Sh 1

Sheaves and exodromy

Talk I : Sheaves and cohomology in topology
Talk II : Why étale sheaves?
Talk III : Constructible sheaves in topology: from monodromy to exodromy
Talk IV : Exodromy for étale sheaves.

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Topology	I	III
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I Sheaves and cohomology in topology

Slogan Sheaves are versatile objects that know about cohomology, fundamental groups, (homotopy type), and variation thereof in families.

§1 Examples of sheaves

Throughout, X is a topological space.

Def Write $\mathrm{Open}(X)$ for the poset of open subsets $U \subseteq X$.

- A presheaf is a functor $P : \mathrm{Open}(X)^{op} \to \mathrm{Set}$ (Ab, $\mathcal{J}$, ...)
- A sheaf on X is a presheaf $\mathcal{F}$ such that for any open $U \subseteq X$ and any open covering $U = \bigcup_{i \in I} U_i$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

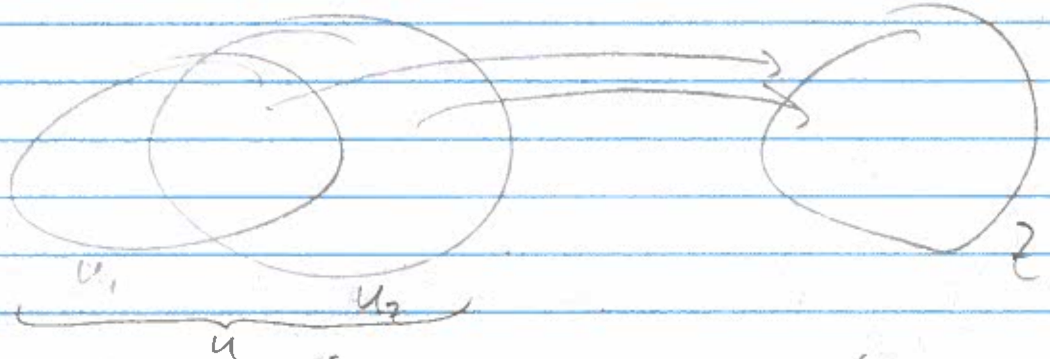
is an equaliser. That is, given $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there is a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$).

Main examples:

Ex 1 Let Z be a topological space, and define

$$h_Z : \mathrm{Open}(X)^{op} \to \underline{Set}$$
$$U \longmapsto \mathrm{Hom}(U, Z).$$

This is a sheaf:



To define a map $U \xrightarrow{f} Z$, it suffices to define $U_i \xrightarrow{f_i} Z$ for all i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$.

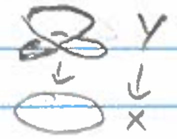
Variants: • $C^\infty(X, \mathbb{R})$ if X is a manifold • $\mathcal{O}_X$ (holomorphic $X \to \mathbb{C}$) if X is a $\mathbb{C}$ mfd
• ...

Ex 2 Let $g: Y \to X$ be a continuous map, and define

$$h_{Y/X} : \mathrm{Open}(X)^{op} \to \underline{Set}$$
$$U \longmapsto \{\text{cts sections } U \to Y\}$$
$$= \mathrm{Hom}_X(U, Y).$$

This is a sheaf: sections can be constructed locally.

Ex 3 Let $S^1 \to S^1$ be a 2:1 cover:



Then $h_{Y/X}(X) = \emptyset$, but $h_{Y/X}(U)$ has two points for any connected $U \subsetneq X$:

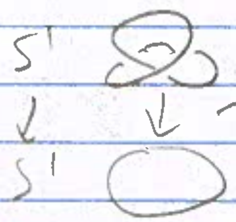
<u>Rmk</u> Ex 1 is a special case of Ex 2: set $Y = X \times Z \xrightarrow{pr_1} X$; then $h_Z = h_{Y/X}$. It is usually denoted h_Y.

<u>Q</u> Are there other examples?

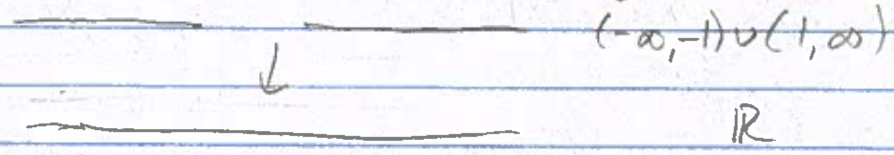
<u>§2 Espace étalé</u>

Recall that $f: Y \to X$ is a <u>local homeomorphism</u> if Y is covered by opens $V \subseteq Y$ such that $f|_V : V \to X$ is a homeomorphism onto an open $U \subseteq X$.

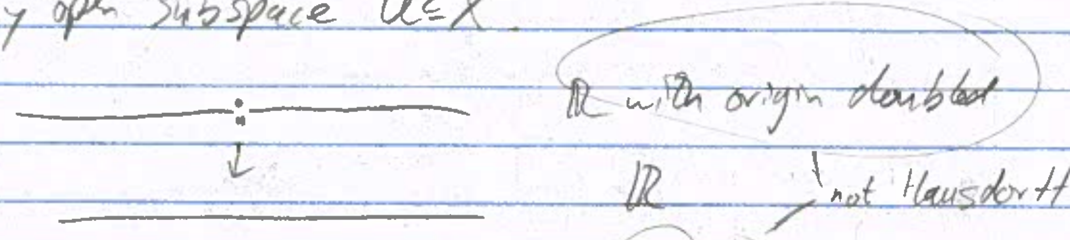
<u>Ex</u> .



- But they don't have to be covering spaces:



- Any open subspace $U \subseteq X$.

- 

- For any $U \subseteq X$, can define $Y = X \amalg_U X \to X$, which is "X with $X \setminus U$ doubled".

<u>Prop</u> The functor

$$\begin{aligned} \underline{\mathrm{LocHomeo}}_{/X} &\longrightarrow \underline{Sh}(X) \\ (Y \to X) &\longmapsto h_Y \end{aligned}$$

is an <u>equivalence of categories</u>.

The inverse $\underline{(P)Sh}(X) \to \underline{LocHomeo}_{/X}$ is the espace étalé, defined as follows: given a (pre)sheaf $\mathcal{F}$, set

$$sp(\mathcal{F}) := \left(\coprod_{U} \mathcal{F}(U)^{disc} \times U \right) \Big/ \sim ,$$

where $\sim$ is generated by $(s, x) \sim (s|_V, x)$ for $x \in V \subseteq U$ and $s \in \mathcal{F}(U)$. That is,

$$sp(\mathcal{F}) = \operatorname{colim} \left(\coprod_{V \subseteq U} \mathcal{F}(U) \times V \overset{\mathcal{F}(U) \to \mathcal{F}(V)}{\underset{V \hookrightarrow U}{\rightrightarrows}} \coprod_{U} \mathcal{F}(U) \times U \right)$$

$$= \int^{U} \mathcal{F}(U) \times U \xrightarrow[\text{local homeo}]{} X$$

(coend)

In general, there is an adjunction

$$\underline{PSh}(X) \underset{h_{(-)}}{\overset{sp}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \underline{Top}_{/X} ,$$

i.e.

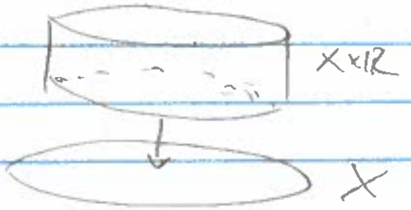
$$Hom_X(sp(\mathcal{F}), Y) \cong Hom_{\underline{PSh}}(\mathcal{F}, h_Y).$$

It becomes an equivalence when restricting to

$$\underline{Sh}(X) \underset{h_{(-)}}{\overset{sp}{\rightleftarrows}} \underline{LocHomeo}_{/X} .$$

Rmk For general $\mathcal{F}$, the $sp(\mathcal{F})$ is horrendous.

Ex Let $\mathcal{F} = h_{\mathbb{R}\times X/X}$:



Then $\mathcal{F}(U)$ is massive for each $U \subseteq X$, and $\mathcal{F}(U) \to \mathcal{F}(V)$ usually not injective for $V \subseteq U$ (so $sp(\mathcal{F})$ is highly non-Hausdorff).

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Def The composite $PSh(X) \xrightarrow{sp} LocHomeo \to Sh(X)$ is the sheafification.
$\mathcal{F} \mapsto \mathcal{F}^\#$ (The unit of the adjunction gives a natural map $\mathcal{F} \to \mathcal{F}^\#$.)

Ex If $A \in \underline{Set}$ and $\mathcal{F}: U \mapsto A$ is the presheaf of constant functions $U \to A$, then $sp(\mathcal{F}) = A \times X \to X$, and $\mathcal{F}^\#$ is the sheaf $\underline{A}$ of continuous (i.e. locally constant) functions $U \to A$. It is called the constant sheaf.

Exc Show that $Y \to X$ is a covering space iff h_Y is locally constant: there is an open cover $\bigcup U_i = X$ and sets A_i such that $h_Y|_{U_i} \cong \underline{A_i}$ for all i. If X is connected, you may take all A_i the same.

§3 Functoriality

Given a continuous map $f: Y \to X$, we get an adjunction

$$Sh(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} Sh(Y),$$

i.e.

$$Hom_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{Sh(X)}(\mathcal{F}, f_*\mathcal{G}).$$

Properties:

- $(f_*\mathcal{G})(U) = \mathcal{G}(f^{-1}(U))$ — easy for sheaves
- $sp(f^*\mathcal{F}) = sp(\mathcal{F}) \times_X Y$ — easy for espaces étalés.

Ex For $i: \{x\} \hookrightarrow X$ the set $i^*\mathcal{F}$ is the stalk $\mathcal{F}_x = \underset{x \in U}{\varinjlim} \mathcal{F}(U)$.
It equals the fibre $sp(\mathcal{F}) \times_X \{x\}$.

Important question:

Q What does $(R)f_*$ do on espaces étalés? Stalks?

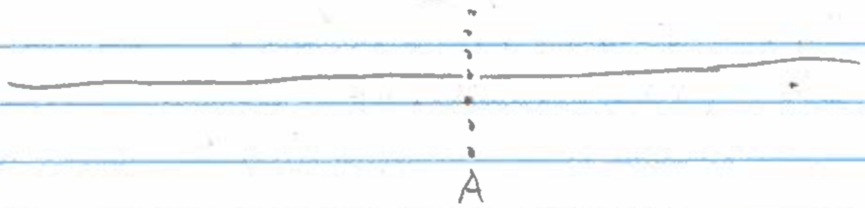
Ex Let $i: \{0\} \hookrightarrow \mathbb{R}$. What is $\mathcal{F} = i_*\underline{A}$ for a set A?

As a sheaf: $i_*\underline{A}(U) = \underline{A}(i^{-1}(U)) = \underline{A}(U \cap \{0\}) = Hom(U \cap \{0\}, A)$
$= \begin{cases} A & 0 \in U \\ * & 0 \notin U. \end{cases}$

Then $sp(i_* A) = \int^{U} \mathcal{F}(U)\times U$ is "generated" by $\mathcal{F}(\mathbb{R})\times\mathbb{R}$ $= A\times\mathbb{R}$ and $\mathcal{F}(\mathbb{R}\setminus\{0\})\times(\mathbb{R}\setminus\{0\}) = * \times \mathbb{R}\setminus\{0\}$, modulo

$$\mathcal{F}(\mathbb{R})\times\mathbb{R}\setminus\{0\} \rightrightarrows (\mathcal{F}(\mathbb{R})\times\mathbb{R}) \amalg (\mathcal{F}(\mathbb{R}\setminus\{0\})\times\mathbb{R}\setminus\{0\})$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$A\times\mathbb{R}\setminus\{0\} \rightrightarrows (A\times\mathbb{R}) \amalg (*\times\mathbb{R}\setminus\{0\})$$

So we take A copies of $\mathbb{R}$ and identify them on $\mathbb{R}\setminus\{0\}$.



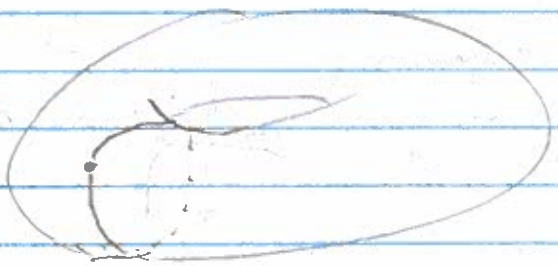
(Talk 3: we will study constructible sheaves $\approx$ finitely presented sheaves).

So f_* is tricky to understand on espaces étalés.

§4 Cohomology

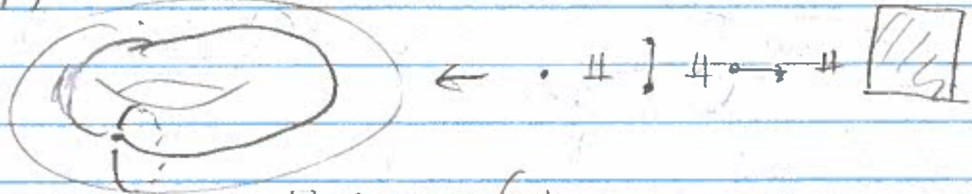
There are many definitions of cohomology.

- $H^i_{sing}(X,\mathbb{R})$: (linear dual of $H_i^{sing}(X,\mathbb{R})$)



X locally path connected (locally contractible?)

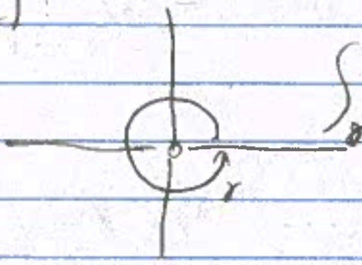
- $H^i_{cell}(X,\mathbb{R})$:



X cw complex

- $[X, K(\mathbb{Z}, i)]$ $\qquad$ X paracompact Hausdorff?

- $H^i_{dR}(X, \mathbb{R})$



$\int_\gamma \frac{dz}{z} = 2\pi i$ $\qquad$ X manifold

Thm They produce "the same" answer (for suitable X).

Pf Secretly via sheaf cohomology! (Or maybe Čech cohomology)

Scholium Sheaf cohomology is the "correct" cohomology in the biggest generality. (See also Talk 2)

From now on, $\underline{Sh}(X)$ will denote sheaves of abelian groups.

Def Sheaf cohomology $R^i(X,-) : \underline{Sh}(X) \to \underline{Ab}$ is the i^{th} derived functor of $\mathcal{F} \mapsto \mathcal{F}(X)$ (global sections, also denoted $\Gamma(X, \mathcal{F})$).

Def Given $f: Y \to X$ in $\underline{Top}$ define $R^i f_* : \underline{Sh}(Y) \to \underline{Sh}(X)$ as the i^{th} derived functor of f_*.

(Choose an injective resolution $\mathcal{F} \to I^0 \to I^1 \to \dots$ and take $R^i F(\mathcal{F}) = H^i(F(I^\bullet))$.)

Ex For $f: X \to pt$, we get $\underline{Sh}(pt) \simeq \underline{Ab}$, and $R^i f_* = H^i(X,-) = \mathrm{Ext}^i(\mathbb{Z},-)$.

So $R^i f_*$ is a relative version of sheaf cohomology.

⚠ Like f_*, $R^i f_*$ does not commute with g^*. So the stalks $(R^i f_* \mathcal{F})_x$ for $x \in X$ may differ from $H^i(Y_x, \mathcal{F}|_{Y_x})$.

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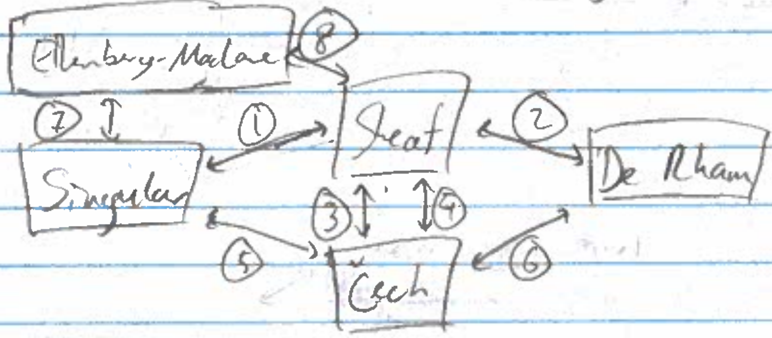
Comparison between cohomologies:



① If X is locally contractible then

$$H^i_{sing}(X, A) \cong H^i(X, \underline{A})$$

for any $A \in \underline{Ab}$.

<u>Idea</u> Let $C^i_{sing}(A)$ be the sheafification of $U \mapsto C^i_{sing}(U, A)$. Show that

$$0 \to \underline{A} \to C^0_{sing}(\underline{A}) \to \cdots$$

is a flabby resolution of $\underline{A}$, and that

$$C^\bullet_{sing}(X, \underline{A}) \to \Gamma(X, C^\bullet_{sing}(\underline{A}))$$

is a quasi-isomorphism.

② If X is a (paracompact) manifold, then

$$H^i_{dR}(X) \cong H^i(X, \underline{\mathbb{R}}).$$

<u>Idea</u> Show that the de Rham complex

$$0 \to \underline{\mathbb{R}} \to \Omega^0_X \to \Omega^1_X \to \cdots$$

is exact (as sheaves!) and that each C^∞_X-module $\mathcal{M}$ is acyclic (uses partition of unity).

③ If X is paracompact Hausdorff, then

$$\check{H}^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X, \mathcal{F})$$

for all $\mathcal{F} \in Sh(X)$.
(A bit tricky.)

④+⑤ If X has an open cover $\mathcal{U} = \{U_i \hookrightarrow X\}_{i \in I}$ such that each $U_{i_0} \cap \dots \cap U_{i_k}$ is a (disjoint union of) contractible space(s), then

$$\check{H}^i(\mathcal{U}, \underline{A}) \xrightarrow{\sim} H^i(X, \underline{A})$$
$$\downarrow \wr$$
$$H^i_{sing}(X, A)$$

(Very easy.)

⑥ If X is a (paracompact) manifold, then

$$\check{H}^i(X, \underline{\mathbb{R}}) \xrightarrow{\sim} H^i_{dR}(X).$$

(Same as ④+⑤).

⑦ If X is a CW complex, then

$$[X, K(A, i)] \xrightarrow{\sim} H^i_{sing}(X, A)$$

for all $A \in \underline{Ab}$.

⑧ If X is paracompact Hausdorff, then

$$[X, K(A, i)] \xrightarrow{\sim} H^i(X, \underline{A})$$

for all $A \in \underline{Ab}$.

(Lurie: $[X, K] \xrightarrow{\sim} \pi_0 R\Gamma(X, \underline{K})$ for all $K \in \underline{Kan}$.)

Maybe the example?

§5 Relative cohomology

If $f: Y \to X$ then $R^i f_* \underline{\mathbb{Z}}$ is a relative version of $H^i(X, \underline{\mathbb{Z}})$, measuring how the fibres change.

Ex Let $m > n > 0$, and consider the projection

$$f: \mathbb{R}^m - \{0\} \to \mathbb{R}^n.$$
$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n).$$

Then $R^i f_* \underline{\mathbb{Z}}$ is the sheafification of $U \mapsto H^i(f^{-1}(U), \underline{\mathbb{Z}})$.
For $U \subseteq \mathbb{R}^n$ contractible we get

$$H^i(f^{-1}(U), \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=m-1,\ 0\in U \\ 0 & \text{else} \end{cases}$$

Thus,

$$R^i f_* \underline{\mathbb{Z}} = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_0 & i=m-1 \\ 0 & \text{else} \end{cases}$$

($\mathbb{Z}_0$: skyscraper sheaf at $0 \in \mathbb{R}^n$)

⚠ The stalks of $R^i f_* \underline{\mathbb{Z}}$ do not agree with

$$H^i(Y_x, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=m-n-1,\ x=0 \\ 0 & \text{else}. \end{cases}$$

But this property does hold for $R^i f_! \mathcal{F}$, the derived functor of

$$f_!: \underline{Sh}(Y) \to \underline{Sh}(X)$$

given by $(f_! \mathcal{F})(U) = \{ s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid \mathrm{Supp}(s) \to U \text{ proper} \}$.

Ex In the example above, we get

$$R^i f_! \underline{\mathbb{Z}} = \begin{cases} \mathbb{Z} & i=m-n \\ \mathbb{Z}_0 & i=1 \\ 0 & \text{else}. \end{cases}$$

and $H^i_c(Y_x, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & i=m-n \\ \mathbb{Z} & i=1,\ x=0 \\ 0 & \text{else}. \end{cases}$

Rmk We get a "relative Poincaré duality": in the example above,

$$R\underline{Hom}_X(Rf_!\underline{\mathbb{Z}}, \underline{\mathbb{Z}}[n-m]) \cong Rf_*\underline{\mathbb{Z}}.$$

($\underline{\mathbb{Z}}[n-m]$: $\mathbb{Z}$ in degree $m-n$)

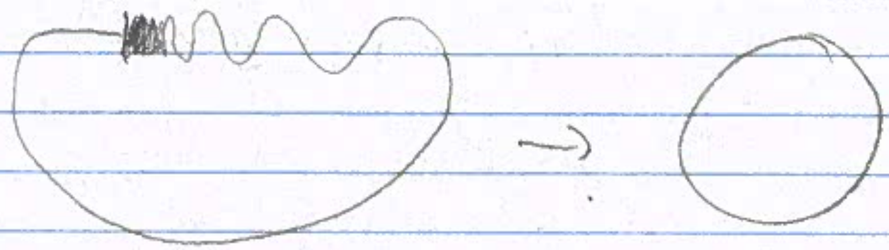
Indeed, $R\underline{Hom}_X(\underline{\mathbb{Z}}[n-m], \underline{\mathbb{Z}}[n-m]) = \underline{\mathbb{Z}}$ and $R\underline{Hom}_X(\underline{\mathbb{Z}}_0[-1], \underline{\mathbb{Z}}[n-m])$
$= R\underline{Hom}_X(\underline{\mathbb{Z}}_0, \underline{\mathbb{Z}})[n+1-m] = \underline{\mathbb{Z}}_0[1-m]$.

(NTS: $\underline{Ext}^i_X(\underline{\mathbb{Z}}_0, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}_0 & i=n \\ 0 & \text{else} \end{cases}$)

Also versions for singular varieties, more general sheaves than $\underline{\mathbb{Z}}$, ...

Time permitting:

Ex Let X be the "Warsaw circle".



Then $X \to S^1$ induces isomorphisms

$$H^i(S^1, \underline{A}) \xrightarrow{\sim} H^i(X, \underline{A}).$$

They also agree with $[X, K(A,i)]$ and $\check{H}^i(X, A)$ since X is compact Hausdorff.

But $H^i_{sing}(X, A) = \begin{cases} A & i=0 \\ 0 & \text{else} \end{cases}$, so H_{sing} gives the wrong answer.