Constructible sheaves in topology

(from monodromy to exodromy)

§1 Introduction

In talk I, we saw equivalences

$$\begin{tikzcd} \underline{Sh}(X) \arrow[r, "\sim"] \arrow[d, phantom, "\cup"] & \underline{LocHomeo}_{/X} \arrow[d, phantom, "\cup"] \\ \underline{LC}(X) \arrow[r, "\sim"] & \underline{Cov}_{/X} \end{tikzcd}$$

└ locally constant sheaves

But we also have:

Thm (Monodromy correspondence)
If X is locally path connected and semi-locally simply connected, then

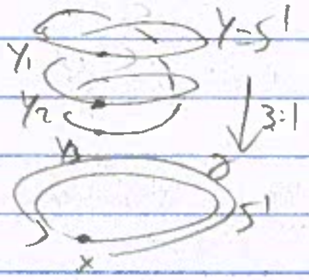
$$\begin{aligned} \underline{Cov}_{/X} &\xrightarrow{\sim} \mathrm{Fun}(\Pi_1(X), \underline{Set}) \\ (Y \to X) &\longmapsto (x \mapsto Y_x) \end{aligned}$$

where $\Pi_1(X)$ is the fundamental groupoid:

- objects are points $x \in X$
- morphisms are homotopy classes of paths $x \to y$.

Ex If X is also path connected, then $B\pi_1(X,x) \hookrightarrow \Pi_1(X)$ is an equivalence of categories for any $x \in X$, so $\mathrm{Fun}(\Pi_1(X), \underline{Set}) \simeq \pi_1(X,x)$-$\underline{S}$

Ex

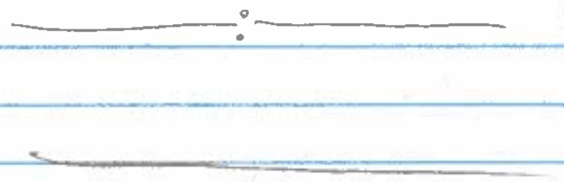


$\longleftrightarrow \{Y_1, Y_2, Y_3\}$ where $\mathbb{Z}$ acts by
$Y_i \mapsto Y_{i+1 \bmod 3}$.

Q Is there a generalisation?

$$\begin{tikzcd}
\boxed{?} \arrow[r, "\sim"] & \underline{Sh}(X) \arrow[r, "\sim"] & \underline{LocHomeo}_{/X} \\
\mathrm{Fun}(\Pi_1(X), Set) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\sim"] & \underline{LC}(X) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\sim"] & \underline{Cov}_{/X} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Ex The line with two origins



$$\begin{tikzcd}
\mathbb{R} \coprod_{\mathbb{R}\setminus\{0\}} \mathbb{R} \arrow[d] \\
\mathbb{R}
\end{tikzcd}$$

wants to correspond to the diagram

$$\{\cdot\} \leftarrow \{:\} \rightarrow \{\cdot\}$$

of sets, i.e. a functor from

$$\bullet \leftarrow \bullet \rightarrow \bullet$$

to Set. We'll make this precise, but only for constructible sheaves.

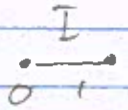
§2 Stratifications

Def Let P be a poset. The Alexandroff topology on P is the topology whose opens are the upwards closed sets ("cosieves"): $x \in U$ and $x \leq y$ implies $y \in U$.

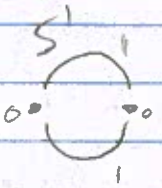
Ex In $P = [1] = \{0 \to 1\}$, the opens are $\emptyset$, $\{1\}$, and $\{0,1\}$ (Sierpinski space)

Def A stratification of a topological space X over a poset P is a continuous map $f: X \to P$.

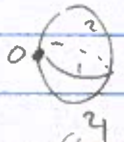
Ex

- 

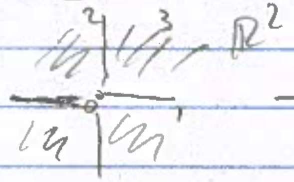
$\rightarrow [1] = \{0 \to 1\}$

- 

$\rightarrow [1] = \{0 \to 1\}$

- 

$\longrightarrow [2] = \{0 \to 1 \to 2\}$

- 

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$

- We also allow

$\rightarrow [1] = \{0 \to 1\}$

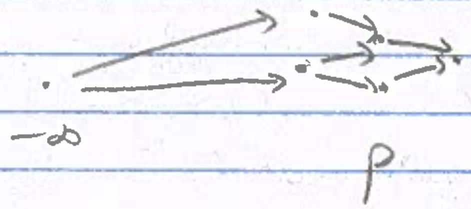
So the closure $\overline{X_p}$ need not be a union of Strata ("axiom of the frontier")

Notation $X_p = f^{-1}(p)$, $X_{\geq p} = f^{-1}(P_{\geq p})$, etc.

locally closed in X

open in X

Def Let P be a poset. Then the *left cone* $P^{\triangleleft}$ is the poset $P \amalg \{-\infty\}$ where $-\infty < p$ for all $p \in P$.



Ex If $P = [n]$, then $P^{\triangleleft} \cong [n+1]$.

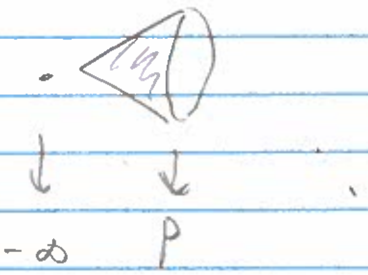
Def The *cone* CX of a topological space X is $(X \times \mathbb{R}_{>0}) \cup \{*\}$, where $U \subseteq CX$ is open iff $U \cap (X \times \mathbb{R}_{>0})$ is open and $* \in U \Rightarrow X \times (0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$.

Rmk "Usually", one defines the cone as $(X\times\mathbb{R}_{\geq 0}) \amalg_{X\times\{0\}} \{*\}$. The continuous map $(X\times\mathbb{R}_{\geq 0}) \amalg_{X\times\{0\}} \{*\} \to CX$ is a homeomorphism if X is compact, but not in general.

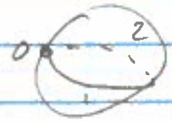
Rmk A stratification $f: X\to P$ induces a stratification

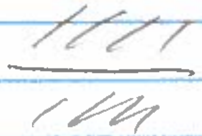
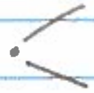
$$f^{\triangleleft} : CX \to P^{\triangleleft}$$
$$* \mapsto -\infty$$
$$(x,t) \mapsto f(x).$$

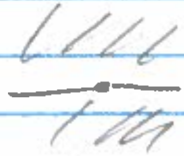
Picture



Def A stratification $f: X\to P$ is conical if every $x\in X$ has an open neighborhood $U\subseteq X_{\geq p}$ (where $p=f(x)$) such that $U\to P_{\geq p}$ is isomorphic to $Z\times CY$ for some $P_{>p}$-stratified space Y and some path connected space Z. Such a U is called a basic neighborhood.

Ex  $\to [2]$ is conical:

- around equator:  $= \mathbb{R}\times$  $= \mathbb{R}\times C(\cdot\ \cdot)$

- around point:  $=$ pt $\times$  $=$ pt $\times\, C($$)$.

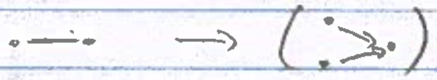
Def A CW complex X is *normal* if the closure $\overline{e^k_\alpha}$ of each cell $B^k \to e^k_\alpha$ is a (finite) subcomplex. It is *regular* if moreover $B^k \to \overline{e^k_\alpha}$ is a homeomorphism.

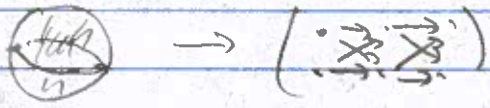
Ex •  (∂B^2 attached to an interior point of e^1). not normal

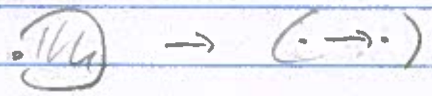
•  (S^n made from 2 cells): normal but not regular.

•  (S^n made from 2 cells in each dimension): regular.

Rmk A normal CW complex X is naturally stratified by its *face poset* $F(X)$ of closed cells under inclusion.

Ex • 

• 

• 

Lemma If X is *regular*, then $X \to F(X)$ is conical.

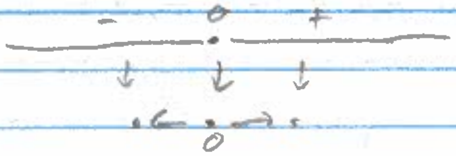
Lemma If X is *regular*, then $X \cong |N(F(X))|$ (homeomorphic!).
The posets arising in this way are exactly those such that $|N(P_{<p})| \cong S^{ht(p)-1}$ for all $p \in P$.

Ex The S^n with 2 cells shows that "normal" is not enough.
Thm (Nocera-Volpe, 2021) Whitney stratifications are conical.

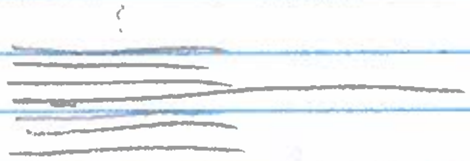
§3 Constructible Sheaves

Def Let $X \to P$ be a stratification and $F \in Sh(X)$. Then F is *P-constructib*
if $F|_{X_p}$ is locally constant for all $p \in P$.

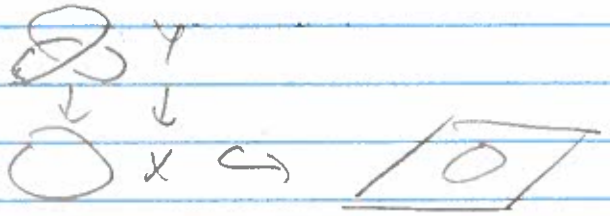
Ex Let $X\to P$ be $\mathbb{R}\to(\overset{-}{\cdot}\leftarrow\overset{0}{\cdot}\rightarrow\overset{+}{\cdot})$:



Then the following are P-constructible:

-  $\mathbb{R}\coprod_{\mathbb{R}\setminus 0}\mathbb{R}$
-  $\mathbb{R}\setminus\{0\}$
-  $(\mathbb{Z}\times\mathbb{R}_{<0})\coprod_{0\times\mathbb{R}_{<0}}0\times\mathbb{R}$.

Exc
- Show that $\mathbb{R}\coprod_{\mathbb{R}\setminus 0}\mathbb{R}$ is the espace étalé of $i_*(*\amalg *)$ where $i: 0\hookrightarrow\mathbb{R}$ is the inclusion.
- Try to draw a picture of $i_* h_Y$ where $i: S^1\hookrightarrow\mathbb{R}^2$ and $Y\to X=S^1$ is the 2:1 cover



Can you construct it from glueing opens along smaller opens?

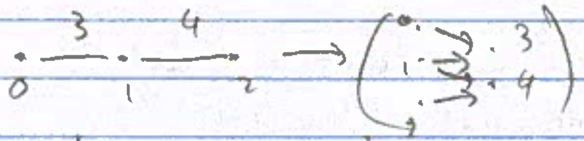
Motivation:

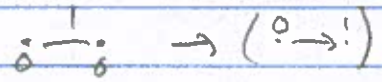
Thm (Grothendieck et. al.)
If $f: X\to Y$ is a morphism of $\mathbb{C}$-varieties, then $R^i f_*\mathbb{Z}$ (or $R^i f_*\mathcal{F}$ for any $\mathcal{F}\in\underline{\mathrm{Cons}}(X^{an})$) is <u>constructible</u> for all i.

Ex For $f:\mathbb{C}^m\setminus\{0\}\to\mathbb{C}^n$ with $m>n>0$, we saw $R^i f_*\mathbb{Z}\cong\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}_0 & i=2m-1\\ 0 & \text{else.}\end{cases}$

§4 Exit path categories

Def A Stratification $f: I \to P$ on $I = [0,1]$ is bounded monotone if $\mathrm{Im}(f)$ is finite and $f: I \to P$ is monotone.

Ex. •  No: $3 > 1$.

•  No: $1 > 0$.

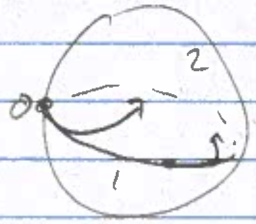
•  $\to (\mathbb{N} \cup \{\infty\})$: No, not bounded.

•  $\to [2]$ Yes.

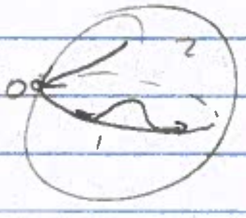
They all look like $[0,a_0], (a_0,a_1], (a_1,a_2], \dots, (a_{n-1},a_n]$.

Def Let $X \xrightarrow{f} P$ be a Stratification. An exit path (w.r.t. f) is a path $\gamma: I \to X$ such that $I \to P$ is bounded monotone.

Ex



✓



X

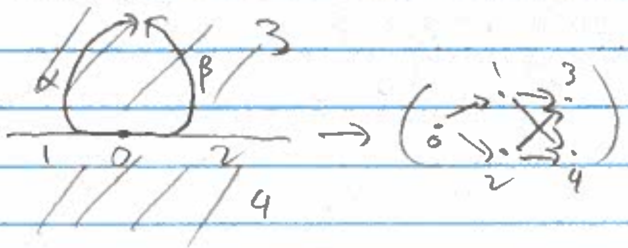
You're not allowed (back) into lower (dimensional) strata.

Def Let $\alpha, \beta: I \to X$ be exit paths from x to y. A homotopy of exit paths is a map

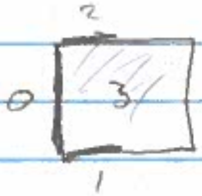
$$h: I^2 \to X$$

such that $h|_{\partial I^2} = x \overset{\beta}{\boxed{\quad}} y$ each $h(t,-): I \to X$ is an exit path, and the Stratification $I^2 \to P$ is finite polyhedral (all $I^2_{\leq p}$ are closed polygons).

Ex



The exit paths α, β are homotopic, but you need to "skip" a stratum. A homotopy looks something like



Def The <u>exit path category</u> $\underline{\mathrm{Exit}_P(X)}$ has

- objects: points $x \in X$
- morphisms: homotopy classes of exit paths $x \rightsquigarrow y$.

It has a natural functor $\underline{\mathrm{Exit}_P(X)} \to P$

$$x \longmapsto f(x)$$
$$(x \rightsquigarrow y) \longmapsto (f(x) \leq f(y))$$

Ex (0) If $P = *$, then $\underline{\mathrm{Exit}_P}(X) \simeq \Pi_1(X)$.

(1) For  $\to \{0 \to 1\}$, we get

$$x_i \longmapsto 0$$
$$\text{else} \longmapsto 1$$

$$\begin{tikzcd}
\underline{\mathrm{Exit}_P(X)} \simeq \Bigg\{ & & x_1 \arrow[dl] \arrow[dr] & & x_2 \arrow[dl] \arrow[dr] & & \cdots & x_n \arrow[dl] \arrow[dr] & & \Bigg\} \arrow[r] & \Bigg\{ 0 \arrow[d] \\
& \bullet & & \bullet & & \bullet & \cdots \, , \; \bullet & & \bullet & & 1 \Bigg\}
\end{tikzcd}$$

(2) For  $\to \{0 \to 1 \to 2\}$,

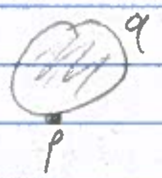
we get

$$\begin{tikzcd}
\underline{\mathrm{Exit}_P(S^2)} \simeq \Bigg\{ & \bullet \arrow[r, shift left, "\ell"] \arrow[r, shift right, "r"'] \arrow[rr, bend left] \arrow[drr, bend right] & \bullet \arrow[r, "u"] \arrow[dr, "d"'] & \bullet \\
& & & \bullet & \Bigg\} \\
& & \{0 \arrow[r] & 1 \arrow[r] & 2\}
\end{tikzcd}$$

where $u\ell = ur$
$d\ell = dr$.

Prop If X is a regular CW complex, then $\mathrm{Exit}_P(X) \to P$ is an equivalence.

Idea $\mathrm{Hom}(e_p, e_q) = *$ since $e_p \cup e_q$ is contractible if $p \leq q$.



Ex The example above again shows that normal is not enough.

§5 Exodromy

Thm (MacPherson (unpublished), Treumann (2009), Lurie (2016?), Curry-Patel (2016?), Porta-Teyssier (2022?), ...)
Let $X \to P$ be a conical stratification. Then

$$\underline{\mathrm{Cons}}_P(X) \simeq \mathrm{Fun}(\underline{\mathrm{Exit}}_P(X), \underline{\mathrm{Set}})$$
$$\mathcal{F} \longmapsto (x \mapsto \mathcal{F}_x)$$
$$(U \mapsto \lim_{x \in \underline{\mathrm{Exit}}_P(U)} F(x)) \longleftarrow F.$$

Pf (Sketch)
Define $\underline{E}_X$ as the category of pairs (U, x) with $x \in U$, and

$$\mathrm{Hom}((U,x),(V,y)) = \begin{cases} \underline{\mathrm{Exit}}_P(U)(x,y) & V \subseteq U \\ \emptyset & \text{else.} \end{cases}$$

It has projections

$$\begin{tikzcd} & \underline{E}_X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathrm{Open}(X)^{op} & & \underline{\mathrm{Exit}}_P(X). \end{tikzcd}$$

This defines functors $a^* : \underline{\mathrm{PSh}}(X) \to \mathrm{Fun}(\underline{E}_X, \underline{\mathrm{Set}}) \xleftarrow{b^*} \mathrm{Fun}(\underline{\mathrm{Exit}}_P(X), \mathrm{Se}$

$$\mathcal{F} \mapsto ((U,x) \mapsto \mathcal{F}(U))$$
$$((U,x) \mapsto F(x)) \longleftarrow F,$$

which by general nonsense have adjoints

$$\begin{tikzcd} PSh(X) \arrow[r, shift left, "a^*"] \arrow[r, phantom, "\perp"] & Fun(\underline{Ex}_X, \underline{Set}) \arrow[l, shift left, "a_* = Ran_a"] \arrow[r, shift right, "\perp"'] & Fun(\underline{Exit_P}(X), \underline{Set}) \arrow[l, shift right, "b_! = Lan_b"'] \arrow[l, shift left, "b^*"] \end{tikzcd}.$$

This gives an adjunction

$$\begin{tikzcd} \underline{PSh}(X) \arrow[r, shift left, "\Psi"] \arrow[r, phantom, "\perp"] & Fun(\underline{Exit_P}(X), \underline{Set}) \arrow[l, shift left, "\Phi"] \end{tikzcd}.$$

Then $U \mapsto \underline{Exit_P}(U)$ is a cosheaf ("van Kampen theorem"), so $U \mapsto \lim_{x \in \underline{Exit_P}(U)} F(x)$ is a sheaf, i.e. Φ lands in $\underline{Sh}(X)$.

If U is a basic open neighborhood of $x \in X$, then x is initial in $\underline{Exit_P}(U)$, so $\Phi(F)(U) = \lim_{y \in \underline{Exit_P}(U)} F(y) = F(x)$. Since basic open form a neighborhood basis around x, we get $\Phi(F)_x = F(x)$, i.e. $\Psi \circ \Phi \xrightarrow{\sim} id$. This also shows that $\Phi(F)(U) \xrightarrow{\sim} \Phi(F)(V)$ for $V \subset U$ both basic, so $\Phi(F)|_{X_p}$ is locally constant, i.e. $\Phi(F)$ is constructible.

Finally, $\Psi(\mathcal{F})(x) = \mathcal{F}_x = \underset{\substack{\rightarrow \\ x \in U \\ \text{basic}}}{\operatorname{colim}} \mathcal{F}(U) = \mathcal{F}(U)$ since a constructible sheaf is constant on basic opens, so

$$\mathcal{F}(U) \to (\Phi \circ \Psi)(\mathcal{F})(U)$$

is an isomorphism for all basic opens U. Then it is an isomorphism since both sides are sheaves. ☐

<u>Cor</u> If $X \to P$ is a conical stratification, then $\pi_1(X) = B(\underline{Exit_P}(X))$ \ "invert everything"

<u>Pf</u> A constructible sheaf is locally constant iff all maps $\mathcal{F}_x \to \mathcal{F}_y$ for $x \to y$ in $\underline{Exit_P}(X)$ are invertible. ☐

<u>Ex</u>  $S^1 \to \{0 \to 1\}$ gives $\underline{Exit_P}(S^1) = (\bullet \rightrightarrows \bullet)$. Inverting everything gives $B\mathbb{Z}$.