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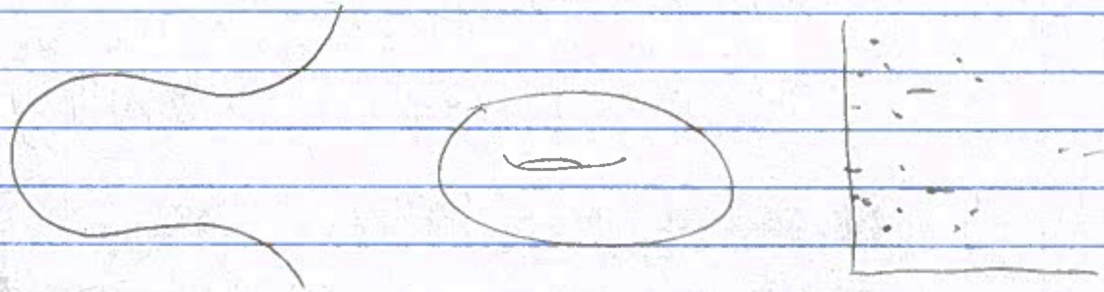
II Why étale cohomology?

(and what is it?)

§1 Motivation

Algebraic geometry studies the solutions to polynomial equations:

Ex The solutions to $y^2 = x^3 - 3x + 7$:



over $\mathbb{R}$ over $\mathbb{C}$ over $\mathbb{F}_{31} = \mathbb{Z}/31\mathbb{Z}$.

The story of étale cohomology starts with varieties over finite fields.

Rmk Recall: for each prime p and each $n \in \mathbb{Z}_{>0}$, there is a unique field extension $\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^n}$ of degree n.

A very crude geometric invariant: given an $\mathbb{F}_q$-variety X, count $|X(\mathbb{F}_q)|$.

Ex • If $X = \mathbb{A}^n$, then $|X(\mathbb{F}_q)| = |\mathbb{F}_q^n| = q^n$.

• If $X = \mathbb{P}^n = \frac{\mathbb{A}^{n+1} - \{0\}}{\mathbb{A}^1 - \{0\}}$, then $|X(\mathbb{F}_q)| = \frac{q^{n+1} - 1}{q - 1} = q^n + q^{n-1} + \dots + q + 1$

Ah, this is a "cellular decomposition" $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \dots \amalg \mathbb{A}^1 \amalg \mathbb{A}^0$.

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<u>Thm</u> (Hasse (1936) for $g=1$, Weil (1948) in general)
If C is smooth projective of genus g over $\mathbb{F}_q$, then
$$\big|\,|C(\mathbb{F}_q)| - q - 1\,\big| \leq 2g\sqrt{q}.$$

<u>Pf</u> Exercise V.1.10 in Hartshorne. $\square$

But what does it mean?

<u>Def</u> If X is a variety over $\mathbb{F}_q$, set
$$Z(X,t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right) \in \mathbb{Q}[\![t]\!].$$
The zeta function $\zeta(X,s)$ is $Z(X, q^{-s})$.

<u>Lemma</u> $$\zeta(X,s) = \prod_{\substack{x\in X \\ \text{closed}}} \frac{1}{1-|k(x)|^{-s}}.$$
$k(x)$ — minimal field of definition

So this really like a zeta function: if $X = \operatorname{Spec}\mathbb{Z}$, the same formula gives the Riemann zeta function.

<u>Ex</u>. For $X = \mathbb{A}^n$, we get
$$\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r} = \sum_{r=1}^{\infty} q^{nr}\frac{t^r}{r} = -\log(1-q^n t),$$
so
$$Z(X,t) = \frac{1}{1-q^n t}.$$

. For $X = \mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$, we therefore get
$$Z(X,t) = Z(\mathbb{A}^n,t)\cdot Z(\mathbb{A}^{n-1},t)\cdot \ldots \cdot Z(\mathbb{A}^0,t)$$
$$= \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}.$$

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- If C is smooth projective of genus g, then Riemann-Roch implies
$$Z(C,t) = \frac{P(t)}{(1-t)(1-qt)}$$
for $P \in \mathbb{Z}[t]_{2g}$. Serre duality gives a functional equation
$$Z(C, \tfrac{1}{qt}) = (t\sqrt{q})^{2-2g} Z(C,t).$$
Taking $\zeta(C,s) = Z(C, q^{-s})$ gives the more familiar
$$\zeta(C, 1-s) = (q^{\frac{1}{2}-s})^{2-2g} \zeta(C,s).$$
Then Hasse-Weil's bound
$$\big| |C(\mathbb{F}_q)| - q - 1 \big| \leq 2g\sqrt{q}$$
is equivalent to the Riemann hypothesis: all zeroes of $\zeta(X,s)$ lie on $\mathrm{Re}\, s = \frac{1}{2}$.

Cor All roots α of $P(t)$ have $|\alpha| = q^{\frac{1}{2}}$.

Conj (Weil 1949)
If X is smooth projective of dimension n, then
$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)},$$
where $P_i \in \mathbb{Z}[t]$ and all roots α of P_i have $|\alpha| = q^{\frac{i}{2}}$.
Moreover,
$$Z(X, \tfrac{1}{q^n t}) = \pm (t q^{\frac{n}{2}})^{\chi} Z(X,t),$$
where $\chi = \Delta_X \cdot \Delta_X$ is the "topological Euler characteristic".
The degrees $\deg P_i$ should be the "Betti numbers".

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History
- Rationality: Dwork (1960) using p-adic analysis.
- Functional equation: Grothendieck et al. (SGA I–V, 1960s)
- Riemann Hypothesis: Deligne (1974).

§2 So where is the cohomology?

"Modern" formulation of Weil's conjectures (minus RH):

Conj There exists a "good" cohomology theory for smooth projective varieties over $\mathbb{F}_q$, such that:

(1) $H^i(X)$ is finite-dimensional, and 0 if $i \notin [0, 2n]$.

(2) $H^{2n}(X) = K$, and the cup product

$$H^i(X) \times H^{2n-i}(X) \xrightarrow{\cup} H^{2n}(X) \xrightarrow{\sim} K$$

is a perfect pairing.

(3) $H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$.

(4) Cycle class maps $CH^i(X) \to H^{2i}(X)$

codimension i subvarieties (possibly singular!)

(5) Lefschetz hyperplane theorem.

Rmk How is this related to ζ-functions?

An $\mathbb{F}_q$-variety X has a Frobenius $F: X \to X$ by

$$[x_0 : \ldots : x_m] \mapsto [x_0^q : \ldots : x_m^q].$$

The miracle is that this lands in X again!
If $f(x_0, \dots, x_m) = 0$, then $f(x_0^q, \dots, x_m^q) = f(x_0, \dots, x_m)^q = 0$, since $(-)^q$ is a ring homomorphism.

Moreover, $X(\mathbb{F}_{q^r})$ is the fixed points of F^r (all multiplicity one).

Prop (Lefschetz fixed point theorem)
If H^* is a Weil cohomology theory, and $f: X \to X$ an endomorphism, then

$$\Delta_X \cdot \Gamma = \sum_{i=0}^{2n} (-1)^i \operatorname{tr}(f_* : H^i(X) \to H^i(X)).$$

Pf Same as in topology! Uses Poincaré duality, Künneth, compatibility of cycle class map and cup product, ... $\square$

(If X is smooth but not projective, use H^i_c)

Since $|X(\mathbb{F}_{q^r})| = \Gamma_{F^r} \cdot \Delta_X$, we see that $\zeta(X,s)$ is determined by traces of Frobenius.

Lemma If X is an $\mathbb{F}_q$-variety, then

$$Z(X,t) = \frac{P_1(t) P_3(t) \cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots P_{2n}(t)}$$

where

$$P_i(t) = \det(1 - tF : H^i_c(X) \to H^i_c(X)).$$

Ex For $\mathbb{A}^n$, we get $H^i_c(X) = \begin{cases} k & i = 2n \\ 0 & \text{else} \end{cases}$, and F acts by $q^n : k \to k$. So

$$Z(X,t) = \frac{1}{(1-q^n t)},$$

as before.

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⚠ No functional equation or Riemann hypothesis if X is not smooth projective.

Ex For $X = \mathbb{A}^1 - \{0\}$, we get

$$Z(X,t) = \frac{Z(\mathbb{A}^1,t)}{Z(\{0\},t)} = \frac{(1-t)}{(1-qt)}.$$

So $P_1(t) = (1-t)$ has the wrong weight! (absolute value q^0, not $q^{\frac{1}{2}}$).

§3 Does algebraic geometry "know" cohomology?

For complex manifolds, we saw that there are many ways to define their cohomology:

- H^i_{sing}, H^i_{dR}, H^i, $\check{H}^i$, $[-, K(-,i)]$, ...

Q1 Are any of them defined algebraically?

A Yes, there is algebraic de Rham cohomology.

But over $\mathbb{F}_q$, this gives a torsion cohomology theory ($\mathbb{F}_q$-linear), so computes the wrong Betti numbers.

($\rightsquigarrow$ Crystalline cohomology is a "p-adic lift" of de Rham cohomology.)

We'll take a more "topological" approach.

Q2 (Hopefully easier)
Does algebraic geometry see $\pi_1(X)$?

<u>Ex</u> $\mathbb{C}\setminus\{0\}$ is algebraic, as is its universal cover $\mathbb{C}$. But

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$$

is <u>not</u> algebraic.

So the answer is <u>no</u>. :(

But... it <u>does</u> see finite coverings:

<u>Ex</u> The finite coverings of $\mathbb{C}^\times$ are

$$\begin{aligned} \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ z &\longmapsto z^n \end{aligned}$$

for $n \in \mathbb{Z}_{>0}$.

<u>Thm</u> "Riemann existence theorem", Grauert-Remmert (1958)
Let X be a smooth complex algebraic variety, and $Y \xrightarrow{f} X(\mathbb{C})$ a finite topological covering space. Then the unique complex structure on Y making f a local biholomorphism is algebraic, and f a morphism of complex varieties.

later (~~<u>Cor</u> Algebraic geometry knows the profinite completion $\hat{\pi}_1(X)$~~

~~<u>Ex</u> For $\mathbb{C}^\times$, we get profinite fundamental group $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$.~~)

<u>Def</u> A morphism $f: Y \to X$ of smooth varieties is <u>étale</u> if the Jacobian $Jf: T_yY \to T_{f(y)}X$ is an isomorphism for all $y \in Y$.

Equivalently, f is flat and unramified ($\Omega_{Y/X} = 0$), or f is flat and $Y \xrightarrow[\Delta_{Y/X}]{} Y \times_X Y$ is an open immersion.

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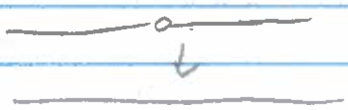
Then finite étale maps $Y \to X$ correspond to finite covering spaces $Y(\mathbb{C}) \to X(\mathbb{C})$.

⚠ In algebraic geometry, finite means in particular proper.

Ex

$$\begin{array}{c} \mathbb{A}^1 - \{0\} \\ \downarrow \\ \mathbb{A}^1 \end{array}$$ is quasi-finite



(all fibres are finite), but not finite.

Reason: $k[x, x^{-1}]$ is not a finitely generated $k[x]$-module.

§4 Galois groups and fundamental groups

Recall: if $\tilde{X} \xrightarrow{p} X$ is a universal cover, then $\pi_1(X)$ can be understood as $p^{-1}(x)$, but this depends on a choice of basepoint $\tilde{x} \in p^{-1}(x)$. More precisely:

Lemma The action of $\pi_1(X, x)$ on $p^{-1}(x)$ by deck transformations is simply transitive. If X is path connected, then

$$\pi_1(X, x) \longrightarrow \mathrm{Aut}_X(\tilde{X})$$

is an isomorphism.

So like in Galois theory, we should define $\pi_1^{et}(X)$ to be

$$\varprojlim_{\substack{Y \to X \\ \text{finite étale} \\ \text{connected}}} \mathrm{Aut}_X(Y).$$

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Ex For $X = \mathbb{C}^\times$, we get a unique cover $Y_n \to X$ of degree n, and

$$\pi_1^{ét}(X) = \varprojlim_n \mathrm{Aut}_X(Y_n) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}},$$

the profinite completion of $\pi_1(X)$.

This is similar to $\mathrm{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t)))$: for every n, there is a unique field extension $\mathbb{C}((t)) \to \mathbb{C}((t^{\frac{1}{n}}))$, and

$$\mathrm{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t))) \cong \hat{\mathbb{Z}}.$$

In fact, $\pi_1^{ét}(\mathrm{Spec}\, k) = \mathrm{Gal}(k^{sep}/k)$ for any field k: connected ~~finite~~ étale covers $Y \to \mathrm{Spec}\, k$ are exactly $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ for finite separable field extensions $k \to L$.

§5 Étale sheaves and étale cohomology

Idea Use étale maps $Y \to X$ to define a topology on X.

Problem These Y are not small enough to be open subsets in X.

Ex For $\underset{\mathbb{C}^\times}{Y} \to \underset{\mathbb{C}^\times}{X}$, $z \mapsto z^2$, there is no Zariski (or étale) open $Z \subseteq Y$ (or $Z \to Y$) such that $Z \to X$ is injective.

Solution (Grothendieck)
Broaden your mind! Just redefine what a topology is.

Def An étale sheaf $\mathcal{F}$ on X is a functor $\mathcal{F}: \mathrm{Et}_X^{op} \to \mathrm{Set}$ such that for every jointly surjective family

$$\{U_i \to U\}_{i \in I}$$

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of maps in $\underline{Et}_X$, the diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equaliser.

Ex If $Y \to X$ is a morphism of varieties (e.g. $Y = Z \times X \to X$), then is

$$h_Y : U \longmapsto \mathrm{Map}_X(U, Y)$$

a sheaf?

There is really something to check!

(It is, but involves some topology plus some algebra.)

⚠ Not every sheaf has an espace étalé (on schemes): Again, they do if you generalise what that means... (algebraic spaces).

Def Étale cohomology $H^i_{et}(X, -)$ is the i^{th} derived functor of $\Gamma(X, -) : \underline{Sh}(X_{et}) \to \underline{Ab}$.

Thm (Artin)
Let X be a $\mathbb{C}$-variety, and A a torsion abelian group. Then

$$H^i_{et}(X, \underline{A}) \xrightarrow{\sim} H^i(X(\mathbb{C}), \underline{A}).$$

Pf Tricky, uses resolution of singularities, Grauert-Remmert, and induction on $\dim X$. $\square$

So we still only get torsion theories! But finally,

Def Let X be a k-variety. Set $H^i_{et}(X, \mathbb{Z}_\ell) := \varprojlim H^i_{et}(X, \mathbb{Z}/\ell^n\mathbb{Z})$.

Q Do the ℓ-adic Betti numbers depend on ℓ?? (open!).