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Exodromy for étale Sheaves

(After Barwick – Glasman – Haine, Wolf, vDdB – Wolf in progress)

§1 Recap

In topology, we have equivalences

$$\begin{array}{ccccc}
 & & Sh(X) & \xleftrightarrow{\sim} & LocHomeo_{/X} \\
 & & \cup & & \cup \\
Fun(Exit_p(X), Set) & \xleftrightarrow{\sim} & Cons_p(X) & & \cdots \\
 & & \cup & & \cup \\
Fun(\Pi_1(X), Set) & \xleftrightarrow{\sim} & LC(X) & \xleftrightarrow{\sim} & Cov_{/X}
\end{array}$$

On the other hand, we briefly discussed π_1^{et}:

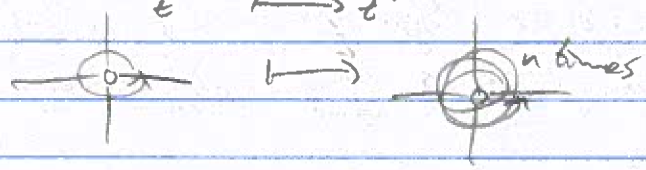
Def Let X be a variety. Then

$$\pi_1^{et} = \varprojlim_{\substack{Y \to X \\ \text{finite étale connected}}} Aut_X(Y).$$

Recall that $Y \xrightarrow{f} X$ is <u>étale</u> if $df : \Omega_{X, f(y)} \to \Omega_{Y,y}$ is an isomorphism for all $y \in Y$ — cotangent space

Ex For $X = \mathbb{A}^1 \setminus \{0\}$, we found a unique finite étale cover

$$\mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 \setminus \{0\}$$
$$z \longmapsto z^n$$



of degree n, so

$$\pi_1^{et}(\mathbb{A}^1 \setminus \{0\}) \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

This is analogous to Galois Theory:

Ex Let $R = \mathbb{C}[[t]] = \{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in \mathbb{C}\}$ be the formal power series, and

$$\mathbb{C}((t)) = \mathrm{Frac}(R) = \{\sum_{i=-\infty}^{\infty} a_i t^i \mid a_i \in \mathbb{C}\}$$

the formal Laurent series. Then $\mathbb{C}((t))$ has a unique extension

$$\mathbb{C}((t)) \hookrightarrow \mathbb{C}((t^{\frac{1}{n}}))$$

of degree n, so

$$\mathrm{Gal}(\overline{\mathbb{C}((t))}/\mathbb{C}((t))) \cong \hat{\mathbb{Z}}.$$

This really looks the same:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathbb{A}^1 \setminus \{0\} & & \mathrm{Spec}\,\mathbb{C}((t^{\frac{1}{n}})) \\ \downarrow (\cdot)^n & & \downarrow \\ \mathbb{A}^1 \setminus \{0\} & & \mathrm{Spec}\,\mathbb{C}((t)) \end{array}$$

Ex The "universal cover" of $\mathrm{Spec}\, k$ is $\mathrm{Spec}\, \bar{k}$.

a point viewed as $\left\{ \frac{\bar{k}}{k} \right.$-variety

In particular, a "base point" of X should be a $\bar{k}$-point, not a k-point (as $\pi_1^{\acute{e}t}(\mathrm{Spec}\, k) \neq 1$ in general). This is called a geometric point.

Lemma (Monodromy equivalence, SGA1)
Let X be a variety and $\bar{x} \to X$ a geometric point. Then

$$\mathrm{Fun}(\pi_1^{\acute{e}t}(X, \bar{x}), \mathrm{Fin}) \simeq \underbrace{\mathrm{Cov}_{/X}}_{\text{finite étale covers}}$$

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(Sketch!)

Pf Define $F: \underline{Cov}_{/X} \to \underline{Fun}(\pi_1^{et}(X,\bar{x}), \underline{Fin})$ by
$(Y \to X) \mapsto (Y_{\bar{x}} \to \bar{x})$.

Note: since $\pi_1^{et}(\bar{x}) = 1$, the étale cover $Y_{\bar{x}}$ is "trivial", ie $\coprod_{i=1}^{n} \bar{x}$.
How to obtain a $\pi_1^{et}(X)$-action? Each connected component of Y will correspond to an orbit, so w.l.o.g. Y is connected.

We say $(Z \to X) \in \underline{Cov}_{/X}$ is Galois if the action of $\mathrm{Aut}_X(Z)$ on $Z_{\bar{x}} = \coprod_{i=1}^{m} \bar{x}$ is transitive. Show that Y has a Galois closure Z: for instance, take any component of $\underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n = \deg(Y \to X)}$ not contained in a diagonal

$$\underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n-1} \xrightarrow{(id, \ldots, id, \Delta, id, \ldots, id)} \underbrace{Y \times_X \cdots \times_X Y}_{n}.$$

If $L = k(\alpha)$, then Galois closure $M = k(\alpha_1, \ldots, \alpha_n)$ conjugates of α; f_α splits completely over M

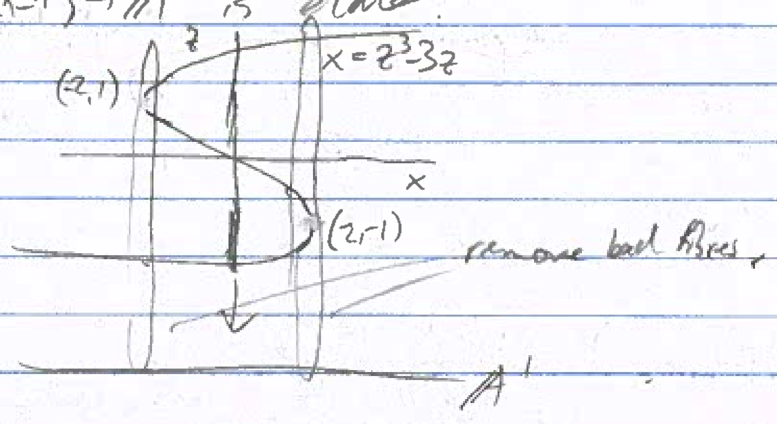
Then $\coprod_{\sigma \in \mathrm{Hom}_X(Z,Y)} Z \xrightarrow{\sim} Z \times_X Y$ and $\mathrm{Aut}_X(Z)$ acts on $Z \times_X Y$, so also on $Y_{\bar{x}} = \bar{x} \times_X Y$ (choose a lift $\bar{x} \to Z$ of $\bar{x} \to X$).

Show F is fully faithful and essentially surjective (pure category theory). □

Maybe easier: use $\mathcal{L}C(X)$ instead of $\underline{Cov}_{/X}$.
(to be defined)

Ex Let $f: \mathbb{A}^1 \to \mathbb{A}^1$, $z \mapsto z^3 - 3z$. Then $df = 3z^2 - 3$ vanishes at $z = \pm 1$, so $f: \mathbb{A}^1 \setminus \{\pm 1\} \to \mathbb{A}^1$ is étale.

Picture



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Recall that it is not finite étale, as it is not a closed map.
(Only quasi-finite étale: all fibres are finite)
But over $\mathbb{A}^1 \setminus \{-2,2\}$, the map

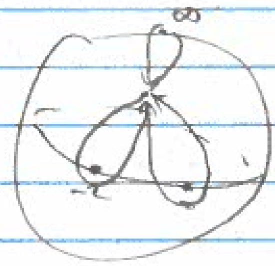
$$\mathbb{A}^1 \setminus \{\pm 1, \pm 2\} \to \mathbb{A}^1 \setminus \{\pm 2\}$$

is finite étale. But it is not Galois: $\mathrm{Aut}_X(Y) = 1$ since every X-automorphism $Y \xrightarrow{\sim} Y$ extends to $\mathbb{A}^1 \to \mathbb{A}^1$, so preserves the ramified points $\{\pm 1\}$.

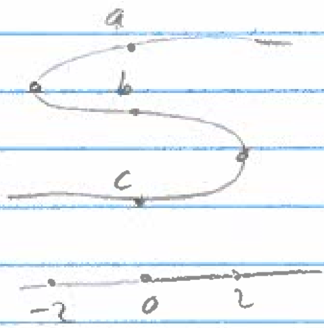
For the Galois closure, it suffices to look at $Y \times_X Y$: it has degree 3 over each factor Y, one of which is $\Delta_Y : Y \to Y \times_X Y$. So the remaining degree 2 part is a 2:1 cover $Z \to Y$, and $\mathrm{Aut}_X(Z) \cong S_3$.
(Using Cardano's formula, we can "compute" Z).

Rmk So we explained $\mathbb{A}^1 \setminus \{\pm 1, \pm 2\} \to \mathbb{A}^1 \setminus \{\pm 2\}$ via $\pi_1^{\acute{e}t}(\mathbb{A}^1 \setminus \{\pm 2\})$.

Actually, over $k = \mathbb{C}$ we know $\pi_1^{\acute{e}t}(\mathbb{A}^1 \setminus \{\pm 2\}) = \widehat{\langle \gamma_2, \gamma_{-2}, \gamma_\infty \mid \gamma_2 \gamma_{-2} \gamma_\infty = 1 \rangle}$



Then we can compute the action of $\gamma_2, \gamma_{-2}, \gamma_\infty$ on $f^{-1}(0)$, say:



picture over $\mathbb{R}$!
Very deceptive.

Monodromy around -2 swaps a and b
2 swaps b and c
∞ rotates: $\gamma_\infty = \gamma_2^{-1}\gamma_{-2}^{-1} = (bc)(ab) = (cba)$.

§2 Étale constructible sheaves

Recall: a presheaf $\mathcal{F}: \underline{Et}_{/X}^{op} \to \underline{Set}$ is a sheaf if

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equaliser for every étale cover $\{U_i \to U\}_{i \in I}$ in $\underline{Et}_{/X}$.

Ex If $Y \to X$ is any morphism, then

$$h_Y : U \longmapsto \mathrm{Map}_X(U, Y)$$

is an étale sheaf.

Rmk But they don't have "espaces étalés"! (well, they do in algebraic spaces...).

Def An étale sheaf $\mathcal{F}$ is locally constant if there is an étale cover $\{U_i \to X\}_{i \in I}$ and finite sets A_i ($i \in I$) such that

$$\mathcal{F}|_{U_i} \cong h_{A_i \times U_i / U_i} =: \underline{A_i}_{U_i}$$

for all $i \in I$. (If X is connected, all A_i the same.)

Ex If $Y \to X$ is finite étale, then we saw that

$$\coprod_{\sigma \in \mathrm{Hom}_X(Z, Y)} Z \xrightarrow{\sim} Z \times_X Y$$

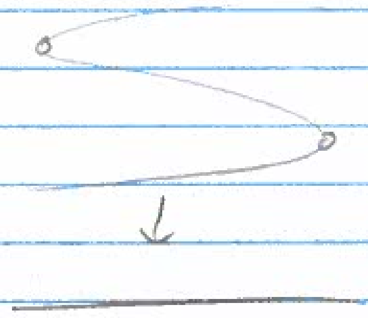
if $Z \to X$ is a Galois closure of $Y \to X$. So h_Y is locally constant.

Lemma $\mathrm{Cov}_{/X} \to \underline{LC}(X)$ is an equivalence.

Pf The inverse construction is called "Galois descent". □

Def An étale sheaf $\mathcal{F}$ is <u>constructible</u> if there exists a finite (algebraic) stratification $f : X \to P$ such that each $\mathcal{F}|_{X_p}$ is locally constant.

Ex We already saw one! If



$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{\pm 1\} = Y & & z \\ \downarrow & & \downarrow \\ \mathbb{A}^1 = X & & z^3 - 3z \end{array},$$

then h_Y is constructible:

- over $\mathbb{A}^1 \setminus \{\pm 2\}$ it is locally constant (with fibre size 3)
- over 2 and -2 it is an isomorphism (but not in a neighbourhood).

In fact:

Lemma If $f : Y \to X$ is an étale morphism of varieties, then h_Y is constructible.

Idea Show that the locus $\{x \in X \mid \deg(Y_x \to x) = n\}$ is locally closed for all $n \in \mathbb{N}$ (if f separated). $\square$

Lemma Let $\mathcal{F}$ be an étale sheaf. Then $\mathcal{F}$ is canonically a colimit of h_Y for $Y \to X$ étale ($\mathcal{F} = \int^Y \mathcal{F}(Y) \times Y$). Moreover, TFAE:

(1) $\mathcal{F}$ is constructible;
(2) $\mathcal{F}$ is a finite colimit of h_Y for $Y \to X$ étale;
(3) $\operatorname{Hom}_{Sh(X_{et})}(\mathcal{F}, -)$ preserves colimits ($\mathcal{F}$ is "compact" or "finitely presented").

("Espace étalé without the espace")

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§3 Exodromy

<u>Thm</u> (Barwick–Glasman–Haine, 2018 preprint/book draft)
Let X be a variety (or qcqs scheme). Then

$$\underline{\mathrm{Cons}}(X) \simeq \mathrm{Fun}^{\mathrm{cts}}(\underline{\mathrm{Gal}}(X), \mathrm{Fin}).$$

(They have a version for constructible sheaves of spaces)
"stacks in ∞-groupoids"

Here, $\underline{\mathrm{Gal}}(X)$ is the Galois category:

- Points are geometric points $\bar{x} \to X$
- morphisms are

$$\begin{tikzcd} \mathrm{Spec}\,\mathcal{O}^{sh}_{X,\bar{x}} \arrow[rr] \arrow[dr] & & \mathrm{Spec}\,\mathcal{O}^{sh}_{X,\bar{y}} \arrow[dl] \\ & X & \end{tikzcd}$$

See p. 9

Endowed with a profinite topology (they don't give an explicit model – only defined by universal properties.)

Ex

<u>Rmk</u> In topology, a key input is that X is semi locally simply connected (e.g. locally contractible) (monodromy) or conically stratified (which implies: if $x \in U$ is a basic open, then $x \in \underline{\mathrm{Exit}}_P(U)$ is initial:


contractible, say
$- \sqcup P$
).

<u>Problem</u> The étale topology does not have small enough opens.

Eg. if $U \to X$ is étale, then the image of $\pi_1^{et}(U) \to \pi_1^{et}(X)$ has finite index.

<u>Solution</u> The pro-étale topology: also allow infinite towers

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Ex For $\mathbb{A}^1 \setminus \{0\}$, we get a pro-étale cover

universal cover??

$$\begin{array}{ccc} \varprojlim_n Y_n & = & \operatorname{Spec} k[t^{\pm\frac{1}{\infty}}] \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ Y_n = \mathbb{A}^1 \setminus \{0\} & = & \operatorname{Spec} k[t^{\pm\frac{1}{n}}] \\ \downarrow (-)^n & & \downarrow \\ X = \mathbb{A}^1 \setminus \{0\} & = & \operatorname{Spec} k[t^{\pm 1}] \end{array}$$

not a variety!

Ex Pro-étale Sheaves on $\operatorname{Spec} \bar{k}$ = Condensed Sets!
(A generalisation of topological spaces, but with better categorical properties)

Thm (Wolf, 2022)
Let X be a variety (or qcqs scheme). Then

$$\operatorname{Sh}(X_{proét}) \simeq \operatorname{Fun}^{cts}(\underline{\operatorname{Gal}(X)}, \underline{\operatorname{Cond}}).$$

All pro-étale sheaves!!!

Thm (vDdB-Wolf, in progress)
The same without qcqs (also applies to weil-étale sheaves), with much easier proof.

Ideas (BGH):
* Reduce to fixed finite stratification
* Induction on number of strata — reduce to $X \xrightarrow{f} [1]$
$U = f^{-1}(1)$
$Z = f^{-1}(0)$
* Use "Artin gluing" (easy) plus "Gabber oriented base change theorem"

Technical, but in some way "classical" (except for the oo-topoi ...)

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· (Wolf): bootstrap from BGH.

"All of this is easy if you know ∞-topoi + étale homotopy theory".
(but even Scholze is confused by these papers).

· (vDdB-Wolf): turn it around: prove the pro-étale version first.

* Bhatt-Scholze constructed "weakly contractible" opens.
* Mimic the topological proof using these.

Q What even is a "profinite category"?

· An object of $\underline{\mathrm{Pro}}(\mathrm{Cat}^{\mathrm{fin}})$?
· A category object in $\underline{\mathrm{Pro}}(\mathrm{fin})$?
· something else?

vDdB-Wolf work with condensed categories from the start:
a "sheaf of categories" (= stack in categories) on $\underline{\mathrm{CHaus}}$
(or actually $\underline{\mathrm{Ext}}$, $\underline{\mathrm{Disc}}$).

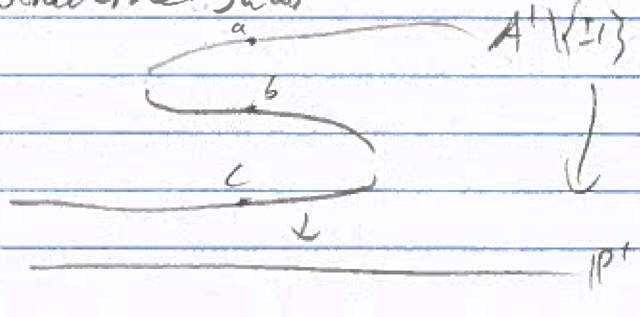
Ex Let $X = \mathbb{P}^1_{\mathbb{C}}$ with stratification

Then

$\underline{\mathrm{Gal}}_p(X) =$ $\widehat{\mathbb{Z}}/\sigma_{-2}$, $\widehat{\mathbb{Z}}/\sigma_{2}$, $\widehat{\mathbb{Z}}/\sigma_{\infty}$ $\to$ $\langle \sigma_{-2}, \sigma_2, \sigma_\infty \mid \sigma_{-2}\sigma_2\sigma_\infty = 1 \rangle^\wedge = \widehat{G}$

$\left(= \underline{\mathrm{Cons}}_p(X)^\wedge \right)$

The constructible sheaf



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corresponds to the diagram

$$\begin{tikzcd}
\{c\} \arrow[dr, Rrightarrow] & & \\
\{a\} \arrow[r, Rrightarrow] & \{a,b,c\} \arrow[loop right] & \begin{array}{l} \gamma_{-2} = (ab) \\ \gamma_2 = (bc) \\ \gamma_\infty = (cba) \end{array} \\
\emptyset \arrow[ur, Rrightarrow] & &
\end{tikzcd}$$

(Here, $g \in \hat{G}/\langle\gamma_{-2}\rangle$ acts by $c \mapsto g(c)$, which is well-defined as γ_{-2} fixes c.)