], Section II.5.10.

4.11 The formulation of Theorem 4.9 is not optimal, in that it does not say whether there is a functorial isomorphism. Here are some comments about this.

Functoriality in the group A is clear from the construction. Given a hyperresolution $X_\bullet$ as above, the isomorphism $H^i_{\mathrm{cond}}(S,\underline{A}) \xrightarrow{\sim} \mathscr{H}^i[\underline{A}(X_0) \to \underline{A}(X_1) \to \underline{A}(X_2) \to \cdots]$ is essentially a definition, as $\mathbb{Z}[X_\bullet]$ is a projective resolution of $\mathbb{Z}[S]$. On the other hand, it follows from the method outlined in Remark 3.4 that the isomorphism $H^i(S,A_S) \xrightarrow{\sim} \mathscr{H}^i[\underline{A}(X_0) \to \underline{A}(X_1) \to \underline{A}(X_2) \to \cdots]$ is also functorial in A.

It can be shown that the isomorphism $H^i(S,A_S) \cong H^i_{\mathrm{cond}}(S,\underline{A})$ is also functorial in the space S. The main point here is that, given a continuous map $S \to T$ of compact Hausdorff spaces, we can construct a commutative diagram

$$\begin{array}{ccc} X_\bullet & \longrightarrow & Y_\bullet \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ S & \longrightarrow & T \end{array}$$

such that $a\colon X_\bullet \to S$ and $b\colon Y_\bullet \to T$ are hypercoverings with all terms X_n and Y_n extremally disconnected Hausdorff spaces. We omit the details.

5 A result on hypercoverings of a point

5.1 For $[n] \in \Delta$ the presheaf $h_{[n]} = \mathrm{Mor}(-,[n])\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set, which is called $\Delta[n] \in \mathrm{Simp(Set)}$. By Yoneda, if X is any simplicial set then

$$\mathrm{Mor}_{\mathrm{Simp(Set)}}\big(\Delta[n], X\big) = X_n \,.$$

In what follows we shall denote $\Delta[0]$ by $*$; it is the constant simplicial set which is the 1-point set $*$ in each degree.

A pointed simplicial set is a simplicial set $X_\bullet$ together with a morphism $* \to X_\bullet$. By the above, to give such a morphism is the same as giving a 0-simplex $* \in X_0$.

Similarly, by a pointed simplicial space we mean a simplicial topological space $X_\bullet$ together with a morphism $* \to X_\bullet$. (The latter is of course automatically continuous in each degree.)

5.2 Let n be a natural number. Consider the category $(\Delta/[n{+}1])_{\leq n}$, whose objects are the morphisms $\varphi\colon [k] \to [n+1]$ in Δ with $k \leq n$, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} [k] & \xrightarrow{\ \varphi\ } & \\ {\scriptstyle\alpha}\downarrow & & [n+1] \\ [k'] & \xrightarrow[\ \varphi'\ ]{} & \end{array}$$

Let $Y_\bullet$ be an object of $\mathrm{Simp}_n(\mathrm{Top})$, i.e., an n-truncated simplicial space. Then $Y_\bullet$ defines a functor $(\Delta/[n{+}1])^{\mathrm{op}}_{\leq n} \to \mathrm{Top}$ by sending $\varphi\colon [k] \to [n{+}1]$ to Y_k and α as above to the induced map $Y(\alpha)\colon Y_{k'} \to Y_k$. The term in degree $n+1$ of $\mathrm{cosk}_n(Y_\bullet)$ is given by

$$(\mathrm{cosk}_n Y_\bullet)_{n+1} = \lim_{(\Delta/[n+1])^{\mathrm{op}}_{\leq n}} Y_k \,. \tag{5.2.1}$$

Every $\varphi\colon [k] \to [n+1]$ with $k \leq n$ factors through one of the morphisms $d_i^{n+1}\colon [n] \to [n+1]$ ($i = 0, \ldots, n+1$), and among all maps φ, the d_i^{n+1} are the only maps whose image has cardinality n. If the image of $\varphi\colon [k] \to [n+1]$ has cardinality at most $n-1$ then it factors through one of the maps $d_j^{n+1} \circ d_i^n\colon [n-1] \to [n+1]$ for some $0 \leq j \leq n+1$ and $0 \leq i \leq n$. These compositions satisfy the identities

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \qquad \text{for } 0 \leq i < j \leq n+1$$

and no other relations.

5.3 Proposition. *In the above situation,*

$$(\mathrm{cosk}_n Y_\bullet)_{n+1} = \big\{ (y_0, \ldots, y_{n+1}) \in Y_n^{n+2} \;\big|\; d_i^n(y_j) = d_{j-1}^n(y_i) \quad \textit{for all } 0 \leq i < j \leq n+1 \big\}\,.$$

For $n = 1$ we have seen this in Example 2.6(2). For a proof, see [4], Lemma 0186 or Section 7.

5.4 Consider a pointed simplicial topological space $(X_\bullet, *)$. We are interested in the existence of continuous maps

$$t_n \colon X_n \to X_{n+1}$$

for all $n \geq 0$, such that

$$\begin{cases} d_0^{n+1} \circ t_n = \mathrm{id}\,; \\ d_i^{n+1} \circ t_n = t_{n-1} \circ d_{i-1}^n \quad \text{if } n \geq 1 \text{ and } i \in \{1, \dots, n+1\}; \\ d_1^1 \circ t_0 = *. \end{cases} \tag{5.4.1}$$

(The last condition may be viewed as the special case $n = 0$ of the second condition. For this, set $X_{-1} = *$ and let $t_{-1} \colon * \to X_0$ be the map given by the chosen 0-simplex.)

5.5 Proposition. *Let $X_\bullet \in \mathrm{Simp(CHaus)}$ be a hypercovering of the 1-point space $*$ such that all X_n are extremally disconnected. Then for every base point $* \in X_0$, there exist continuous maps $t_n \colon X_n \to X_{n+1}$ for $n \geq 0$ that satisfy the conditions* (5.4.1).

Proof. We construct the maps $t_n \colon X_n \to X_{n+1}$ by induction. By assumption, the map

$$(d_0^1, d_1^1) \colon X_1 \to X_0 \times X_0$$

is surjective, and since X_0 is an object of EDCH there exists a continuous map $t_0 \colon X_0 \to X_1$ such that $d_0^1 \circ t_0 = \mathrm{id}$ and $d_1^1 \circ t_0 = *$.

Assume, then, that $n \geq 1$ and that $t_0, \dots, t_{n-1}$ have been constructed. Consider the continuous map

$$\gamma \colon X_n \to X_n \times X_n \times \cdots \times X_n \qquad (n+2 \text{ factors})$$

given by

$$x \mapsto \big(x, t_{n-1} d_0^n(x), \dots, t_{n-1} d_n^n(x)\big).$$

To prove the existence of a map t_n with the desired properties follows, it suffices to show that γ factors through $(\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1} \subset (X_n)^{n+2}$. Indeed, because $X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$ is surjective and X_n is extremally disconnected, we then obtain a continuous map $t_n \colon X_n \to X_{n+1}$ such that the composition

$$X_n \xrightarrow{t_n} X_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$$

is the map γ, which means precisely that t_n satisfies all required relations.

Using the description of $(\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$ given in Proposition 5.3, all that remains is to verify the following relations:

$$d_0^n\big(t_{n-1} d_{j-1}^n(x)\big) = d_{j-1}^n(x) \qquad \text{for } 1 \leq j \leq n+1; \tag{5.5.1}$$

$$d_i^n\big(t_{n-1} d_{j-1}^n(x)\big) = d_{j-1}^n\big(t_{n-1} d_{i-1}^n(x)\big) \qquad \text{for } 1 \leq i < j \leq n+1. \tag{5.5.2}$$

The first is immediate from the identity $d_0^n \circ t_{n-1} = \mathrm{id}$. For (5.5.2) we use the identity $d_{i-1}^{n-1} \circ d_{j-1}^n = d_{j-2}^{n-1} \circ d_i^n$; together with the relations (5.4.1) for t_{n-1} this gives

$$d_i^n\big(t_{n-1} d_{j-1}^n(x)\big) = t_{n-2} d_{i-1}^{n-1} d_{j-1}^n(x) = t_{n-2} d_{j-2}^{n-1} d_i^n(x) = d_{j-1}^n\big(t_{n-1} d_{i-1}^n(x)\big),$$

as desired. □

6 Condensed cohomology with real coefficients

6.1 If X is a topogical space, write $C(X,\mathbb{R}) = \{\text{continuous functions } X \to \mathbb{R}\}$. For X compact Hausdorff this is a Banach space with respect to the sup-norm.

We denote by $\underline{\mathbb{R}}$ the condensed abelian group given by $\mathbb{R}$ with its Euclidean topology. The main result of this section is that for a compact Hausdorff space S we have

$$H^i_{\text{cond}}(S,\underline{\mathbb{R}}) = 0$$

for all $i > 0$. Clausen and Scholze in fact prove something more precise. To state this, we take a hypercovering $a\colon X_\bullet \to S$ as in 4.8, with all X_n extremally disconnected Hausdorff spaces. By Proposition 4.7,

$$H^i_{\text{cond}}(S,\underline{\mathbb{R}}) \xrightarrow{\sim} \mathscr{H}^i[C(X_0,\mathbb{R}) \xrightarrow{d} C(X_1,\mathbb{R}) \xrightarrow{d} \cdots]$$

6.2 Theorem. *Let assumptions and notation be as above. If $\varepsilon > 0$ and $f \in C(X_i,\mathbb{R})$ (for $i \geq 1$) is a function with $df = 0$, there exists a function $g \in C(X_{i-1},\mathbb{R})$ with $dg = f$ and $\|g\| \leq (1+\varepsilon)\cdot\|f\|$.*

6.3 Remark. The theorem implies that condensed cohomology with coefficients in $\underline{\mathbb{R}}$ on a compact Hausdorff space again agrees with sheaf cohomology. For this, recall that on a paracompact Hausdorff space S the sheaf $\mathscr{C}_S$ of continuous $\mathbb{R}$-valued functions is soft; hence its cohomology in degrees $i \geq 1$ vanishes.

However, the proof of the theorem is not based on a comparison between condensed cohomology and sheaf cohomology, as in the case of discrete coefficients. The main problem here is that if we try to apply Corollary 3.6 with $\mathscr{F} = \mathscr{C}_S$, the sheaves $a_p^{-1}\mathscr{C}_S$ that appear are not the same as the sheaves $\mathscr{C}_{X_p}$ of continuous $\mathbb{R}$-valued functions on X_p.

Theorem 6.2 will be deduced from the following claim.

6.4 Claim. *Fix $i \geq 1$ and $c > 0$. For any $\varphi \in C(X_i,\mathbb{R})$ with $d\varphi = 0$, there exists a function $h \in C(X_{i-1},\mathbb{R})$ such that*

$$\|h\| \leq (1+c)\cdot\|\varphi\| \qquad \textit{and} \qquad \|\varphi - dh\| \leq c\cdot\|\varphi\|\,.$$

6.5 We first explain how the claim gives the theorem. For this, let $\varepsilon > 0$, and let $f \in C(X_i,\mathbb{R})$ be a function with $df = 0$. Choose $c \in (0,1)$ such that $(1+c)/(1-c) < 1+\varepsilon$. We proceed inductively:

- Take $\varphi_0 = f$. The claim gives a function h_0 with $\|h_0\| \leq (1+c)\cdot\|f\|$ and $\|\varphi_0 - dh_0\| \leq c\cdot\|f\|$.
- Next take $\varphi_1 = \varphi_0 - dh_0$. The claim gives a function h_1 with $\|h_1\| \leq (1+c)\cdot\|\varphi_1\| \leq c(1+c)\cdot\|f\|$ and $\|\varphi_1 - dh_1\| \leq c\cdot\|\varphi_1\| \leq c^2\cdot\|f\|$.
- Next take $\varphi_2 = \varphi_1 - dh_1$. The claim gives a function h_2 with $\|h_2\| \leq (1+c)\cdot\|\varphi_2\| \leq c^2(1+c)\cdot\|f\|$ and $\|\varphi_2 - dh_2\| \leq c\cdot\|\varphi_2\| \leq c^3\cdot\|f\|$.
- Continue inductively.

Because $C(X_{n-1},\mathbb{R})$ is complete, we can take

$$g = \lim_{n\to\infty} (h_0 + h_1 + \cdots + h_n).$$

By construction,

$$\|g\| \le (1+c)(1+c+c^2+\cdots)\cdot\|f\| = \frac{1+c}{1-c}\cdot\|f\| \le (1+\varepsilon)\cdot\|f\|,$$

and $\|f - dg\| = 0$, which means that $f = dg$.

6.6 *Reduction to the case* $S = \{\mathrm{pt}\}$. Assume Theorem 6.2 is true if S is a point. We deduce from this the general case.

Let S be compact Hausdorff, $a\colon X_\bullet \to S$ a hypercovering by extremally disconnected compact Hausdorff spaces. For $s \in S$, write $X_\bullet(s) = a^{-1}\{s\}$, which is a hypercovering of $\{s\}$ by extremally disconnected compact Hausdorff spaces.

Our goal is to prove Claim 6.4. Fix $i \ge 1$ and $c > 0$, and let $\varphi \in C(X_i,\mathbb{R})$ be a nonzero function with $d\varphi = 0$. For $s \in S$, let φ_s be the restriction of φ to $X_i(s) \subset X_i$. The theorem (for $S = \{s\}$) then gives the existence of a continuous function $\psi_s\colon X_{i-1}(s) \to \mathbb{R}$ with $d\psi_s = \varphi_s$ and $\|\psi_s\| \le (1+c)\cdot\|\varphi_s\|$. By Tietze's extension theorem, there exists a continuous function $\Psi_s\colon X_{i-1} \to \mathbb{R}$ that extends ψ_s and such that $\|\Psi_s\| = \|\psi_s\|$.

We can choose an open neighbourhood U_s of s in S such that

$$\|\varphi - d\Psi_s\|_{a_i^{-1}(U_s)} < c\cdot\|\varphi\|.$$

(Note that $W = \{x \in X_i \mid |(\varphi - d\Psi_s)(x)| \ge c\cdot\|\varphi\|\}$ is closed in X_i by continuity of $\varphi - d\Psi_s$. Because $a_i\colon X_i \to S$ is proper, $a_i(W) \subset S$ is closed, and $s \notin a_i(W)$. Now take $U_s = S \setminus a_i(W)$.)

Because S is compact, finitely many of the sets U_s, say $U_{s_1},\dots,U_{s_m}$, cover S. Now choose a partition of unity, $1 = \sum_{i=1}^m \rho_i$ with $\mathrm{Supp}(\rho_i) \subset U_{s_i}$ and such that the ρ_i take values in $\mathbb{R}_{\ge 0}$. Then $h = \sum \rho_i\Psi_{s_i}\colon X_{i-1} \to \mathbb{R}$ is a continuous function with $\|h\| \le (1+c)\cdot\|\varphi\|$ and $\|\varphi - dh\| \le c\cdot\|\varphi\|$. This proves Claim 6.4.

6.7 It remains to prove Theorem 6.2 in case $S = \{s\}$ is a point. Note that $H^i_{\mathrm{cond}}(\{s\},\mathbb{R}) = 0$ for all $i > 0$ (because $\{s\}$ is extremally disconnected); but we need the quantative version of this result. This follows from Proposition 5.5. Indeed, choosing any base point $* \in X_0$, that proposition gives the existence of continuous maps $t_n\colon X_n \to X_{n+1}$ satisfying the conditions given in Section 5.4. Now let $f\colon X_n \to \mathbb{R}$, with $n \ge 1$, be a continuous function such that $df = 0$. This just means that for every $x \in X_{n+1}$ we have the relation

$$\sum_{i=0}^{n+1} (-1)^i \cdot f(d_i^{n+1}x) = 0. \tag{6.7.1}$$

Now define $g = f \circ t_{n-1} : X_{n-1} \to \mathbb{R}$. For $y \in X_n$ we then have

$$
\begin{aligned}
dg(y) &= \sum_{j=0}^{n} (-1)^j \cdot g(d_j^n y) \\
&= \sum_{j=0}^{n} (-1)^j \cdot f(t_{n-1} d_j^n y) \\
&= \sum_{j=0}^{n} (-1)^j \cdot f(d_{j+1}^{n+1} t_n y) \\
&= \sum_{i=1}^{n+1} (-1)^{i+1} \cdot f(d_i^{n+1} t_n y) \\
&= f(d_0^{n+1} t_n y) && \text{by (6.7.1)} \\
&= f(y).
\end{aligned}
$$

This shows that $dg = f$, and it is clear from the definition of g that $\|g\| \le \|f\|$.

This completes the proof of Theorem 6.2.

7 Appendix: details for Proposition 5.3

7.1 We place ourselves in the situation considered in Section 5.2: n is a natural number, and $Y_\bullet$ is an n-truncated simplicial space. Define

$$C = \left\{ (y_0, \dots, y_{n+1}) \in Y_n^{n+2} \;\middle|\; d_i^n(y_j) = d_{j-1}^n(y_i) \quad \text{for all } 0 \le i < j \le n+1 \right\}.$$

It is clear from (5.2.1) that there is a canonical continuous map

$$\theta : (\operatorname{cosk}_n Y_\bullet)_{n+1} \to C\,.$$

Our claim is that this map is an isomorphism. To prove this, we construct an inverse.

7.2 Every morphism $[k] \to [n+1]$ in Δ has a unique factorisation

$$[k] \twoheadrightarrow [m] \hookrightarrow [n+1]$$

as a surjective map followed by an injective map. Moreover, every commutative diagram

(7.2.1)
$$
\begin{array}{ccc}
[k] & \searrow^{\varphi} & \\
{\scriptstyle\alpha}\downarrow & & [n] \\
[k'] & \nearrow_{\varphi'} &
\end{array}
$$

can be uniquely extended to a commutative diagram

$$
\begin{array}{ccccc}
[k] & \twoheadrightarrow & [m] & \searrow & \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & [n] \\
[k'] & \twoheadrightarrow & [m'] & \nearrow &
\end{array}
$$

7.3 To construct an inverse of θ we have to assign to every element $(y_0,\ldots,y_{n+1}) \in C$ a collection of elements

$$(y_\varphi \in Y_k)_{\substack{\varphi\colon [k]\to[n+1]\\ \text{with } k\le n}}$$

in such a manner that for every diagram (7.2.1) we have the relation $\alpha^*(y_{\varphi'}) = y_\varphi$. By the above remarks, it suffices to define y_φ for all injective maps φ and to check the desired relation for all injective maps α.

If $\varphi\colon [k] \to [n+1]$ is injective with $k \le n$, choose a factorisation of φ as a composition

$$[k] \xhookrightarrow{\rho} [n] \xrightarrow{d_i^{n+1}} [n+1]$$

and define $y_\varphi = \rho^*(y_i)$. We have to show that this is independent of choices. If $k = n$ this is clear. If $k \le n-1$, suppose

$$[k] \xhookrightarrow{\sigma} [n] \xrightarrow{d_j^{n+1}} [n+1]$$

is another factorisation of φ. Withour loss of generality we have $0 \le i < j \le n+1$. (If $i = j$ then $\rho = \sigma$ and there is nothing to prove.) Then there exists a commutative diagram

$$\begin{array}{ccccccc} & & & & [n] & & \\ & & & \overset{d_i^n}{\nearrow} & & \overset{d_j^{n+1}}{\searrow} & \\ [k] & \xhookrightarrow{\tau} & [n-1] & & & & [n+1] \\ & & & \underset{d_{j-1}^n}{\searrow} & & \underset{d_i^{n+1}}{\nearrow} & \\ & & & & [n] & & \end{array}$$

such that $d_i^n \circ \tau = \sigma$ and $d_{j-1}^n \circ \tau = \rho$. Because we have imposed the relation $d_i^n(y_j) = d_{j-1}^n(y_i)$, it follows that $\rho^*(y_i) = \tau^*\big(d_{j-1}^n(y_i)\big) = \tau^*\big(d_i^n(y_j)\big) = \sigma^*(y_j)$, as desired.

It is clear now that for any diagram (7.2.1) with α, φ and φ' injective, we have $y_\varphi = \alpha^*(y_{\varphi'})$: if $\varphi' = d_i^{n+1} \circ \rho$ with $\rho\colon [k'] \hookrightarrow [n]$ then $\varphi = d_i^{n+1} \circ (\rho \circ \alpha)$ is a factorisation of φ, and therefore $y_\varphi = (\rho\circ\alpha)^*(y_i) = \alpha^*\big(\rho^*(y_i)\big) = \alpha^*(y_{\varphi'})$.

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