Distances, edges & domains

Advanced Geometry, part I

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Distances, edges and domains

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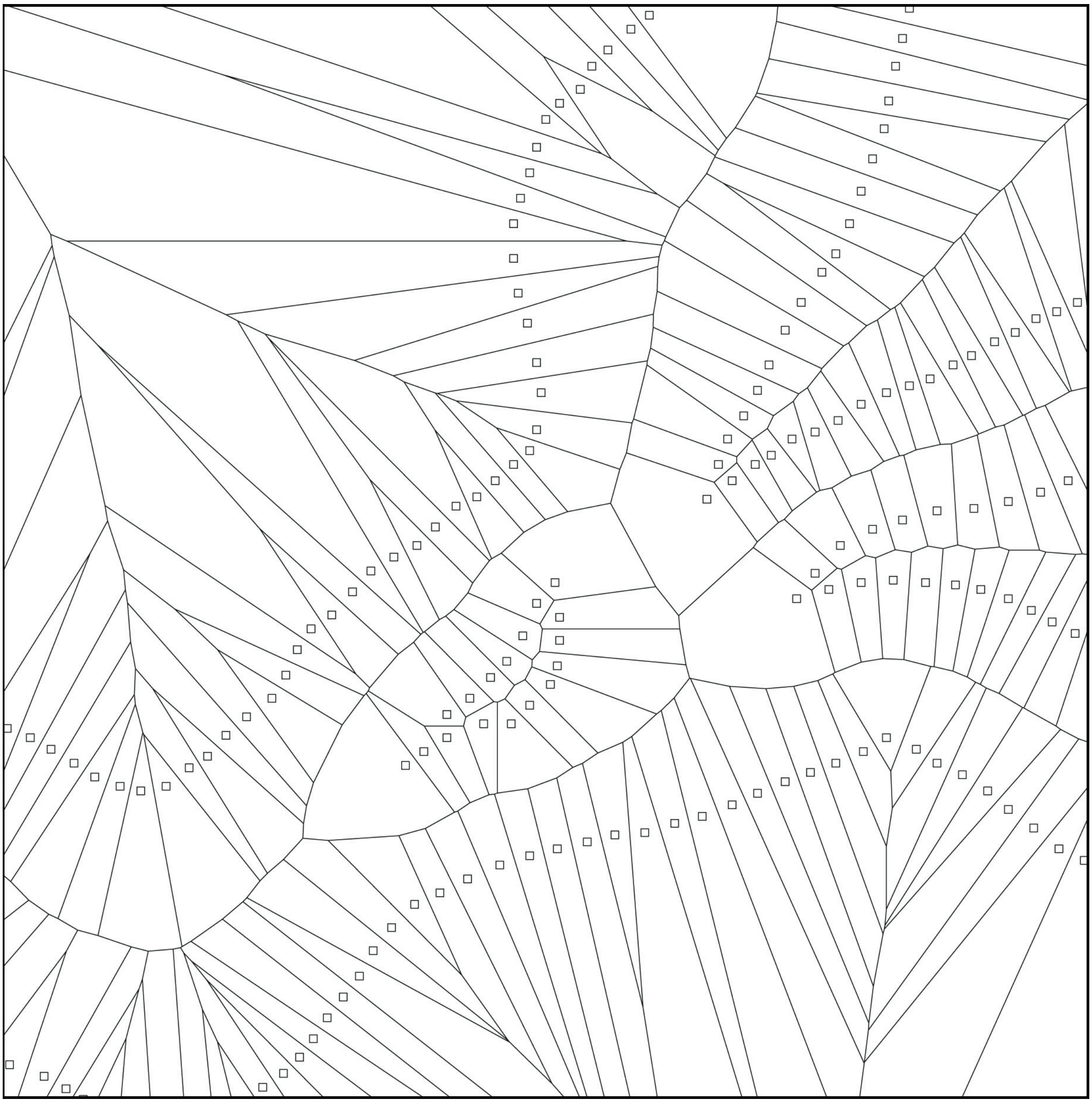
Chapter 1
Voronoi-diagrams

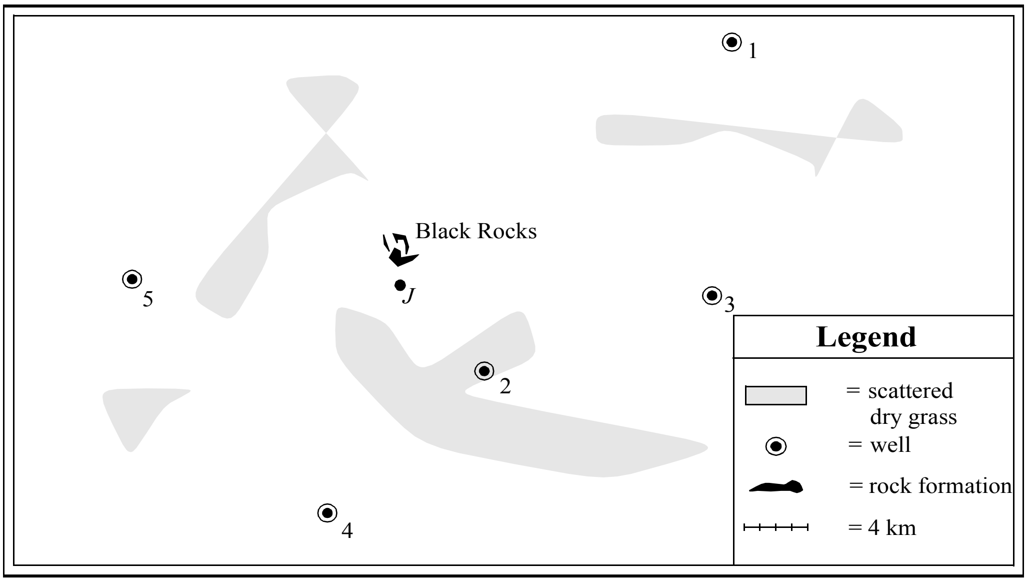


In this first chapter you will encounter a certain way of partitioning an area, which has many applications.
We will start with simple cases, but make sure you keep an eye on the figure on the front page. That is also an example of one of the partitions in this chapter!

You will have to sketch quite a lot. You can do that in this book. For some exercises special worksheets are included.
Often you will need a protractor, ruler and compass.
You may also have to make sketches and graphs in your notebook while working out the solutions.

1: In the desert

Below you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at *J*. You are very thirsty and you only brought this map with you.



1 a. To which well would you go for water?

That choice was not difficult. Of course you would go to the nearest well.

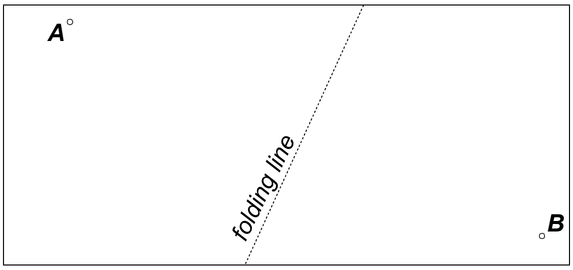
- **b.** Point out two other places from where you would also go to well 2. Choose them far apart from each other.
- **c.** Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest.
- **d.** What can you do when you are standing exactly on the edge of two different domains?
- **e.** Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells.
- **f.** In reality the desert is much larger than is shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?
- **g.** The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?
- **h.** What kind of lines are the edges? Straight? Curved?

In this exercise you just partitioned an area according to the *nearest-neighbour-principle*. Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then.
Next we will investigate the simple case of two wells, or actually two points, since we might not be dealing with *wells* in other applications.

2: The edges between two domains

folding

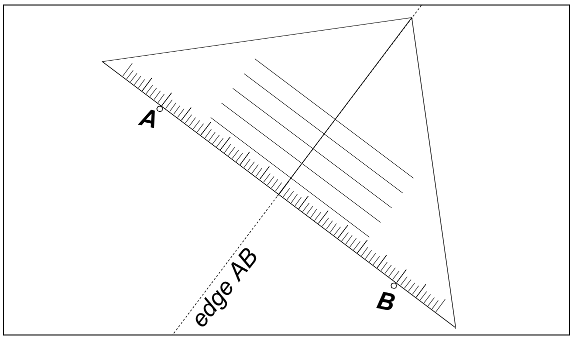
A simple case with two wells is shown here. We neglect the dimensions of the wells themselves, i.e.: we pretend they have no size at all. In the figure: the *points A* and *B*.



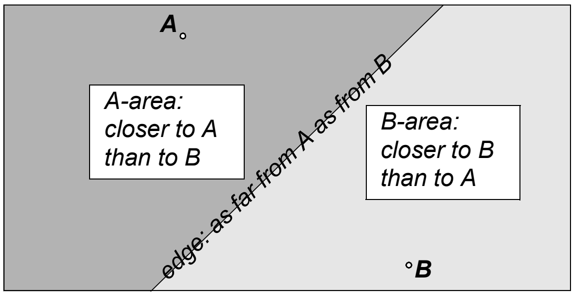
On paper the edge between the domains of *A* and *B* is easy to find, namely by folding the paper so that *A* lies on top of *B*. This folding line is the edge between the areas belonging to *A* and *B*.

protractor

There is also another method to find this edge easily and fast: with the protractor. See the figure on the right. *A* and *B* are both at the same distance from the middle of the protractor.



In this figure the areas round *A* and *B* have different colors.
To the *A*-domain applies:
distance to A < distance to B.
To the *B*-domain applies:
distance to A > distance to B.
Only on the edge applies:
distance to A = distance to B.



halfplane

Actually, you should imagine that there is more than just the sketched part: everything continues unlimited in all directions. The two domains, determined in such a way, are infinitely large and are bounded by a straight line. The name for such an domain is *halfplane*. Include the edge as part of both halfplanes.
In the figure the domains of *A* and *B* are therefore both halfplanes. These halfplanes overlap each other on the edge.

2 The edge is often called *conflict line*. A good name? Why?

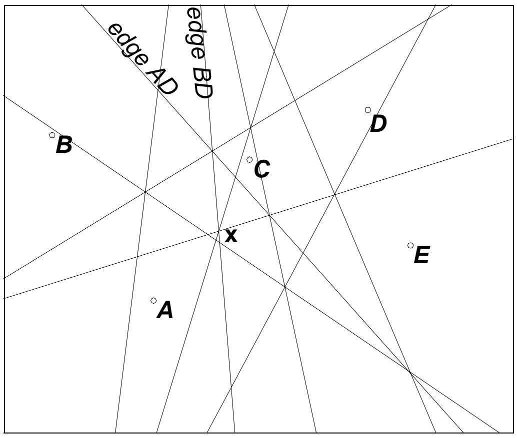
3: More points, more edges

Through folding, we will now investigate a situation with four points.
Also take worksheet A; that is page 121 of this book.

3 For each pair of points we determine the edge by folding.
a. First compute how many folds are necessary and then proceed with the actual folding. Try to do this as precise as possible; for instance hold the piece of paper up to the light. Use the folded lines to sketch the partition of the area.
b. While folding, a lot of intersections of the folding lines arise. Nevertheless there are different kinds of intersections. What differences do you notice?
c. One fold turns out to be redundant. What causes that?

excluding technique

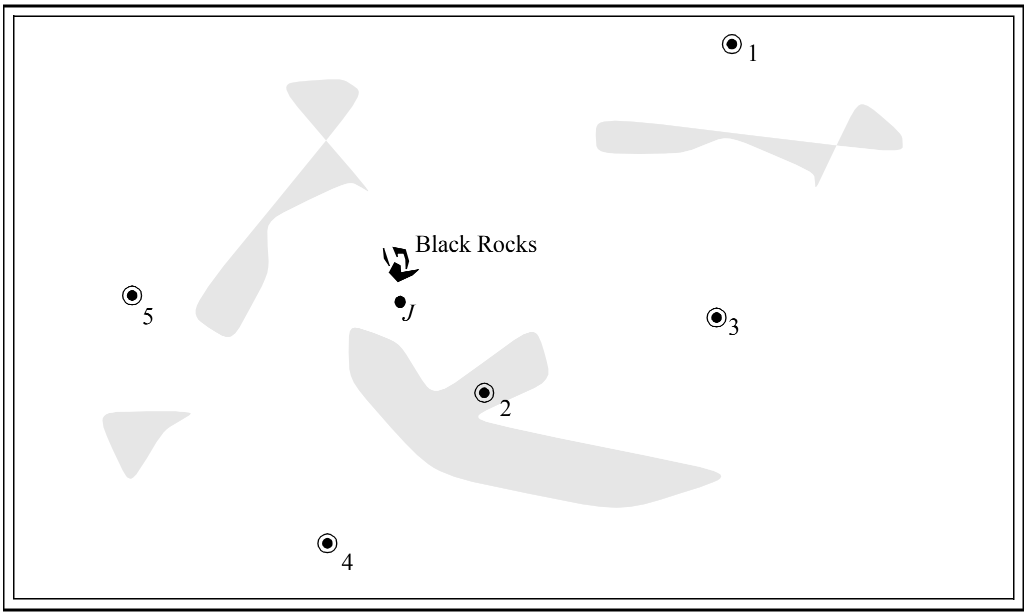
Here a situation with five points where the edges for all pairs of points are shown.

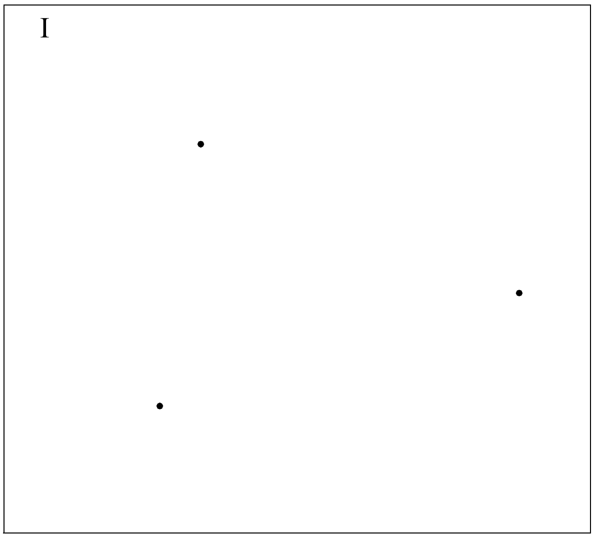


4 **a.** How many edges are there?
b. Of the cross **x** near the edge *BD* you know for certain: it certainly does not belong to *B*. How can you tell?
c. Use other lines to exclude other possible owners of **x**. In the end one remains. Which one?
d. Try to find out for the other areas to what center they belong and with this excluding technique finish the partition using five colors.

with the protractor

5 Now sketch, using the protractor method, the exact edges round the wells on the desert map.

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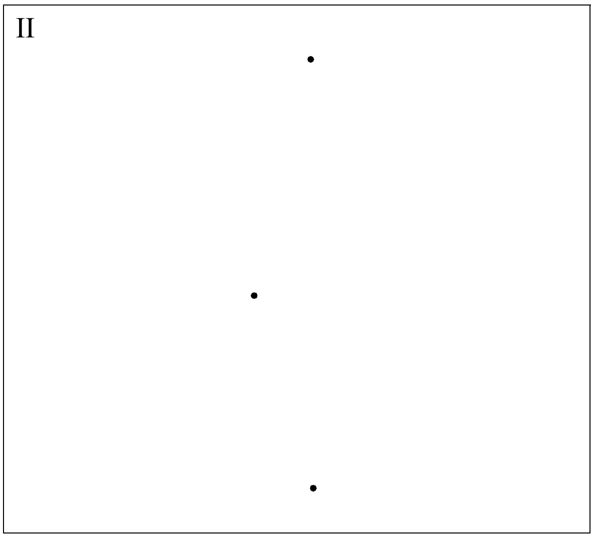
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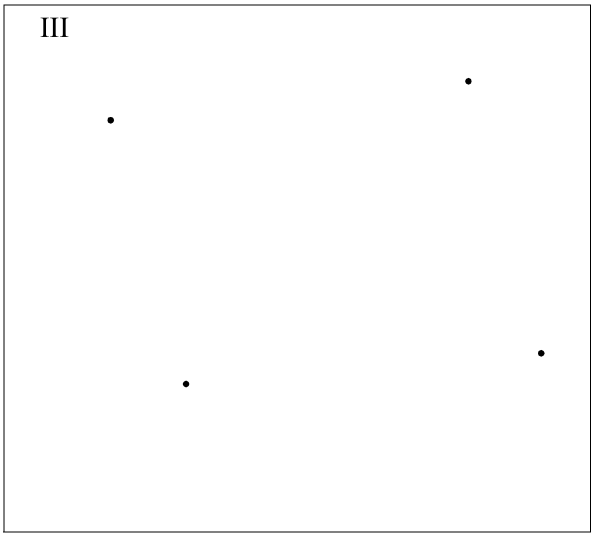
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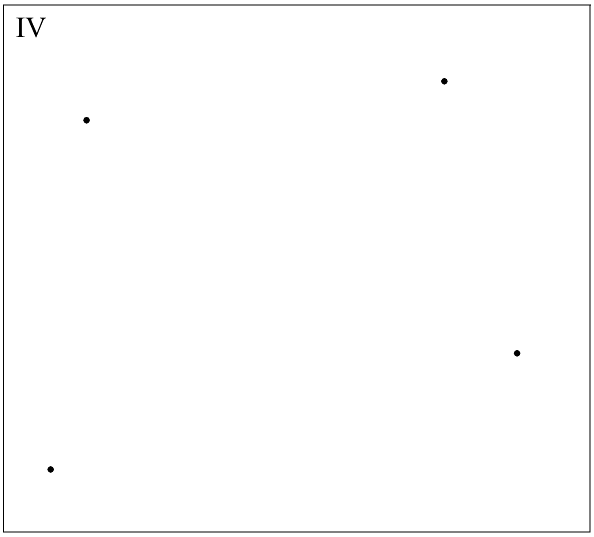
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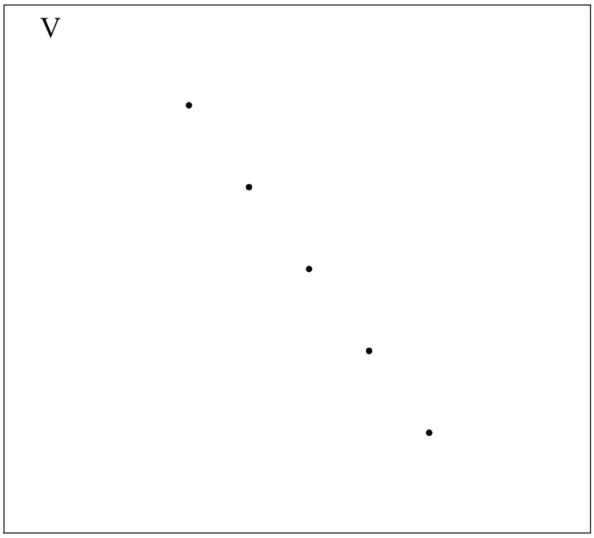
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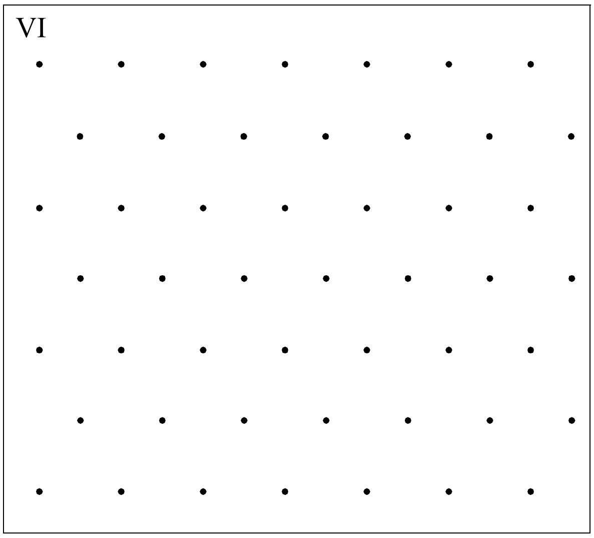

VI

4: Voronoi-diagrams: centers, edges, cells

nearest-neighbor-principle

In the preceding we made partitions of an area according to the '*nearest-neighbour-principle*'.
We now discuss some more terminology.

centers

The points around which everything evolves (in this example the wells) will now be called *centers*. Throughout this book we will always assume we have a finite number of centers.

Voronoi-diagram

The figure of edges is called the *Voronoi-diagram* belonging to the centers
Another name is *edge diagram.*

Voronoi-cell

The area that belongs to a center, is called a *Voronoi-cell*, or, in short, *cell* of that center.

vertices

A Voronoi-cell is bounded by straight lines or by segments of straight lines. The points where several lines converge, are called the *vertices* of the Voronoi-diagram. (Singular: *vertex*).

history

Voronoi-diagrams are named after the mathematician Voronoi. He (in 1908) and Dirichlet (in 1850) used these diagrams in a pure mathematical problem, the investigation of positive definite square forms. In 1911, Thiessen used the same sort of diagrams while determining quantities of precipitation in an area, while only measuring at a small number of points. In meteorology, geography and archaeology the term Thiessen-polytope instead of Voronoi-cell became established.

6 **a.** On the previous page you can see six situations. Each dot represents a center. Sketch the edge diagrams for these situations.
b. In situation I you find one point in the middle where three edges converge. What can you say about the distances of that point to the centers?
c. Does situation II also have such a point?
d. In the situations III and IV only one center is not in the same place. However, the Voronoi-diagrams differ considerably. Try to indicate the cause of that difference.
e. In situation V the centers lie on one line and thus the diagram is fairly easy to draw. What can you say about the mutual position of the edges and the shape of the Voronoi-cells?
f. Situation VI has lots of centers. But thanks to the regularity, sketching of the edge diagram is again a simple affair. Once one cell is known, the rest follows automatically.
Do you know anything in nature which has this pattern as a partition?

infinitely large cells

7 Nowhere is it said that a Voronoi-cell is enclosed on all sides by (segments of) lines. In fact, some cells are infinitely large, even though that is not visible in the picture.
a. How many infinite cells are there in the well example on page 5?
b. In situations with two or three wells there are only infinite cells. Now sketch two situations with four centers. In one situation all cells must be infinitely large, in the other situation not all cells are infinitely large.
c. Describe a situation with twenty centers and twenty infinite cells.
d. Where do you expect the infinite cells to occur in a Voronoi-diagram?

5: Three countries meeting; empty circles

Below you see a redivision of the Netherlands as a Voronoi-diagram*. The centers are the province capitals.

three-countries-point

On each of the vertices of the Voronoi-diagram three cells converge. In such cases we talk about a *three-countries-point*, even if the context does not talk about countries.

8 **a.** What do you know of the distances of the 'three-countries-point' between the cities of Middelburg, Den Haag and Den Bosch to those three cities?

b. Put you compass point in that three-countries-point. Now draw a circle through those three cities with this three-countries-point as its center.

c. Now put your compass point somewhere on the edge between Zwolle and Arnhem, but not in a vertex of the diagram. Sketch a circle with this point as center, which passes through Arnhem.

largest empty circles

What you just sketched, are examples of *largest empty circles*.
A largest empty circle in a Voronoi-diagram is a circle in which no centers lie and on which lies at least one center.

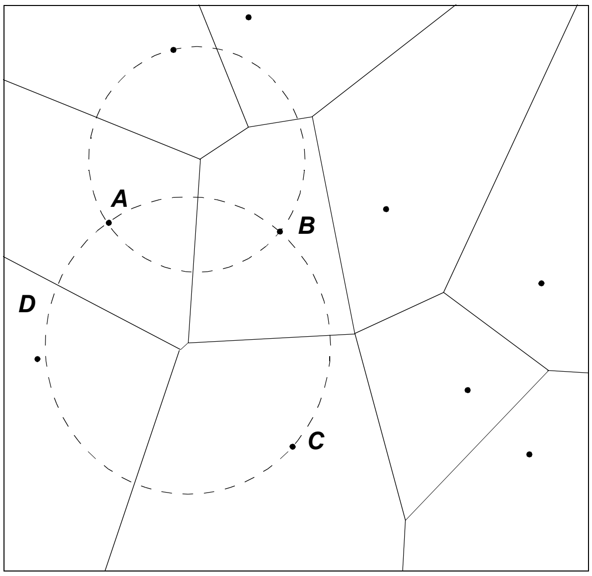
The name *largest empty circle* is chosen well: if you enlarge such a circle around its center just a tiny bit, the interior of the circle would not be empty anymore: for sure there will be one or more centers inside.

* There is a map on page 123 which shows the official division in provinces in dotted lines.

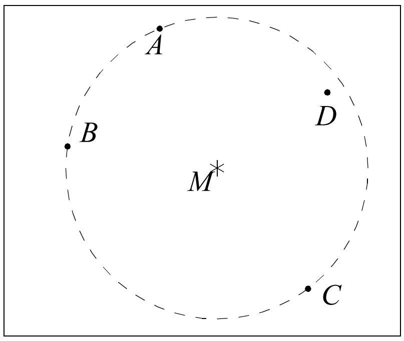
9 a. In this Voronoi-diagram two largest empty circles are already sketched. Mark their centers.

b. Sketch several largest empty circles:
- one with *four* centers on the circle,
- one with *two* centers on the circle,
- one with *one* center on the circle.

c. What can you say in general about the number of centers on a largest empty circle round a three-countries-point?



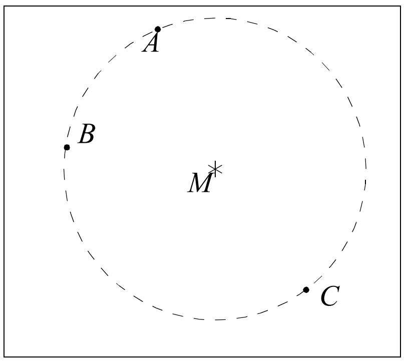
10 a. On the right you see a situation with four centers. The centers are the black dots. The little star at M is the center of the circle through the centers A, B and C. Can M represent the three-countries-point of *the cells around* A, B and C? Why? Or why not?



b. Here you see the same figure, only the center D is left out. Sketch the Voronoi-diagram. Be careful: M is not a center itself, but you can make good use of M in some way.

c. Now add center D yourself and expand the Voronoi-diagram, but do it in such a way that M becomes a four-countries-point.

d. Three-countries-points are very normal, four-countries-points are special. Explain why.

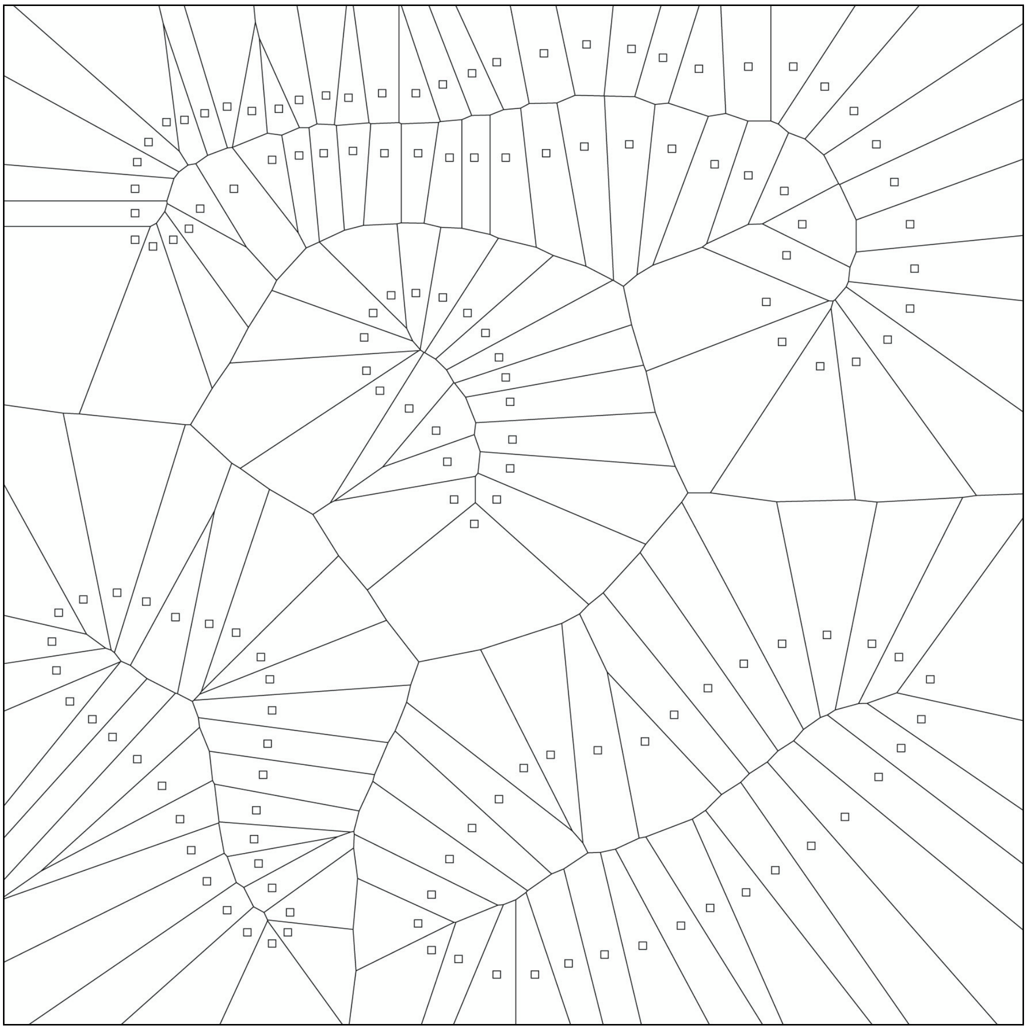


11 On the next page you will find a Voronoi-diagram of which the centers lie on the coast of four islands.
Parts of the Voronoi-diagram have actually become edges between domains around those islands.

a. Mark those edges with a color. A partition in four domains arises.

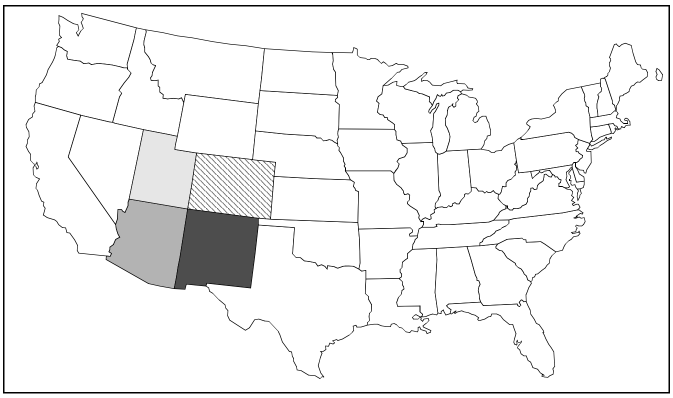
b. You could also talk about three-countries-points in the last partition. Now sketch a couple of largest empty circles, which just touch three of the islands. Where should you place their centers?

four islands



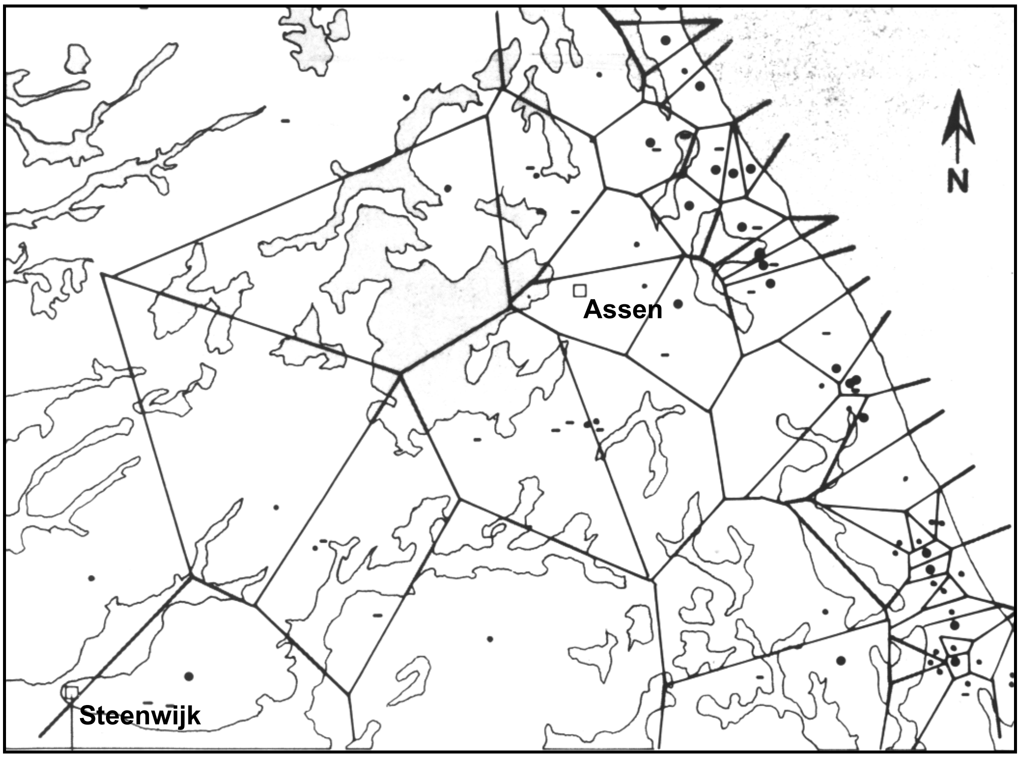
true four-countries-point

True four-countries-points between countries occur very rarely in real life. On the right one is shown, between de states Utah, Colorado, Arizona and New Mexico in the USA.
If you know another one, speak up!



12 Of course there is no Voronoi-diagram for this map of the United States.
Sketch a situation of centers and a Voronoi-diagram yourself, for which,
- only four-countries-points occur and no three-countries-points
- and for which there is a square cell
- and for which the edges lie in each other's extension, are parallels, or are perpendicular.

6: Chambered tombs in Drenthe



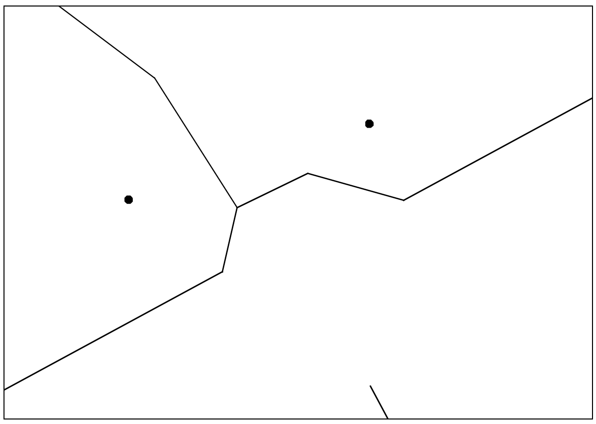
A partition of the eastern part of the Drents plateau in imaginary territories. Centers here represent groups of chambered tombs. Some chambered tombs were used as a storeroom for bones and skulls over a period of 600 years. The Voronoi-diagram shows a possible partitioning of the area. Archeologists often research whether such partitions correspond to the distribution of pottery in an area. This could provide indications about the social and economical structure in former days.

13 a. Observe: the cell in which Assen lies and the one north of it have centers which lie symmetrical in relation to the edge. Why that symmetry?

b. Do the centers lie symmetrical everywhere in relation to the borders? Is this necessarily so for a Voronoi-diagram?

c. The cell southwest of Assen contains several dots. Which dot is used for making the Voronoi-diagram?

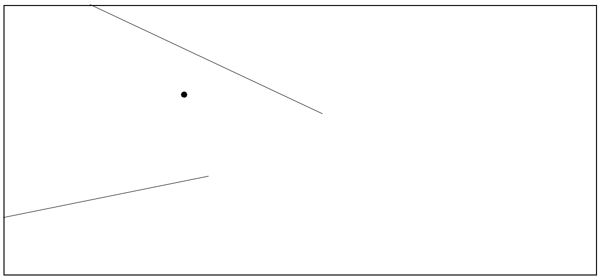
14 Reasoning with symmetry could also help to fill up an incomplete map of centers and edges. On the right an incomplete Voronoi-diagram is shown.
Complete the diagram.



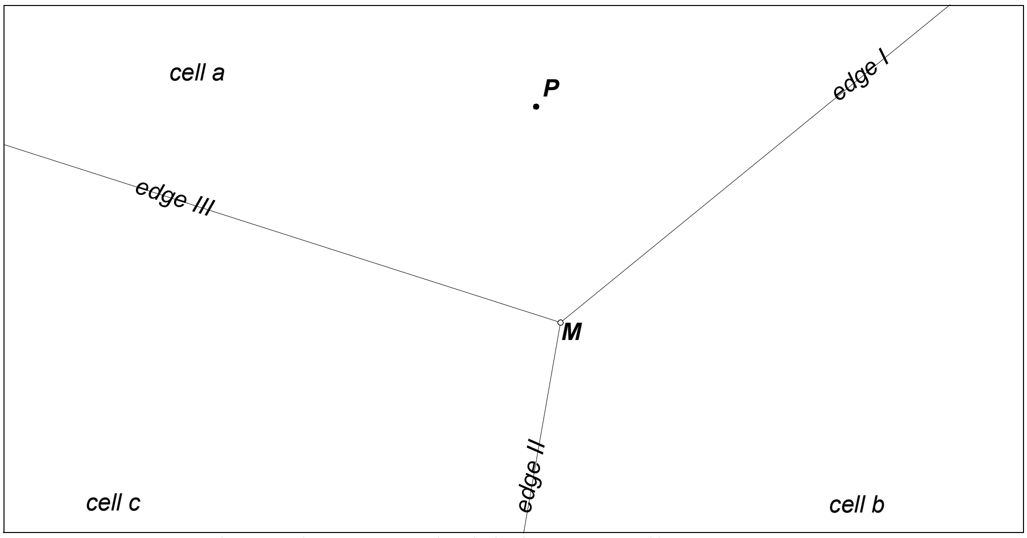
reflection

Voronoi-centers of adjacent cells are always each other's mirror image in relation to their Voronoi-edge. So you can recover missing centers by reflecting in an edge!
We will illustrate this technique with the following two examples.

15 This figure shows only one center. Since two edges are (partially) indicated, there have to be two other centers and a third edge. Finish the sketch accurately.



16 Below, the edges of a Voronoi-diagram with three centers are given.
In cell *a* lies point *P*.
Try to work as precise as possible in this exercise, or else you will get into trouble. You can do the reflection exactly using your protractor. See page 6.



a. *P is certainly not the center which belongs to cell a!*
You can verify this by reflecting *P* in edge I; name the reflection P_1.
Then reflect P_1 in edge II. Name the reflection P_2.
Finally, reflect P_2 in edge III. Name the reflection *Q*.
Why is it not possible that *P* is the center of cell *a*?

b. Sketch the middle of the line *PQ* and name it *R*. Now also reflect *R* successively in the three edges; in this way point *S* arises. What do you notice about this ultimate point *S*?

c. The found point *R* (or *S*) could be the center of cell *a*, but that is not necessarily true. Another option, for example, is a point that lies in the middle of *R* and the indicated three-countries-point *M*. Verify this by repeated reflection.

reconstruction-problem

The final result of exercise **16** is surprising. Using the method given above, you are apparently able to reconstruct possible centers, without knowing one of them.

Of course the question is: Why does this work so well? Several clues are given in *exploration exercise A*, on page 16, so you can get to the bottom of this.

Summary of chapter 1

This chapter was a provisional exploration of Voronoi-diagrams.
Furthermore several concepts were discussed.

nearest-neighbour-principle

A Voronoi-diagram arises when a number of points are given and the plane is partitioned so you can determine everywhere what the nearest point is. In this fashion a partition in subregions arises. This is called partitioning according to the *nearest-neighbour-principle*.

You will find definitions of the concepts *center, Voronoi-diagram, edge diagram, Voronoi-cell, cell* and *vertex* on page 9.

The Voronoi-cells can be infinitely large. These infinite cells belong to centers which lie close to the side; we shall have to specify this later.

three-countries-points

In general three cells meet in a vertex. Such vertices are called three-countries-points.
To have more than three cells converging in a vertex is possible, but rare.

largest empty circle

A circle in which no centers lie, but on which does lie a center, is called a largest empty circle. Such a circle cannot be enlarged from its center.

reflection, reconstructing problem

If only edges are given the centers can sometimes be recovered by reflection. For this, the fact that the centers lie *symmetrically* with respect to their edge is used.
To find the *centers for a given Voronoi-diagram* is called solving the reconstruction problem.

preview

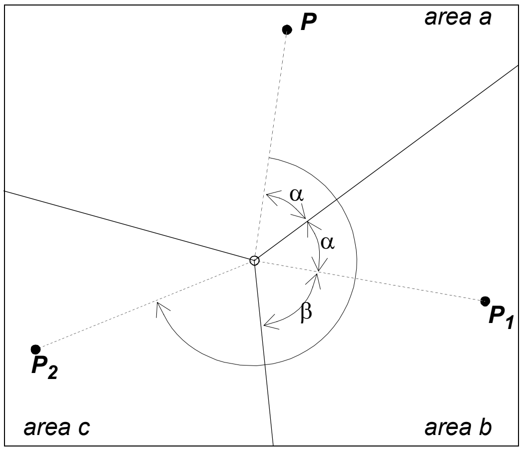
In the next chapters we will first go deeper into the mathematics, which until now we used incidentally. Doing that we will argue more independently and draw less conclusions from measuring figures only.
Nevertheless one of the results will be very practical, namely that we will find a fast way to determine whether a center D is inside of the circle through A, B, C or not, *without determining the actual circle itself*. This will make the construction of a Voronoi-diagram a lot easier.
There is a chapter in which you will learn more about Voronoi-diagrams using a computer program. For this you will need the knowledge in this chapter as well as the next.

Exploration exercise A: Recovering the centers

This exercise combines with exercise **16**.
First of all, make sure that you understand the method used to construct possible centers of a Voronoi-diagram of three cells with a three-countries-point.

Exercise ONE

*Find and describe a argumentation which proves that the method of exercise **16** always works.*



Several hints:

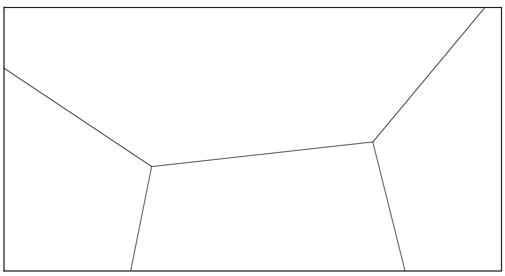
a. The figure on the right already shows P_1 and P_2. The next reflected point would be Q, but for now call this point P_3 and do three more reflections. You will discover something very special about P_6 if you have drawn carefully enough.
b. ***If*** you were sure that indeed P_6 is always the same as P is always true, ***then*** you can conclude that the middle R of PP_3 will end up on top of itself after *three* reflections. Find out why by reflecting the line segment PP_3 three times.
c. But *why* is P_6 equal to P? That's the main question now.
The figure shows several angles. You could also think about reflection as the rotation of the bar MP around the center of rotation M. Compare the intersection angle of MP to MP_2 with the angle of cell b.

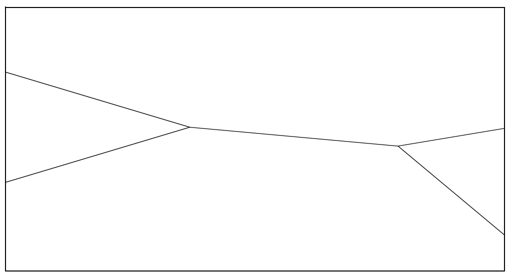
Exercise TWO

Once you found one possible position for the center of cell a, you also know all the other possibilities. Work this out.

Exercise THREE

Find a method to recover the four centers that are not given in the same type of Voronoi-diagrams as shown below.
Hint: Divide and rule.

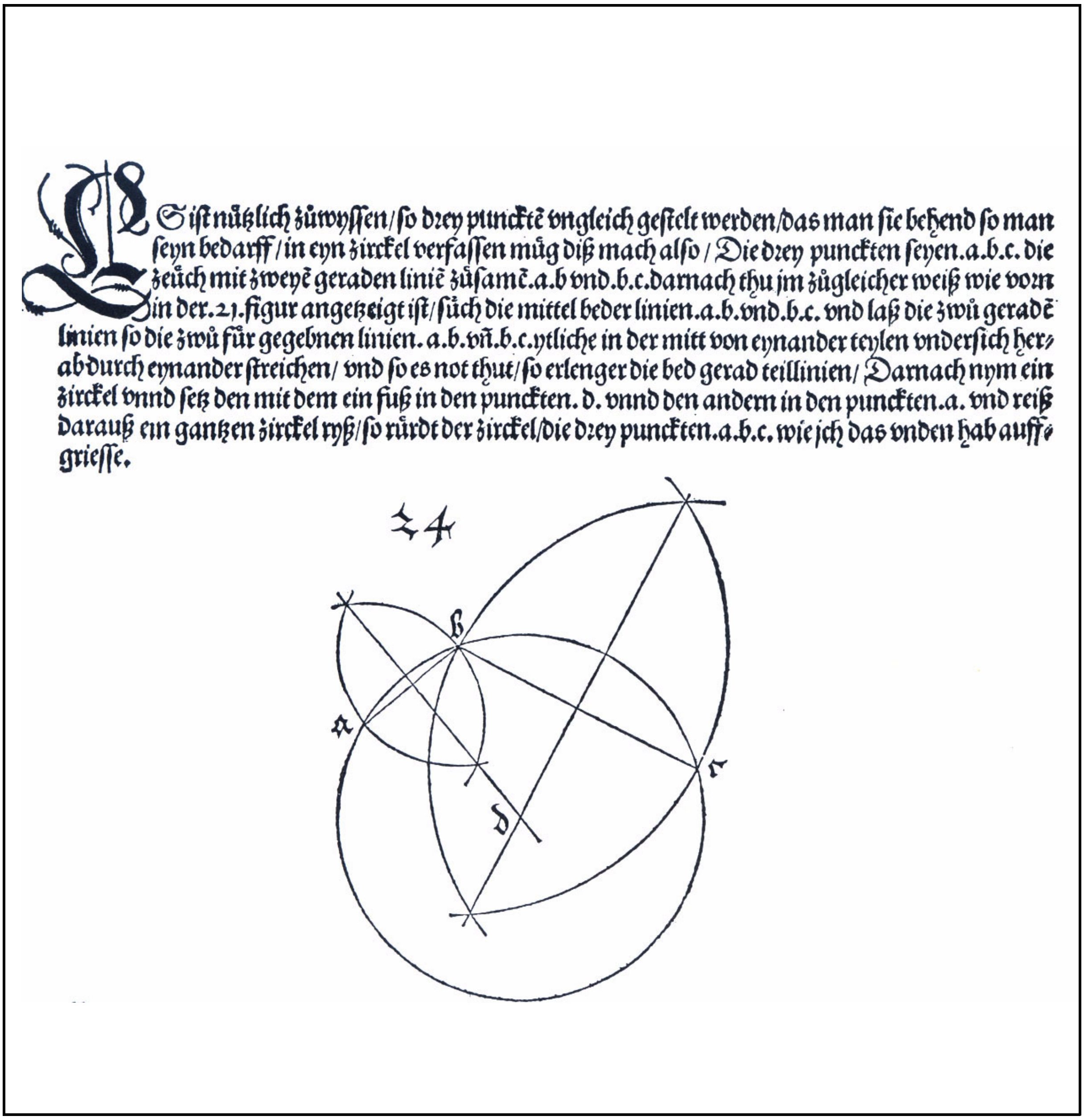




Complete your investigation as follows:

– Make a report of a half to at most one page in which you write down your argumentations for the exercises above.
– Add clear figures. Refer to the figures in your report.

Chapter 2
Reasoning with distances



LEs iſt nützlich zůwyſſen/ ſo drey punckte ungleich geſtelt werden/das man ſie behend ſo man ſeyn bedarff / in eyn zirckel verfaſſen mũg diß mach alſo / Die drey punckten ſeyen.a.b.c. die zeũch mit zweyẽ geraden liniẽ zůſamẽ.a.b vnd.b.c.darnach thu jm zůgleicher weiß wie vorn in der.21.figur angezeigt iſt/ ſůch die mittel beder linien.a.b.vnd.b.c. vnd laß die zwů geradẽ linien ſo die zwů für gegebnen linien. a.b.vñ.b.c.ytliche in der mitt von eynander teylen vnderſich her= ab durch eynander ſtreichen/ vnd ſo es not thut/ ſo erlenger die bed gerad teillinien/ Darnach nym ein zirckel vnnd ſetz den mit dem ein fuß in den punckten. d. vnnd den andern in den punckten.a. vnd reiß darauß ein gantzen zirckel ryß/ ſo rürdt der zirckel/die drey punckten.a.b.c. wie jch das vnden hab auff= griſſe.

In this chapter the reasoning will be a lot more exact than in the previous chapter.
In principle we deduct things from scarcely any given data. We will also think about how that kind of reasoning works and how you can write it down.

The illustration with the text in gothic letters on the front page becomes a reality:

UNTERWEYSUNG DER MESSUNG MIT DEM ZIRCKEL UND RICHTSCHEYT.

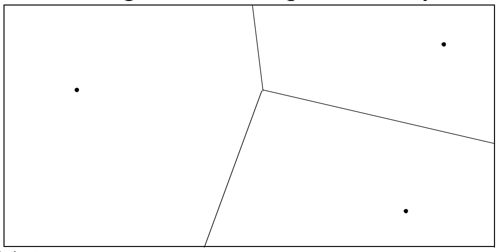
This is a geometry book, published in 1525 for painters and artists, by Albrecht Dürer. The figure is an illustration for what in this chapter will be *theorem 5*. That theorem says that for any triangle there is exactly one circle which passes through the three vertices of the triangle. Dürer uses in his figure the triangle *abc*; there it is indicated how the center of the circle and the circle itself can be determined by ruler and compass only.

7: Introduction: reasoning in geometry

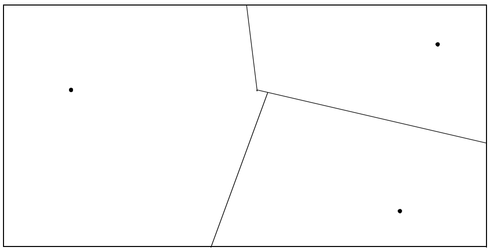
In the last chapter we investigated several things concerning Voronoi-diagrams. We used what was seen in figures and sketches. However, we might have assumed things of which we do not know for sure if they are true.

problem A:

Why does the Voronoi-diagram of three points always looks like this:



and never like this?



problem B

The Voronoi-edge of two points always appears to be a straight line. The folding technique backs up this idea. But why are folding lines always straight? We *have seen* this in many cases. We do not really know why.

In this chapter we will look further into those two questions.

reasoning instead of looking

Now we want to get certainty about these questions by *reasoning,* and not by *looking* at some sketched figures.
This is why this chapter will have a more theoretical character, especially at the end, because at the beginning we will discuss concrete diagrams, but in the latter part *theorems* and *proofs* are discussed.
You might feel as if you are walking on egg shells. This is true, but you will get used to it; moreover, you will start complaining if something is stated without a proof.

1 **a.** In the section '*The edges between two domains*' (page 6) more non-founded properties of the edge between two areas are stated. Which for example?
b. And how do we use those in the section '*Chambered tombs in Drenthe*' (*page 13*)?

8: An argumentation with three-countries-points and circles

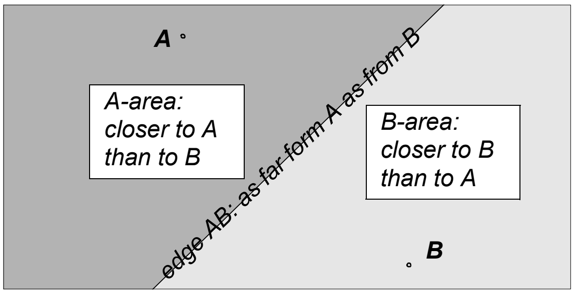
First we tackle problem A of the previous page:

Why do the edges of a Voronoi-diagram with three centers intersect in one point?

You need to look at such questions with a critical eye. Hence:

2 **a.** Sketch a few three-centers situations, which don't even have three edges. Thumb through the examples of the previous chapter, if need be, to get some ideas.
b. What characterizes these situations?

In the remainder of this section we will not reckon with this special case. In the summary we will include the warning!

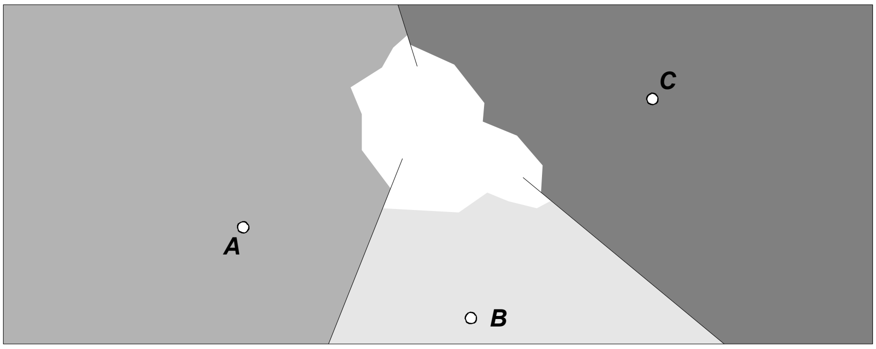


In the previous chapter the Voronoi-edge played the leading role. The figure shows its characteristics
Just the points on the edge have equal distance to A and B. Phrased differently:

property Voronoi-edge

Only to the points *P* on the Voronoi-edge between *A* and *B* applies: ***(distance from P to A) = (distance from P to B)***

We will use this clear form while reasoning with three edges. Below you see a figure in which it is not known if the three edges converge into one point.



3 This exercise will help you find an argumentation for: *the three edges converge in one point.*
a. Indicate the point of intersection of the Voronoi-edge AB and the Voronoi-edge BC and call it M.
b. Write down, using the property of Voronoi-edges, the two accompanying equalities and derive the third equality. Write that one down also.
c. What does that equality say about the point M? (Remember the property once more.)
d. Did you reach the goal of the argumentation?

4 The circle, which has M as center and passes through A, also passes through B and C.

a. How do you know that for certain?
b. In the previous chapter this circle played an important role. What role was that?

critical remarks

While working on exercises **3** and **4** we solved the problem on page 19, and more, so it seems.

5 Try to answer the following questions:
a. Whereupon is the assumption based that such an intersection point *M* exists ?
b. Don't we also (maybe carefully hidden, but nevertheless) use the fact that the edges are straight lines?

This is not as easy as it looks!
However, the argumentation of exercise **3** is beautiful, and we will hold on to it.
From here on, though, we choose to look for more certainty.

We will follow this strategy:
a. Show unimpeachably that the Voronoi-edge of two centers is a straight line.
b. Show that under the condition that the three centers are not on one line, both edges do intersect.

In this fashion we will look for a solid foundation in our argumentation. That has to be found in the properties of the concept of distance, because that is where it all began.

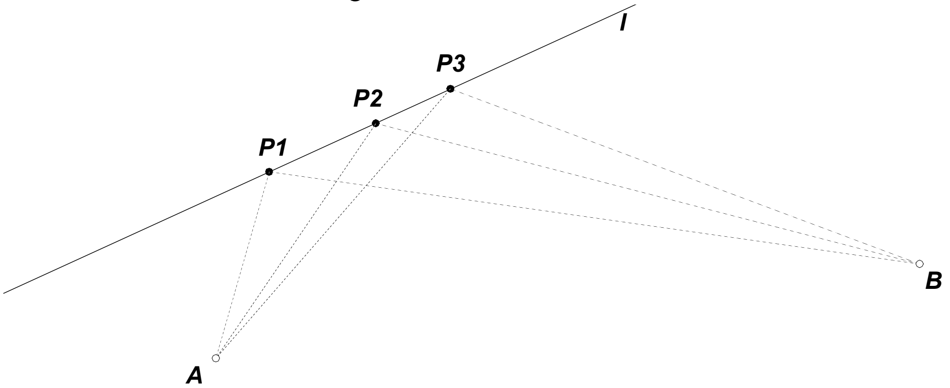
In the following paragraph we will work our way towards this solid foundation, starting with a possibly unexpected problem.

9: Shortest paths and triangle inequality

shortest-path-principle

The shortest path from point A to point B is the straight line, which connects A and B.

6 But what is the shortest path from A to B if in the mean time we also need to go *via line l* like below? We will get to the bottom of this now.



a. Measure which of the three paths from A to B is the shortest.

b. We don't know whether there might be an even shorter path! Here is a pretty trick:

Reflect A in line l, name the reflection A'.
Also connect A' to the points P.

Why does the following now apply:

from A to B via P_1 is as long as *from A' to B via* P_1*?*

c. Now determine, using point A', point Q on l, such that the path via Q leads to the shortest path.

d. Think of a situation where it is of importance to find a shortest path of this kind.

There is much more to discover on finding shortest paths in more complex situations. We will do so in chapter 6.
Now we only establish that the *shortest-path-principle* is the basis of the solution. We will rephrase this principle more precisely.
Since we are talk about *distance* all the time, we first introduce a notation for the distance between two points.

distance notation

From here on we will denote the distance between two points A and B as $d(A, B)$.
Because we are thinking in terms of *comparing* distances, it does not matter whether you think of centimeters on paper or of kilometers in the landscape. $d(A, B)$ is always a non-negative number and you can use it in expressions such as equalities and inequalities. Also expressions like $d(A, B) + d(C, D)$ have meaning.
The d originates from the word *distance*.

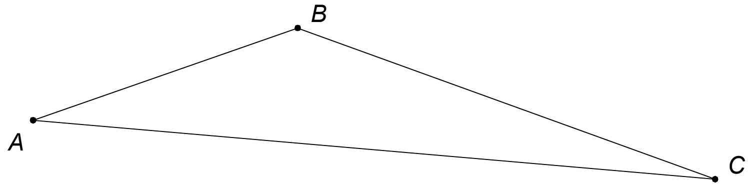
simple properties

7 **a.** Translate in common English what is asserted here:

for points P and Q always holds: $d(Q, P) = d(P, Q)$

b. What can you say about points A and B if $d(A, B) = 0$?

The figure shows three points and their connections:



Next we will describe, using the *d*-notation, that going from A to C via B is a detour when B is not on the line segment AC. This has a name: the triangle inequality.

Triangle inequality

Triangle Inequality

For each set of three points *A*, *B* and *C* holds: *d(A,C)* $\leq$ *d*(*A, B*) + *d*(*B, C*).
The equality occurs only if *B* is on the line segment between *A* and *C* .
In every other case there is a true inequality.

The name *triangle inequality* originates from the fact that the *inequality* holds if A, B and C form a *triangle*.

8 In the figure on the previous page, expanded with *A'* and *Q*, you can apply the triangle inequality to show that *A-Q-B* is the shortest path.
a. To which triangle would you apply it?
b. *Q* is on *A'B*; this is drawn. What can you say about the triangle inequality in this situation?

9 On closer inspection the *triangle inequality* is somewhat more modest than the *shortest-path-principle*.
a. Sketch a situation in which two paths are compared, and where the shortest-path-principle has some meaning, but the triangle equation has not.
b. The argumentation of exercise **6** runs impeccably and yet only the triangle inequality is used? What causes that?

Of course we choose the simplest starting-point.

starting-point

We therefore assume the triangle inequality as an established fact.
From now on, you can refer to it in your argumentations.
Thus our reasonings about distances are no longer up in the air; you now know what you can build on.

10 You could legitimately ask yourself: What is the triangle inequality itself based on? If you asked yourself that question, the following exercise is for you, otherwise not.
a. Good question, difficult answer. Try to think of something yourself on which you would base the triangle inequality.
b. In that case, what would be the next question?

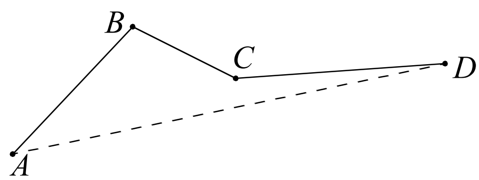
A different notation

Often the notation $|AB|$ is used for the length of a line segment with end points A and B. In this chapter, where everything breathes the air of distances, we use *d(A, B)*.
When we encounter figures with line segments and their lengths, you will also see $|AB|$.

extra exercise

11 Show that for every set of four points A, B, C and D holds:

$$|AD| \leq |AB| + |BC| + |CD| .$$



10: The concept of distance, Pythagorean Theorem

Another very important property of the concept of distance can be expressed as the well-known Pythagorean Theorem for right-angled triangles.

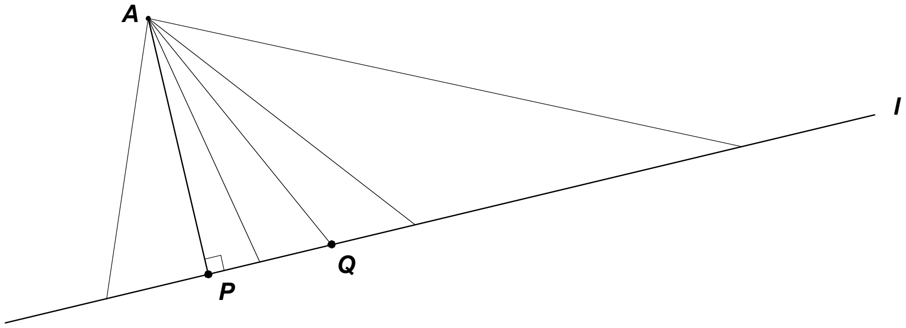
12 Phrase that theorem using the *d*-notation of the previous section. Your phrasing should deal with a triangle *ABC*, of which one angle is right.

We will now use this theorem in order to determine the shortest distance from a point to a line and also to ensure the correctness of the method.

shortest distance to a line

In this figure *A* is a point outside the line *l*. You are probably convinced of the following:

Of all possible connection line segments the line segment, which is perpendicular to l is the shortest.



13 a. Write down– in *d*-notation – what holds for triangle *APQ* according to the Pythagorean Theorem.

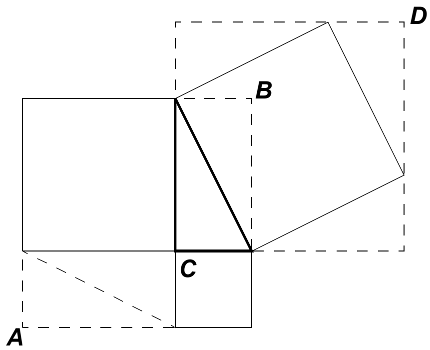
b. How does *d(A, P) < d(A, Q)* result from that?

The Pythagorean Theorem is also one of the fundamentals you can use. You could also prove the Pythagorean Theorem based on more elementary things, but we will also not do this exhaustively. One possibility is outlined below as an 'extra'.

extra, Multatuli

This is the beginning of nr. 116 from the *Ideas* of Multatuli.

I recently found a new proof for the Pythagorean theorem. Here it is. By, as shown in the adjacent figure, constructing six triangles – each equal to the given right-angled triangle – one acquires two equal squares, AB and CD. If one subtracts four triangles of each of these figures, one proves the equality of the remainder on both sides, which was to be shown.



It cannot be done any simpler, or so it seems to me. After finding this proof, I heard of the existence of an article, which discusses this topic. I purchased this little book, but it did not contain my demonstration. Furthermore I deem that none of the therein assimilated proofs is as illustrative and clear as mine.

Up to here the proud writer of the *Max Havelaar.*

The proof of Multatuli leans heavily on the concept area. We did not exactly establish what its properties are. Moreover, it is rather easily assumed that certain parts of the figure are squares. But ok, for now we will join Multatuli.

14 a. Put some more letters in the figure and write down an argumentation which eventually leads to the equality part of the Pythagorean Theorem, expressed in areas of certain squares.

b. What would you have to show in order to conclude that the oblique 'square' is in fact a square.

extra; alternative for exercise 13 a/b.

15 Also without the Pythagorean Theorem you are able to prove that the perpendicular line from A to l provides the shortest distance. Use the following hint and your own inventiveness.

Hint: How do you get from A to A the fastest if you have to go via line l?

The illustration shown below comes from a medieval manuscript.
It was made in the monastery of Mont Saint Michel in Bretagne, when Robert de Torigni was the abbot, during the years 1154 through 1186.
The manuscript contains figures and texts about astronomy; the abacus, bells, and of course mathematics are used in each of those. Presumably a lot is copied from Arabic manuscripts; in the Arabic world of those days a lot more attention was paid to mathematics and science than in Christian Europe.
This picture is of an application of the Pythagorean theorem in archery. You can see the arch, and the word 'sagitta' (arrow) is written at the hypotenuse and the base. You can also see close to the sides: 'filum V pedii', 'filum IIII pedii' and 'altitudo III pedii'. Translated: threads of 5 and 4 foot, a height of 3 foot. It is the well-known 3-4-5 triangle. Would the medieval monk who made the illustration have understood that it was about the meaning of the picture and not about reading the numbers? The proportions of the sides do not match those in the picture!

11: Properties of the perpendicular bisector

Next we will use the triangle inequality to prove that the
Voronoi-edge of the centers A and B
is equal to the
perpendicular bisector of A and B.

important

This section is definitely the hardest of this chapter. Even if you do not catch on to all the details, you will be able to continue with the next section, but make a good effort to try to follow the reasoning. The more argumentations like this you can follow, the easier it will get later on, simply because you have had some training.

First a *definition,* which should be familiar. With definitions in mathematics we establish exactly what is meant.

definition Voronoi-edge

The Voronoi-edge between two points *A* and *B* is the set of points *P* for which hold: *d(P, A) = d(P, B).*

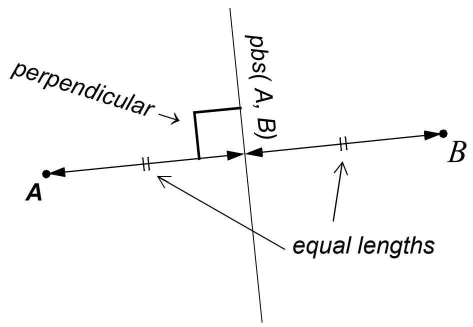
In chapter I you got the big impression that Voronoi-edge of *A* and *B* is exactly the perpendicular bisector of the line segment *AB*. We will now *prove* this. Since we want to start only from definitions and familiar things, we also have to define 'perpendicular bisector'.

definition perpendicular bisector

The perpendicular bisector of line segment *AB* is the line which is passing through the midpoint of *AB* and is perpendicular to *AB.*

pbs(A, B)

We will agree upon a notation for the 'perpendicular bisector of line segment *AB*': *pbs(A, B)*.
The figure displays two characteristics of the perpendicular bisector:



– it is *perpendicular* to the line segment
– it *divides* the line segment *in two.*

What we would love to pose:

statement: equality

The Voronoi-edge of two points *A* and *B* and the perpendicular bisector of the line segment *AB* coincide.

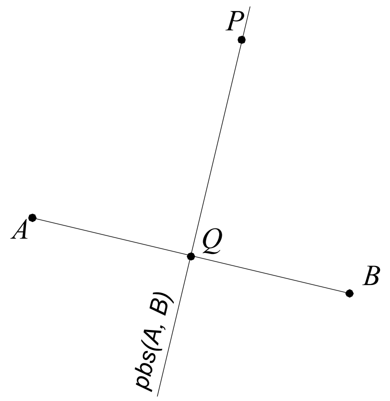
This means quite a lot: not only that all of the points of the perpendicular bisector lie on the Voronoi-edge, but also that the Voronoi-edge does not consist of more points. And vice versa.

That is why two things need to be proven separately:

a. Every point which lies on the perpendicular bisector, is also on the Voronoi-edge.
b. Every point which does *not* lie on the perpendicular bisector, is also *not* on the Voronoi-edge.

We discuss both parts separately.

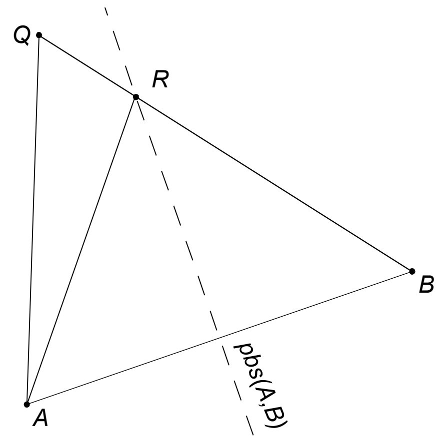
16 Proof of part a:
Every point, which lies on the perpendicular bisector is also on the Voronoi-edge
The figure shows line segment AB and also the *pbs(A, B)*. Q is the middle of AB. P is a point that lies on the perpendicular bisector.



a. Indicate in the figure, in green, the *two* things, which you can use now according to the definition of *pbs(A, B)*.
b. Color the line segments of which you have to prove that they have the same length red.
c. Write down what Pythagoras says about $d(P, A)$ and $d(P, B)$ and derive from that:
$$d(P, A) = d(P, B).$$
d. Did you use both of the characteristics of the perpendicular bisector? Where in the argumentation?

17 We are halfway there, but we still have to do the proof of b:
Every point, which does not lie on the perpendicular bisector, is also not on the Voronoi-edge.

The situation is represented adjacent. Q is not ón *pbs(A, B)*, but on the side of A. BQ will then certainly intersect with the perpendicular bisector, call the point of intersection R. R is certainly not on line segment AQ. This is what you know and what you can use in your proof.



a. Write down in d-notation what you need to prove.
b. Since we have already proven part a, you do know something about point R. Note that as an equality.
c. Also formulate an inequality, which contains Q, R and A.
d. Combine these to obtain the wanted conclusion.

18 Doesn't the figure of the triangle inequality remind you of a problem we encountered earlier?

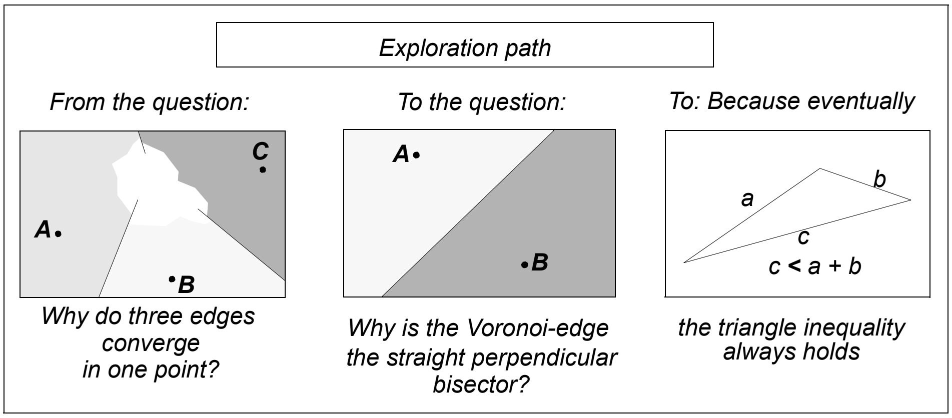
As a matter of fact, we also need to research the possibility that $d(A, Q) > d(B, Q)$. Of course Q is on the other side of B, and this boils down the whole thing to consistently switching the letters A and B. This is no longer interesting.

12: From exploration to logical structure

Introduction to this section

research

In the preceding we explored why the three Voronoi-edges of three points (in general) meet in one point.
We started off with the problem of partitioning an area and then found out that a fundamental property of the concept of distance was of importance.
The exploration developed as follows:

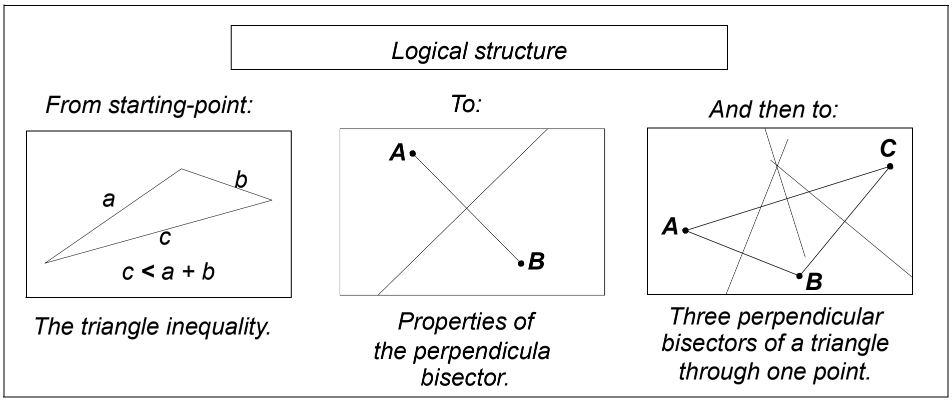


Up to here the exploration phase.

logical structure

Since we are reasoning, the logical structure will be the other way around when we look at it afterwards: first the triangle inequality, and deduct from there that the *Voronoi-edge* and the *perpendicular bisector* coincide and then finally derive from there the statement about the concurrency of the three edges.

In this section we will repeat the whole in that last form. We will abandon the terminology of Voronoi-diagram; we will now make our proof mathematically pure.
In survey like this:



We will formulate the most important statements, which will be proven concisely as theorems. We wil number the theorems, as this will make future reference easier. That is exactly what happens in a logical structure: what is proven before, you can use later.

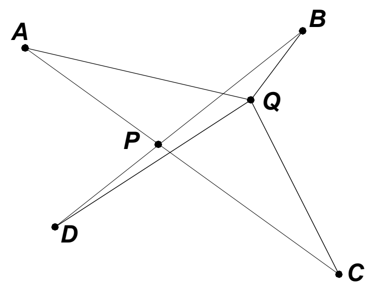
Starting-points: triangle inequality and Pythagoras

The triangle inequality can be proven from other, more primitive starting-points. However, we will not do this. It will be our first theorem.

Theorem 1 **(Triangle inequality)**
For each set of three points *A*, *B* and *C* holds: $d(A,C) \leq d(A, B) + d(B, C)$.
The equal sign occurs only if *B* lies on the line segment *AC*.
In all other cases a real inequality occurs.

Next an exercise in practicing the use of the triangle inequality.

Four points are given: A, B, C and D. P is the point of intersection of the line segments AC and BD. Q is a different point than P.
You need to show that
the four distances from Q to A, B, C and D together are bigger than
the four distances from P to A, B, C and D together.



19 a. Write down the to be proven statement using the d-notation as follows:
To show: d(P, A) + ... ≤ ....
b. Then start the proof with:
Proof: ...
and use (one or more times) the triangle inequality.

For the sake of completeness the Pythagorean theorem is stated below. You saw a proof of it when you did the extra exercises on page 25.

Theorem 2 **(Pythagoras)**
If in a triangle *ABC* angle *B* is right, then this equality holds:
$d(A, C)^2 = d(A, B)^2 + d(B, C)^2$.

The perpendicular bisector

We gave a definition of a perpendicular bisector. We will copy it here.

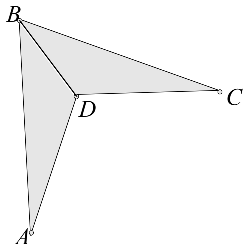
definition perpendicular bisector

The <u>perpendicular bisector</u> of line segment *AB* is the line which passes through the midpoint of *AB* and is perpendicular to *AB*.

The properties of the perpendicular bisector are mentioned in this theorem:

Theorem 3 **The perpendicular bisector of line segment *AB* is the set of points *P* for which hold $d(P, A) = d(P, B)$.**
For points *P* outside of the perpendicular bisector holds:
If $d(P, A) < d(P, B)$, then *P* lies on the *A*-side of *pbs(A, B)*.
If $d(P, A) > d(P, B)$, then *P* lies on the *B*-side of *pbs(A, B)*.

You can apply the theorem in the following problem. Think about this: if two points of a Voronoi-edge are known, you know the entire edge.



20 In this delta wing AB and BC have equal length and also AD and DC are of the same length. Show that BD is perpendicular to AC.

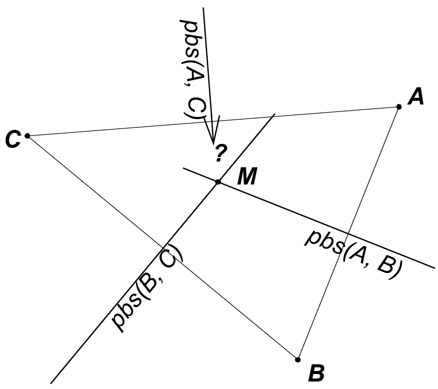
Perpendicular bisectors in the triangle

Theorem 4 **In each triangle *ABC* the perpendicular bisectors of the sides meet in one point.**

21 a. In the section '*An argumentation with three-countries-points and circles*' there was a problem: the three centers were not allowed to lie on one line. How did we get around that here? If the three points do lie on one line, what would happen to the three perpendicular bisectors? Sketch a (complete) figure.

b. How is the three points on one line situation excluded by the wording of the theorem?

Adjacent you see a figure, which illustrates the theorem. M is the point of intersection of the perpendicular bisectors AB and BC. At the bottom you see a scheme, which represents the proof.



22 In fact, this is the proof as it was given in exercise **3**, page 20.

a. Point out exactly how the different parts of the exercise correspond to the ones of the scheme.

b. Step 1 and 1-bis differ from the conclusion step. In which way?

step 1
M on $pbs(A, B)$ Thus $d(M, A) = d(M, B)$

step 1bis
M on $pbs(B, C)$ Thus $d(M, B) = d(M, C)$

↓ ↓

connection step
Thus $d(M, A) = d(M, C)$

↓

conclusion step
$d(M, A) = d(M, C)$ Thus M on $pbs(A, C)$

The circumscribed circle

First the definition.

circumcircle

The circumcircle of a triangle ABC is the triangle's circumscribed circle, i.e. the circle that passes through each of the triangles' three vertices A, B and C.

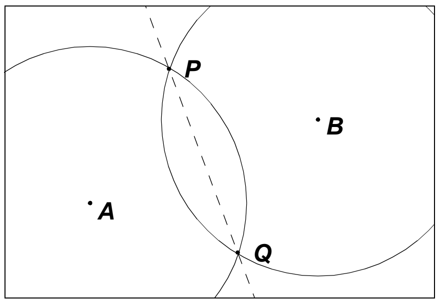
The theorem about the circumcircle is easy to formulate.

Theorem 5 **Each triangle *ABC* has one and only one circumcircle. The center of the circumcircle, i.e. the circumcenter, is the intersection of perpendicular bisectors of the triangle.**

The proof is simple: the point *M*, where the three perpendicular bisectors intersect, has equal distance to each of the three vertices and is the only point with that property.

We will now do some exercises with circles and perpendicular bisectors.

23 This figure shows two circles with equal radii. Their centers are *A* and *B*. Based on which theorem do the intersections *P* and *Q* lie on the perpendicular bisector of line segment *AB*?

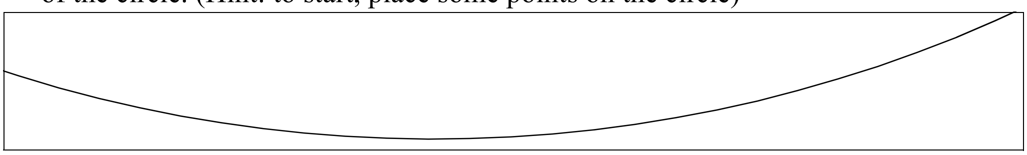


This is a recipe to construct the perpendicular bisector with compass and ruler. Do not use a protractor or the numbers on the ruler.

24 Examine how Dürer used this technique for the front page of this chapter, when finding the circumcircle about the three points *a*, *b* and *c*.

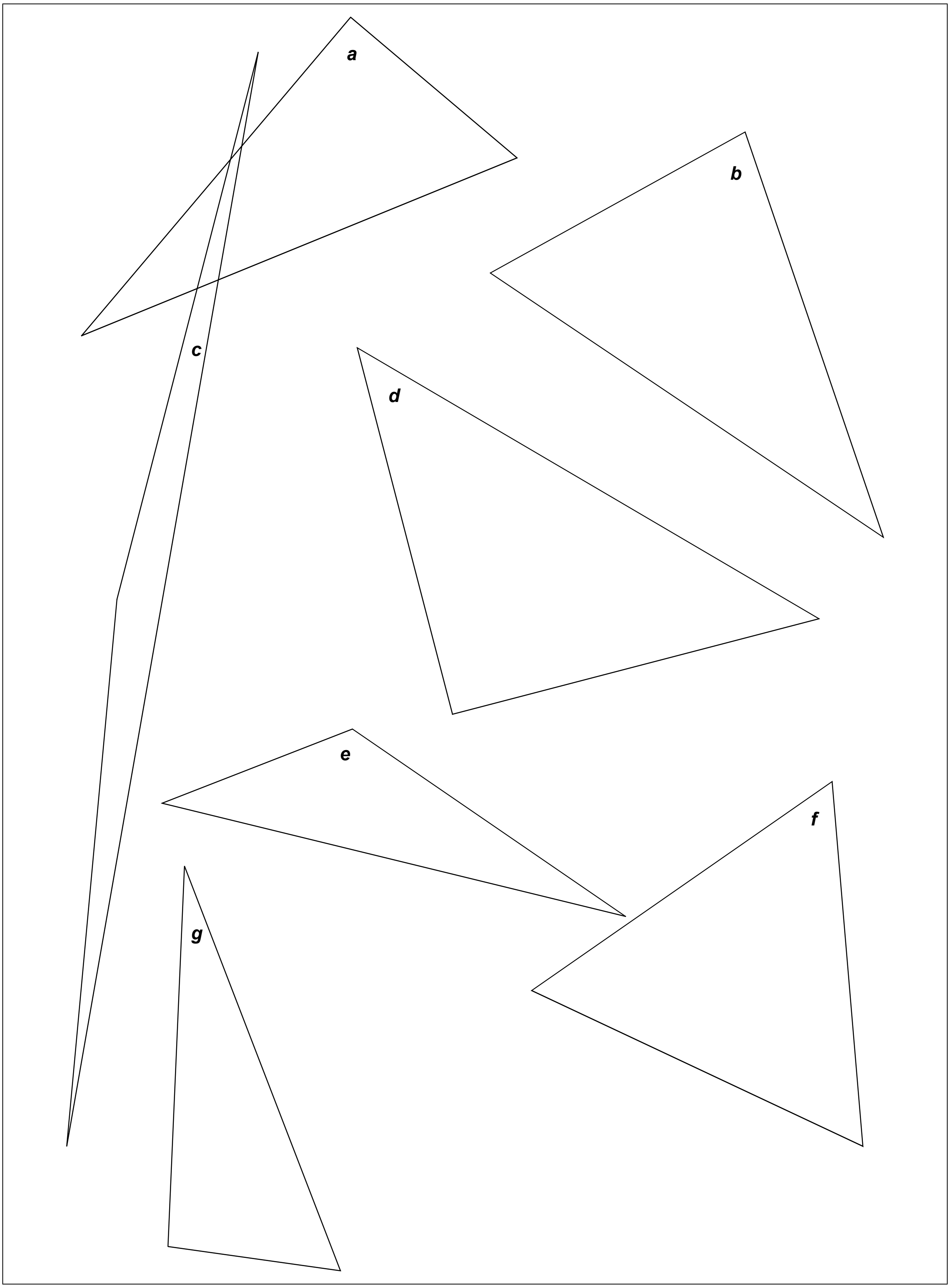
25 On the next page you will see some triangles.
Pick out a few and determine the intersection of the perpendicular bisectors and sketch the circumcircle. In at least one case, use the construction with compass and ruler.
Choose these in such a manner that one center lies *inside*, one center lies *on* and one center lies *outside* the involved triangle. How does this *inside-on-outside* link to the shape of the triangle?

26 Here a part of a circle is given. Determine, using only compass and ruler, the center of the circle. (Hint: to start, place some points on the circle)



Summary of chapter 2

This section as a whole is actually the summary of this chapter!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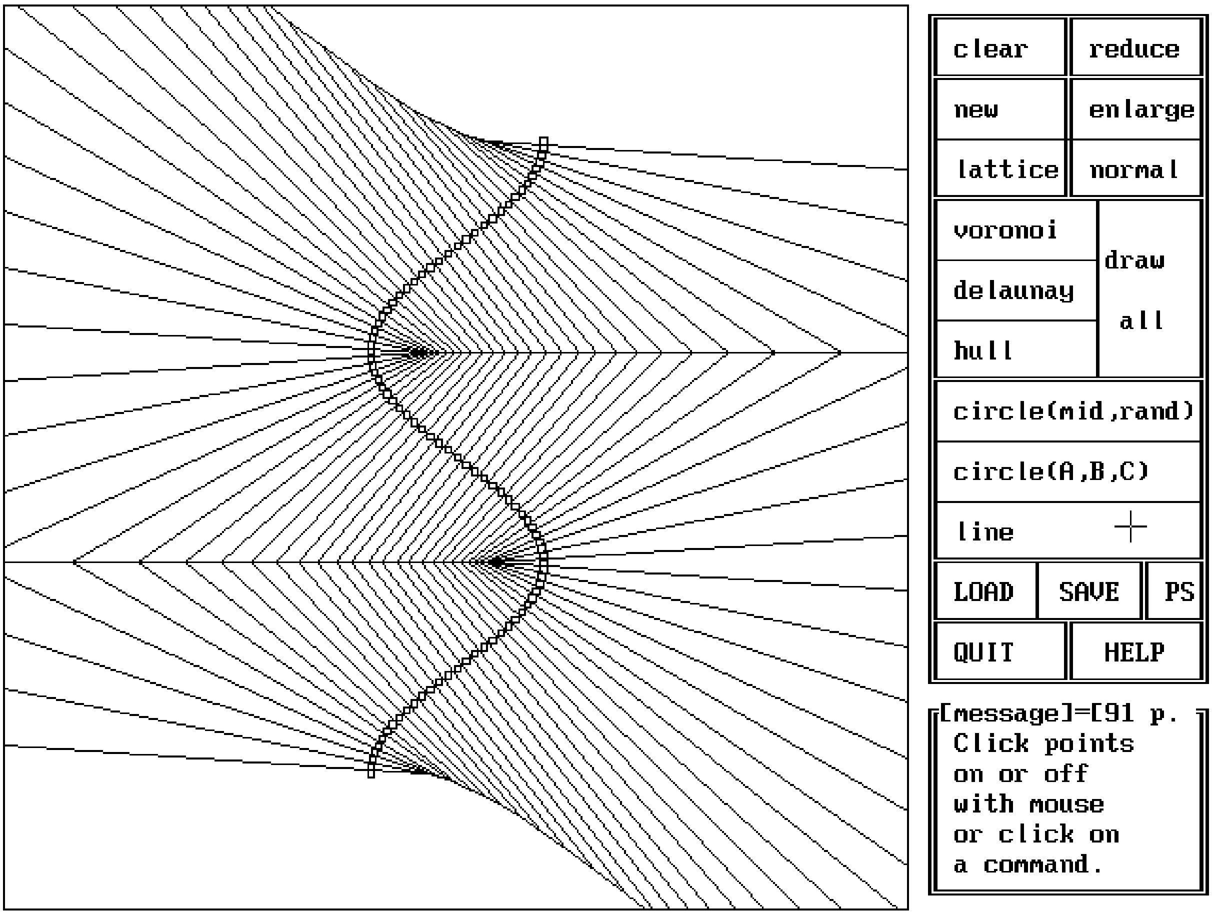
Chapter 3
Computer practical Voronoi-diagrams



Introduction, software availibility
In this chapter we will construct Voronoi-diagrams using the computer. The chapter is of a practical nature: you will try a lot, but prove little to nothing.
Since we will be working with more than 5 or 6 centers rather fast and easily, we can also look at other properties than the ones in chapter 2.
The closing exercises are a preview of what comes later, when we will no longer work with pointlike centers, but with areas as centers.

Voronoi-program
The illustration on the first page of this chapter shows the program used in this chapter.

You can download it for free from
`http://www.fi.uu.nl/wisweb/welcome_en.html`
Look for ‘Voronoi’ in the list.
Download, unzip to a folder of your choice by clicking on *voronoiengels.exe*.
Start the program in your folder by clicking ‘voronoi’.
It is an old program, but it will still work under most Windows-versions.

VoroGlide
If the indicated program is not available or does not work correctly, you can try *VoroGlide* on the internet.
It is a Java-applet, with which most (but not all) things in this chapter can be tried.
It is available at:
`http://wwwpi6.fernuni-hagen.de/GeomLab/VoroGlide/index.html.en`

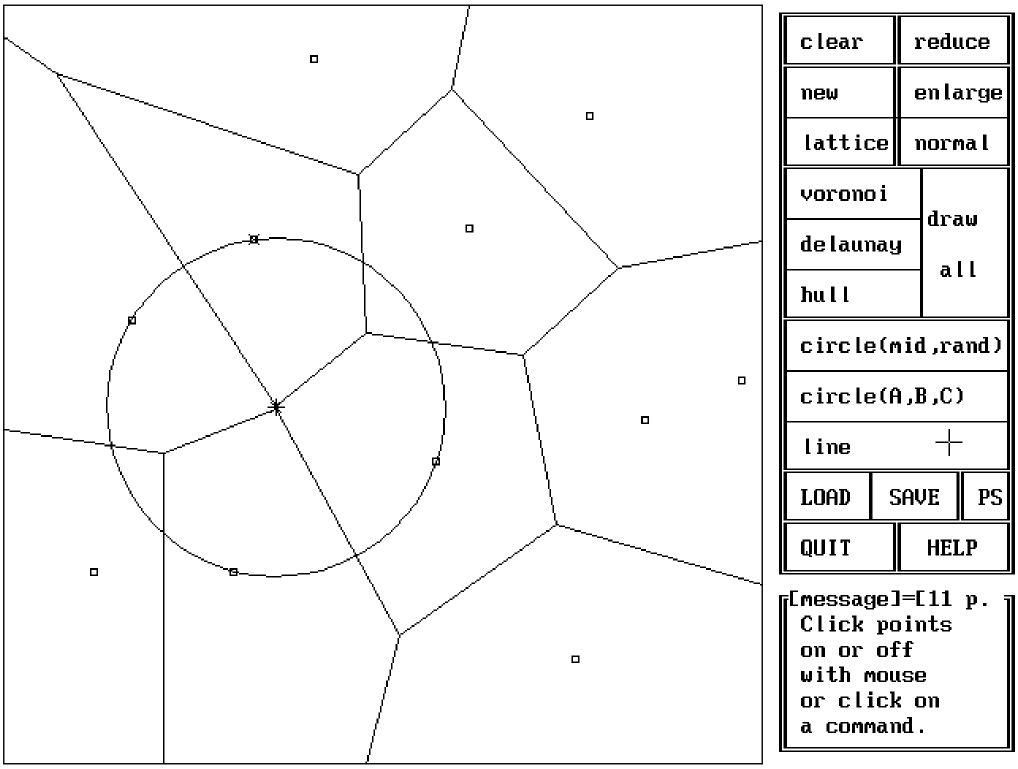
Just start VoroGlide, by clicking on the screen you create vertices, which can be dragged.
There are no ready made figures to load with VoroGlide, there are no tools to draw circles and lines.

13: Introduction

You know that constructing a Voronoi-diagram is based on drawing perpendicular bisectors, but finding the right line segments can be very time-consuming if there are a lot of centers. In applications of Voronoi-diagram we usually have situations with a lot of centers. It is obvious that there is a need for computer programs that can do the time-intensive sketching. We will be working with such a program now.

It depends on the local situation how the program is started. Follow the directions of the experts on the spot. See also the introduction on the opposite page

operation Once the program is running, the screen will look like this:



You operate the program mainly with your mouse; you only need to click with the left button. In the left part of the screen you click points on or off. In the right part you click on the task for the program. That is it! For some assignments you will be given directions on how to use the `report`-part of the screen.

1 a. Make the screen shown above. Indicate some points and then click on the button `voronoi`.

b. Let's test whether the diagram was drawn correctly by sketching some largest empty circles. There are two ways to do this:

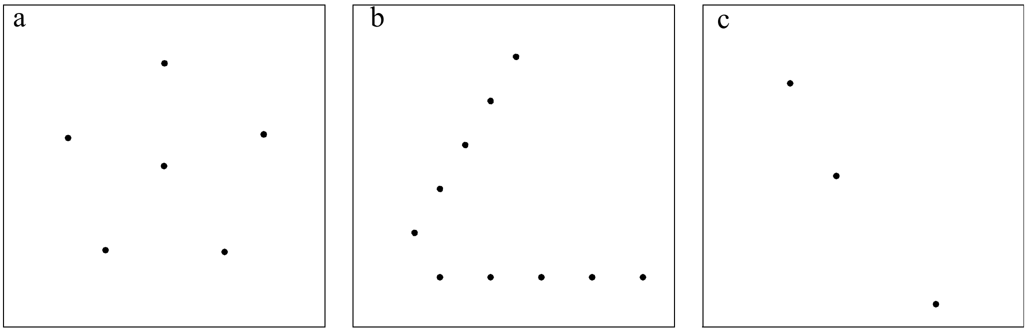
- `circle(mid,rand)`. Use this option to sketch a circle with given center. Let the center be a vertex of the Voronoi-diagram and let the edge-point be a suitable center. Choose in such a fashion that you will get a largest empty circle.
- `circle(A,B,C)`. Use this option for sketching a circle through three given points. The program will construct the front page of chapter 2 in a second

for you.
Choose the points *A, B* and *C* in such a way that you get a largest empty circle. Now the center of the circle is also indicated. Is that a Voronoi-vertex? Does it have to be?

c. The buttons `clean` and `new` don't need any explanation!

Now you know enough to get busy with certain problems.

2 **a.** Let the computer sketch the Voronoi-diagrams according to the following positioning of the centers.

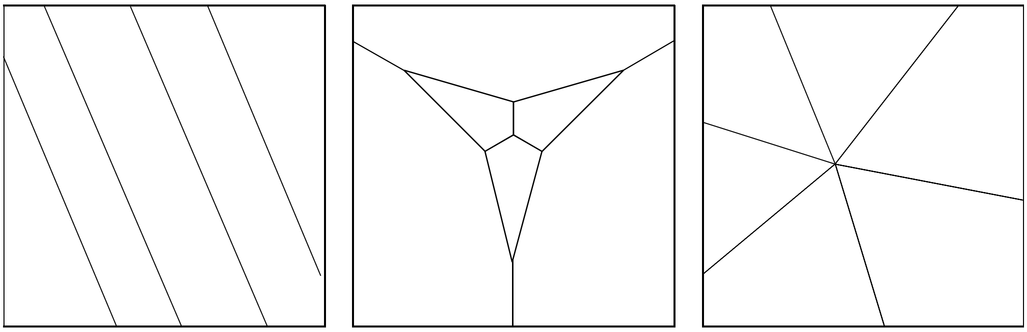


b. If you construct situation b very precisely, a number of edges will be straight behind one another. They form a line. What is that line actually?

c. In situation c you cannot see whether there is a three-country-point or not. Use the option `reduce` to get a good overview of the whole diagram. If necessary, repeat it a number of times. You could also use the option `line` first to get a refined placing of the points.

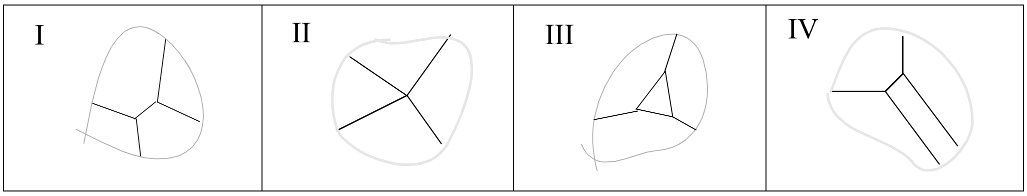
PS. Use `normal` to return, or choose `new` directly.

3 Construct by clever choosing the centers of the Voronoi-diagrams in the figures shown below. Remember that you can also sketch circles and lines first to find out where the centers must be placed. Then indicate in the figures where the centers need to be approximately.

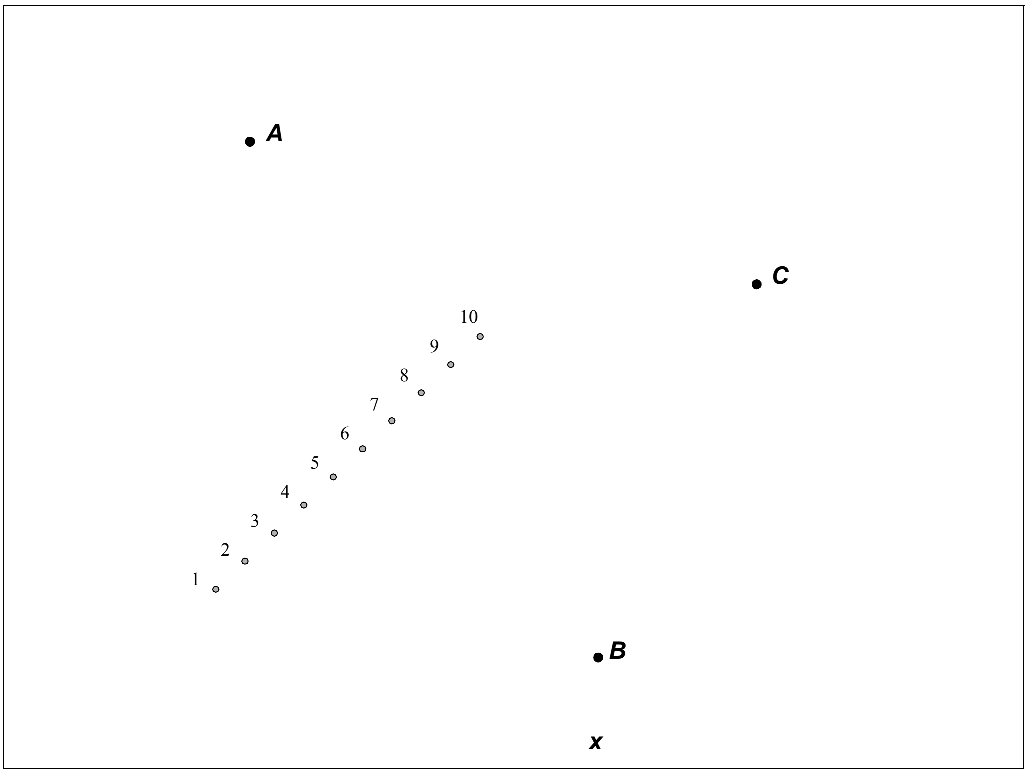


14: The influence of the fourth point

There are only four clearly distinctive diagrams possible for four points, namely these four:



4 Choose approximately three points like the black points *A*, *B* and *C* shown in the figure below. In this exercise we will add a fourth point (*D*) every time and see what the



effect of that point is. (The points 1 through 10 don't play a part until the next exercise.)

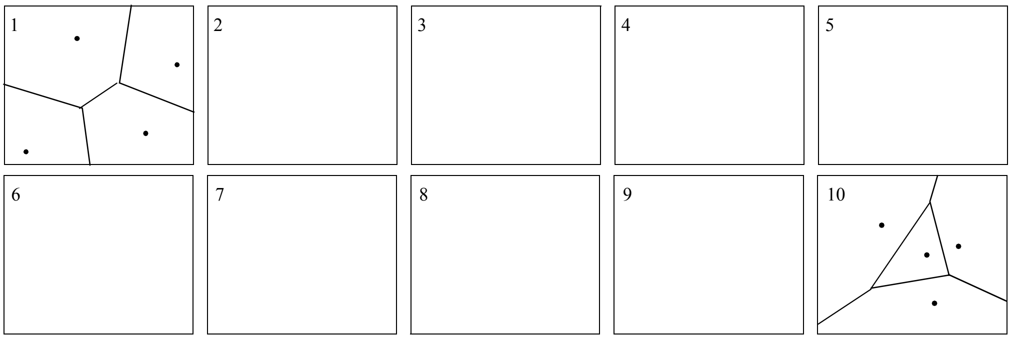
a. Indicate a fourth point in such a way that you get a type III diagram. If you move the point a little, the diagram will remain a type III.

b. Show on the screen all possible places for *D* for which the Voronoi-diagram is of type II.

c. Do the same for type IV.

d. What type arises if *D* lies close to *x*? Use, if necessary, the option `reduce` if it does not become clear.

5 If the fourth point consecutively takes the positions 1, 2, 3, ... , 10, the Voronoi-diagram changes gradually. The beginning and ending of that process are given. Sketch the intermediate stages and determine the type for each state.



6 From the preceding you can see that the transitions from one type to another type take place when point D passes through the circumscribed circle or through one of the three lines (through A and B, through B and C, through C and A).

a. A type belongs to each intermediate stage. Indicate that in the figure with I, II, III or IV.

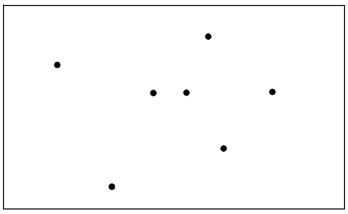
b. If you pick a point D 'at random', the chance for two of the four types is very large, but the chance forthe other two of the four types is very small. Why?

15: Infinitely large cells

Meanwhile you have seen that cells exist which are enclosed on all sides by edges and that there are cells for which this is not the case. Now we will look especially at that last kind of cells.

7 **a.** For starters: make the Voronoi-diagram for the situation on the right. There are two finite cells.

b. Now use the option `reduce` several times. What does the diagram start to look like more and more?



hull

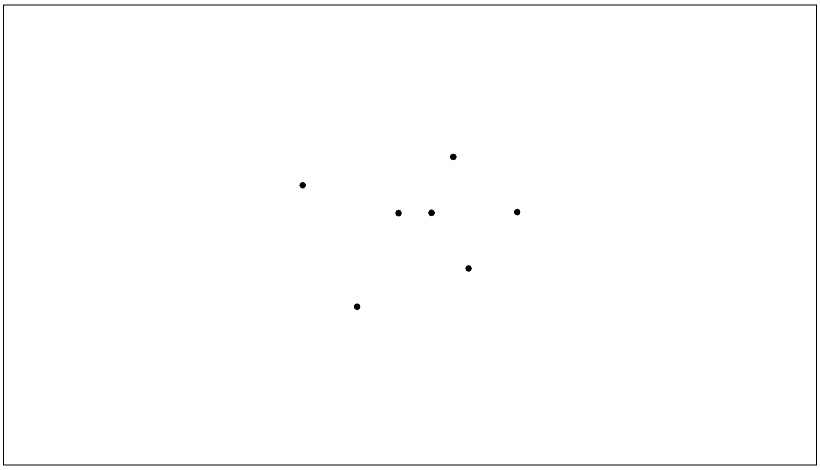
c. Now first choose the option `normal` and second the option `hull`. A red closed line is added now. Imagine that the centers are nails in a board, which still stick out a bit. With what small domestic object could you easily show this line?

d. Add a couple of points *within* the hull, ask for the `voronoi` diagram and again decrease considerably. Does the result differ from exercise **b**? Did you get any new infinite cells?

e. Go back to `normal` and add one point, but do that in such a fashion that you do add an infinite large cell and that the other infinite large cells stay infinitely large.

In this last question you try to add a center to the hull, while the centers already belonging to the old hull also belong to the new one.
If you answered a rubber band to question **c** above then this is what you need to do: make sure that the rubber band does not come off the other nails the band touches.

f. Here we have the situation of question **7 a** again, with some more space around. Indicate with a color the areas in which you can choose a new center in such a way that all the centers which are on the hull do not come loose from the hull.

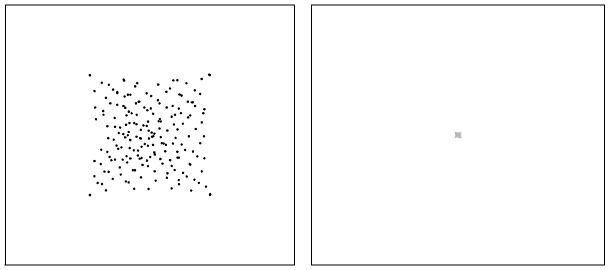


Temporarily conclusion from the preceding:

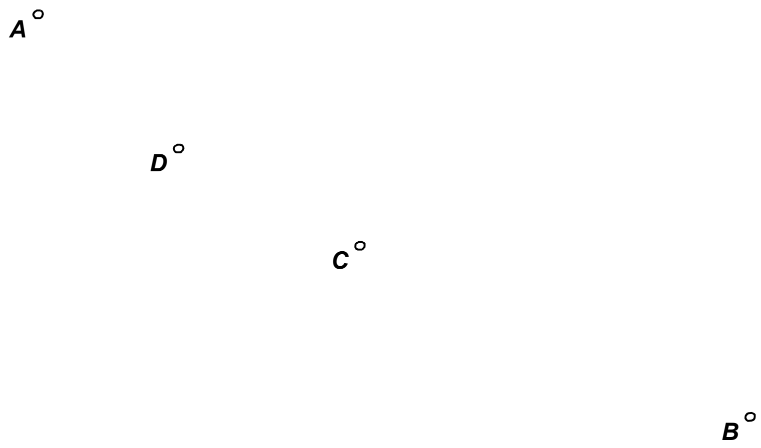
conclusion 1

The centers which are on the hull have infinitely large cells.

8 Here you see a situation where the computer cannot help you to sketch the Voronoi-diagram.
This would take too long. However, you can predict how the diagram will look, if you decrease scale a lot. Sketch that in the figure next to it, where the little gray cloud in the middle represents the group of centers.



9 **a.** Construct this situation, with Voronoi-diagram and hull. Verify, possibly by reduction, whether the cells of *A* and *B* adjoin.



b. Sketch the circle through *A*, *D* and *B* and also the circle through *A*, *C* and *B*. Find their centers, if necessary use the option `reduce`. Why do they need to lie on the Voronoi-edge of *A* and *B* or on its extension?

c. Add within the hull of *A*, *B*, *C* and *D* a new center *E*, but in such a way that the line segment *AB* still belongs to the hull. Again sketch the circle through *A*, *E* and *B*. Do the cells of *A* and *B* still adjoin? Why (or why not)?

We draw another temporary conclusion:

conclusion 2

Two centers which are connected with a line segment of the hull, have adjoining infinitely large cells.

10 Test with several examples whether this conclusion holds.

11 Construct an example with seven centers for which the Voronoi cells of the two furthest apart centers adjoin.

The two conclusions we drew in this paragraph look solid and reliable, but they are based only on your observations. You could prove them using the method of the last chapter. Since it would be a lot of work and would not lead to new insights, and also since we will not build further on these conclusions, we will leave it at this.

16: Extra assignments Voronoi-diagrams

The problems in this paragraph are not connected. You can choose out of one or more of:
– improved provinces for the Netherlands
– river, sea, delta, island
– special situations

instructions for loading and saving
In this paragraph we work with a couple of larger examples.

The names of the figures, which are available through LOAD, often appear in two forms on the screen: with and without a number in the filename.
If your computer is not very fast, choose the version without the number.
The number represents the number of centers used.

Loading
You open the examples we use by clicking the button `LOAD`. The computer will give you a list of possibilities. You click on the name of your choice.

Saving
If you want to save your diagrams and want to use the button SAVE, you will need your own floppy. Let the new name then start with A: (or another drive-letter); then your diagram will be saved on the A:-station.

improved provinces for theNetherlands

12 This assignment deals with the province capitals of the Netherlands. You can get them on the screen in two ways:
a: with worksheet B, page 123. Loosen and hold it in front of the computer screen. Copy the twelve province capitals as precisely as possible. The best way to do this is to copy the worksheet onto transparent paper.
b: Click on the button `LOAD` and click on `nedkaart`.
Construct the Voronoi-diagram in any way you like.
a. Would the Voronoi-cells round Maastricht and Assen adjoin? Investigate this using the buttons `reduce` and `hull`. You could find the answer to this question directly on the map using a ruler. How?
Choose `normal` to get the map back to its original size.

improve the province division

13 a. As you have seen, the edges of the Voronoi-cells do not coincide everywhere with the actual province borders. Now, the inhabitants of Amsterdam of course think than Amsterdam should be the capital of Noord-Holland. Remove Haarlem and add Amsterdam. Does the division improve?
Try to relocate more centers so that the Voronoi-diagram starts to look more and more like the real province division.
b. Another possibility is: build the provinces with more than one cell. You get too many edges, but some province borders will be approximated easier.

improve the railways

14 a. Now choose the button `Delaunay`.
You see blue lines appear between centers.
Which centers are being connected and which not?
Verify that Utrecht is connected to all the province capitals of adjacent provinces.

b. Is the Delaunay-triangulation a good proposition for improving the railroads? Include in your comments what are good improvements and which would be a waste of money.

continuation Delaunay-triangulation

The Delaunay-triangulation gives all the connections between the centers of adjoining cells. Another name for it is: the neighbor diagram.

15 Why does the following apply:

a. There are precisely as many line segments in the Delaunay-triangulation as there are edges in the Voronoi-diagram.

b. An edge of the Voronoi-diagram belongs to each connection in the Delaunay-triangulation. These two are perpendicular to each other, but do not have to intersect.

c. In a Voronoi-diagram with only a three-countries-point (not four-or-more) the number of three-countries-points is equal to the number of triangles in the Delaunay-triangulation.

d. The convex hull is part of the Delaunay-triangulation?

16 Add to the ten small figures of exercise **5** a sketch of the Delaunay-triangulation. Observe: if the Voronoi-diagram changes type, so will the Delaunay-triangulation.

river, sea, delta, island

17 Load the example `river171`. First make the Voronoi-diagram. You could think of a river of which you see two banks and an island in the middle. The idea is to find a route, which lies on a greatest possible distance from the banks.

a. You now see a clearly curved whole line. Why was making a Voronoi-diagram a very good approach here?

b. Sketch a few circles, which just touch both banks. How would you describe the channel in terms of *distances*?

18 Load the example `ndzee199`. With some good will you recognize the North Sea with Great Britain on the left and the Netherlands, Germany, Denmark and Norway on the right.

a. For oil exploitation purposes, the North Sea has been partitioned according to the nearest-neighbor-principle, so it should be divided carefully. Make the Voronoi-diagram.

b. Between England and the Netherlands and between England and Norway you can easily find the edge, which divides the continental plane. For the oil exploitation this edge is actually used.

c. On the group of islands to the northeast of Great Britain, the Shetland-islands, an important British oil harbor is located. Several years ago a mammoth tanker got stranded on the rocks. Check how the North Sea would be partitioned:

– if this group of islands belonged to a Scandinavian country. Before 1472 this was the case, but back then people did not drill for oil.

– if this group of islands did not exist. Also look what the influence is further to the north. (use the button `reduce`).

d. Between Germany and the Netherlands the edge in the sea is hard to find. Still it can be done. Look which cells belong to the Dutch shore centers and which to the German ones. Make a sketch of all edges.

19 You have seen the example `archi176` earlier, in chapter 1.

a. Load it and draw with `circle(mid, rand)` several largest empty circles, which touch three islands.

b. Suppose, you have to navigate through these islands from north to south, but you would like to be as invisible as possible. What route would you choose?

In one of the next chapters we will go further into such lines, which lie at an equal distance from several *areas* (here the banks and the islands).

special locations

20 Load the example `recht98`. The Voronoi-cells will be long strips, you know that by now.

a. Put an extra center a few centimeters from the cell-line. In the remainder of the exercise this center will be referred to as *F* and the line on which all the other centers lie will be called *l*. Show *l* on the screen.

b. Later on you will make the Voronoi-diagram. What will the cell round *F* look like? Test your assumption with the computer.

c. What happens to the cell round *F* if *F* is chosen closer to *l*?

d. You have seen a similar form before. How do we call that form?

e. On each of the edge segments of the cell round *F* lies a point which has the same distance to *F* and *l*. How do you determine such a point?

f. Let such a point be the center of a circle and take *F* as an edge-point. Can you explain that the circle touches *l*?

If you put the points on the straight line really close together, the cell round *F* will attain a more fluent form. The small, straight, pieces of edge, of which that form is built, indicate the direction of the 'tangent line' of the fluent form.

g. Look at such a piece of edge. Two centers belong to it: *F* itself and another one. How is the position of the tangent line linked to the position of those two centers?

21 Load the example `cirkel90`. The Voronoi-diagram here is easy to predict.
- **a.** But add a center *F*, inside or outside of the circle (both at the same time is of course also possible).
- **b.** Investigate elaborately how the position of *F* is linked to the form of the cell round *F*. Write down your findings.

other figures with symmetry

22 The examples `honey182`, `pent`, `vlinder` and `vier` give nice symmetrical figures.
- **a.** If you add to `honey182` the vertices of the Voronoi-diagram as centers to the original position and again ask for the Voronoi-diagram, what would you see? You need to able to think this one out before the computer is done!
- **b.** Try to do something similar to `pent`. This figure has a quinary symmetry.

23 You can also load and look at the examples `orion`, `beren` and `wild`. Orion, Ursa Major and Ursa Minor are constellations.

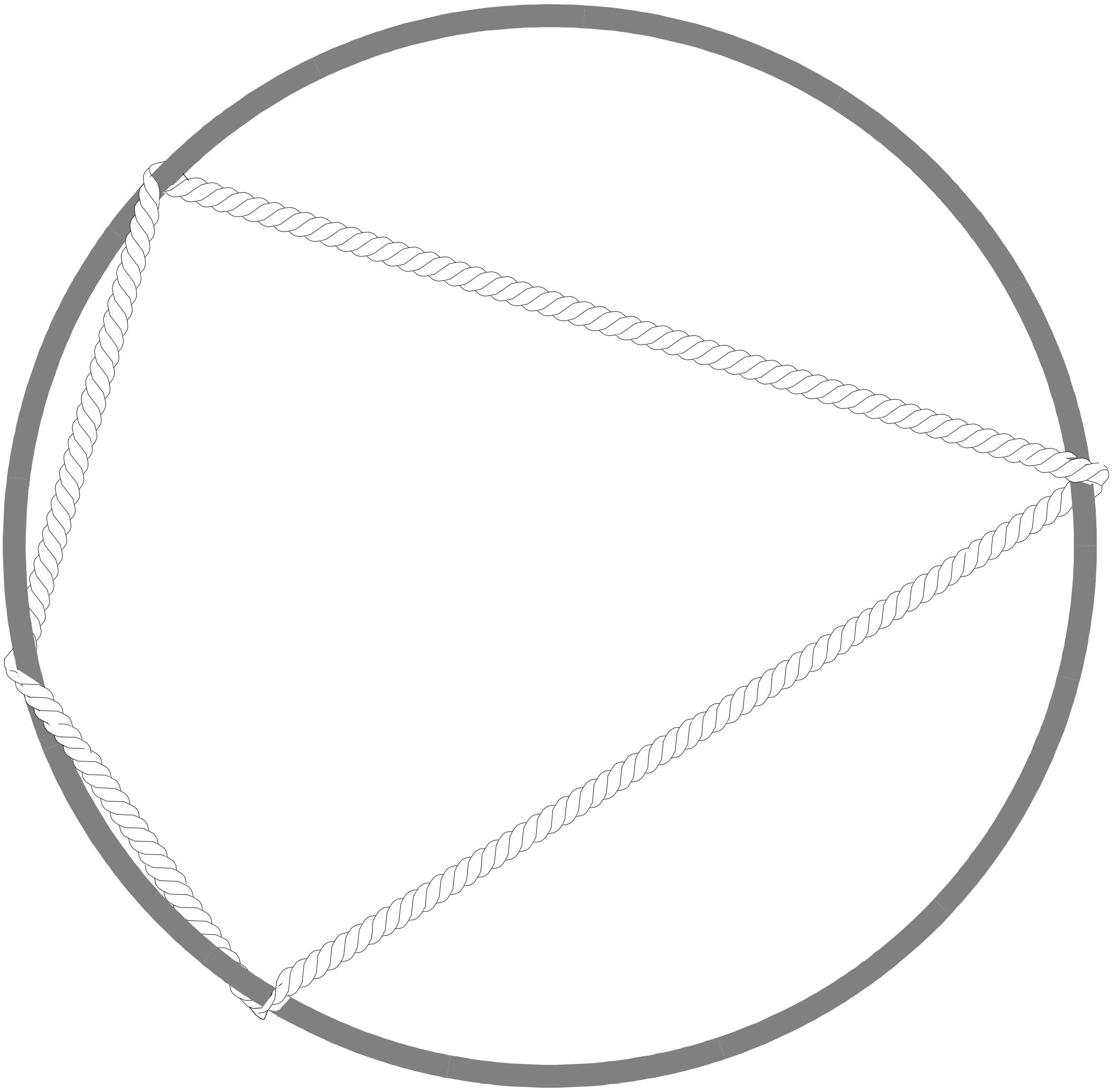
Summary of chapter 3

In this computer practical you were able to try several things you have seen before, like the existence of largest empty circles.
While exploring the influence of a fourth point on a diagram with three centers, it appeared that the circumscribed circle of the triangle of the three centers plays a dominant role. Furthermore, the sides of the triangle were also important.

You also took a more precise look at infinite cells. Their centers turned out to lie on the hull.

If you work with large numbers of points, you could form figures like rivers and islands. The Voronoi-diagram shows the edges between those figures.
One example was a single point against a series of points on a line. It looked like we found a figure which bore an uncanny resemblance to a parabola.

Chapter 4
A special quadrilateral

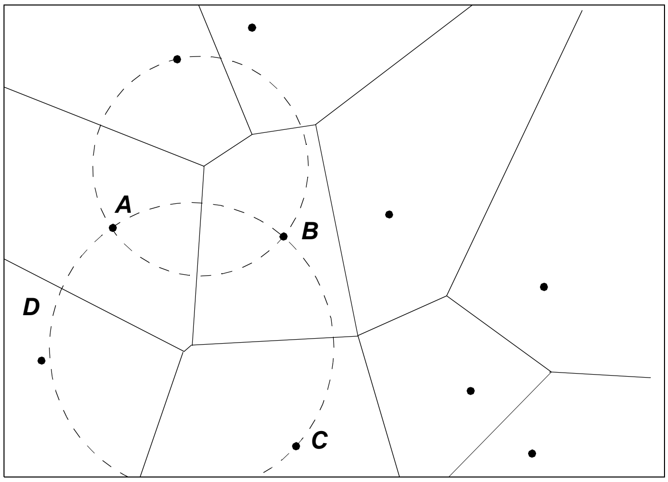


In this chapter we will continue reasoning.

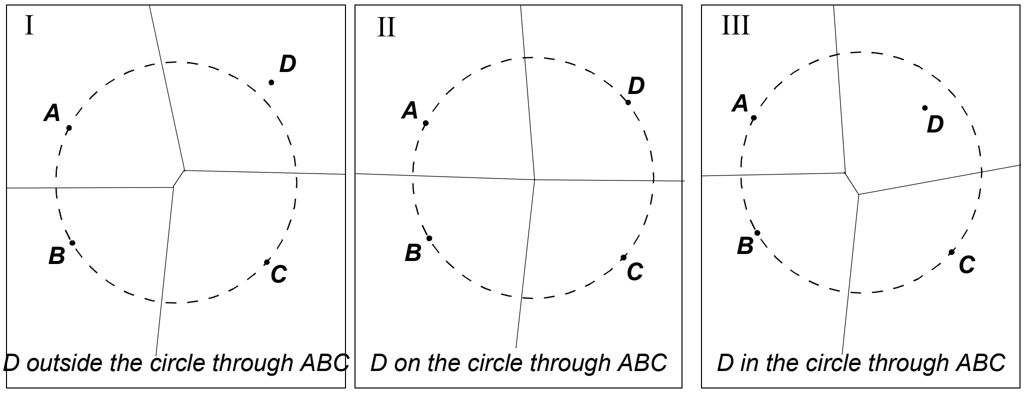
In general we will deduct several things concerning distances and angles from very few given data. We will also think about the process of reasoning itself and how to write down proofs.

17: Cyclic quadrilaterals

In chapter I, dealing with Voronoi-diagrams, you encountered this example, in where point *D* lies just outside the circumcircle of triangle *ABC*. This circle is empty and thus the cells round *A*, *B* and *C* converge into a three-countries-point, which of course is the center of the circumcircle of triangle *ABC*. Thus *D* does not disturb the three-countries-point.



For the position of the fourth point *D,* compared to the circle through the three points *A*, *B* and *C*, there are three options:



(In case III it is possible that *D* lies so close to *B* that the cell round *D* is closed.)

1 **a.** In case II the Voronoi-diagram is very special: there is a four-countries-point. Why?
b. Which vertex of the Voronoi-diagram is the center of the triangle *ABC* in case I?
c. And how about case III?

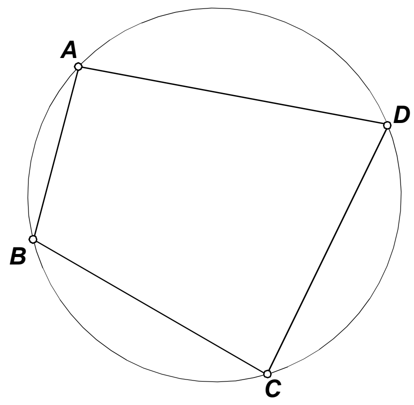
Since the sides of the quadrilateral in the special case are all *chords* of the circle, we call quadrilateral *ABCD* a cyclic quadrilateral. The **definition** is:

definition of cyclic-quadrilateral

A quadrilateral is called a **cyclic quadrilateral** if its vertices lie on one circle.

In this section we shall prove a relation between the sizes of the angles for these special quadrilaterals. Later on we shall use this relation for other purposes than Voronoi-diagrams.

2 On the right you see a sketch of a cyclic quadrilateral $ABCD$. In the figure we didn't actually portray the main property of the circle: that there is a center M and that the line segments MA, MB, etcetera, have equal length.



a. Therefore, sketch the center and the line segments MA, MB, MC and MD.

b. The quadrilateral is now divided in four triangles. The eight angles to the vertices of the quadrilateral are equal two by two. Why?

c. Indicate equal angles with the same symbol; for example, use the symbols $*$, $\circ$, $\times$, $\bullet$. In each vertex you see a different combination of signs. But what do you notice when you compare the sum of the signs of A and C to the sum of B and D?

First a short footnote.
The only correct answer to question **2b** is: because triangle ABM is isosceles.

The *isoscelesty* is given, namely $d(A, M) = d(B, M)$. That you can conclude the *equiangularity* from that is based on an at the moment non-formulated theorem about isosceles triangles. We will not prove this theorem here. We will assimilate it in the summary. Later on, you will draw up a list of these kinds of theorems, which have already been familiar to you for some time.

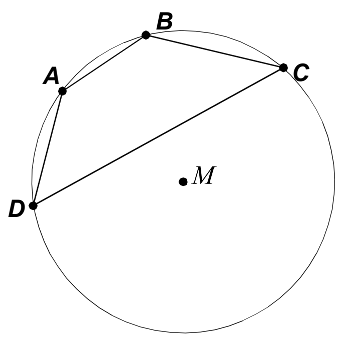
What you have just proven is the following *temporary* theorem:

Temporary theorem of the cyclic quadrilateral

In each cyclic quadrilateral *ABCD* holds: $\angle A + \angle C = \angle B + \angle D$.

It seems that we reached a fine result by smart reasoning. However, there are some annoying issues left.

Problem A The first question you need to ask yourself is:
Is the theorem proven for every cyclic quadrilateral one can think of?
For example, think about a cyclic quadrilateral as shown on the right.
We again looked at just one special figure, which is not representative for all cases. For in this one, the proof above does not hold....

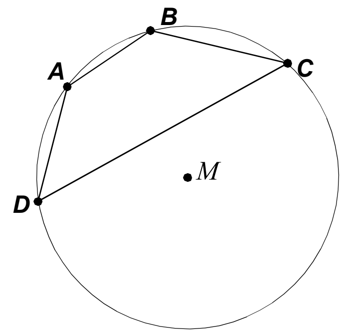


Problem B The temporary theorem only deals with equality of $\angle A + \angle C$ and $\angle B + \angle D$. This is a bit meager. Maybe something can be said about the *size* of $\angle A + \angle C$ and $\angle B + \angle D$.

Problem C Just like in the complete theorem about the perpendicular bisector, theorem 3, page 29, we need to know what happens if D lies inside or outside of the circle, since this is the most frequently occurring case!

18: Scrutinize proving

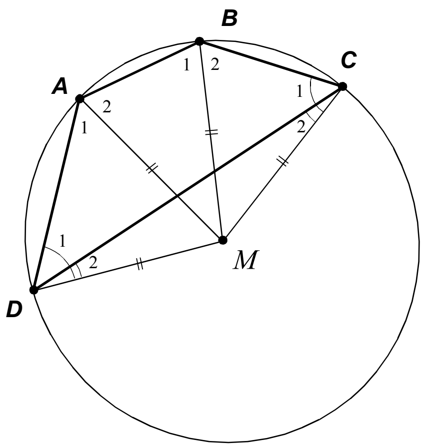
Problem A



3 **a.** What is the essential difference between this cyclic quadrilateral and the one from exercise **2**?
b. Again go over the steps of exercise 2. Where do you need to deviate from exercise **2** for this case?

It is not very difficult to find a proof for this situation. We will draw up this proof in a clearly noted form as an exercise in notation.

This is the accompanying sketch.



Now we can talk easily about all kinds of angles and parts of angles in the situation without referring to them with strange symbols.

4 In the theorem we are proving $\angle A$ plays a role.
We mean by $\angle A$ the vertex angle $\angle DAB$.
The sketch also shows $\angle DAB = \angle A_1 + \angle A_2$.
Here we are not reasoning based on a sketch, but merely showing what we mean with all those letter notations.
a. What does $\angle D$ mean in the theorem? Write down the relation with $\angle D_1$ and $\angle D_2$.
Look carefully at how the little arches are indicated.

The complete proof could start as follows:

$\angle D_1 = \angle A_1$, $\angle A_2 = \angle B_1$, $\angle B_2 = \angle C_1$, $\angle C_2 = \angle D_2$
(since the triangles *DMA*, *AMB*, *BMC* and *CMD* are isosceles)

This is actually
a statement
with a motivation.

In the remainder of the proof we will use the data from the sketch and these four equalities to rewrite the sum of angles $\angle A + \angle C$ step by step to $\angle B + \angle D$. Between the brackets is stated why an equality holds, thus those are again motivations.

$\angle A + \angle C = \angle BAD + \angle DCB$
= (dividing angles)
$(\angle A_1 + \angle A_2) + (\angle C_1 - \angle C_2)$
= (using equal angles)

b. Complete this story.
The last line of this story will be
$= \angle B + \angle D$
(You could - if you get stuck - search by starting at $\angle B + \angle D$ and splitting up the angles)

For the first case (where M lies inside of the cyclic quadrilateral $ABCD$) you could have written down the proof in the same fashion.

5 The sketch would be different, but in the proof you would need to change some details. Which?

case distinction

We are still working on the temporary theorem of the cyclic quadrilateral. We have made a careful distinction between the cases where the center of the circumcircle lies inside or outside the quadrilateral.

6 **a.** Is the temporary theorem of the cyclic quadrilateral now proven for áll possible cyclic quadrilaterals? In other words: are there other situations than those where M lies *inside* respectively *outside of* the quadrilateral?
b. If you find another case, which of the two proofs holds?

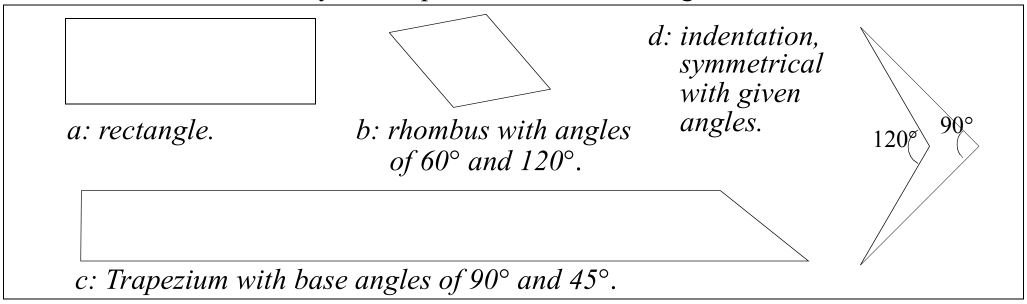
Conclusion from this part for Problem A:
The temporary theorem of the cyclic quadrilateral was put under some pressure, but is eventually saved by adjusting the proof for the other case.
While doing that we also practiced how to write down a proof clearly. We distinguished statements and motivations.
Noting angles with indices was handy for keeping the relation between proof and sketch.
On to problem B.

Problem B
That was:
The temporary theorem only talks about the equality of $\angle A + \angle C$ and $\angle B + \angle D$. This is a bit meager. Maybe something can be said about the size of $\angle A + \angle C$ and $\angle B + \angle D$.

7 **a.** Find out for exercises **2** and **4** what $\angle A + \angle C$ and $\angle B + \angle D$ are, using a protractor. The four results are not very different!
b. What is your statement (still to prove!!) about the sum of opposite angles in a cyclic quadrilateral?

If your statement is right, you can also say something about the total sum of $\angle A + \angle C + \angle B + \angle D$ in such a quadrilateral. Before we will prove that the sum of the four vertex angles in a *cyclic quadrilateral* is 360°, we check whether we need to restrict ourselves to cyclic quadrilaterals.
Namely, we switched from $\angle A + \angle C = \angle B + \angle D = 180°$ to $\angle A + \angle C + \angle B + \angle D = 360°$, but the last thing also holds if we have for instance $\angle A + \angle C = 140°$ and $\angle B + \angle D = 220°$. In other words: for the total sum of angles of 360° we maybe should not restrict ourselves to cyclic quadrilaterals.
We will first try to find out how general $\angle A + \angle C + \angle B + \angle D = 360°$ can be true.

8 Determine the total internal sum of angles for these examples. These are a few special cases for which it is easy to compute and determine angles.



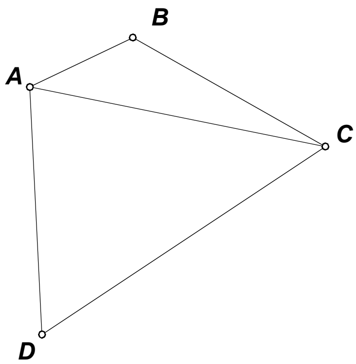
(Note that in d the angle of 120° is not an inside-angle of the quadrilateral.)

Let's formulate the statement first as a theorem, the proof will follow.

Theorem 6 **In each quadrilateral the sum of angles is equal to 360**°.

You will give the proof in the form which just has been shown. For the kernel of the proof you of course will need to know where you need to look, but you do already know that the sum of angles in a triangle is 180°.
This is something you learnt earlier which you can use now. In the summary we will assimilate this fact as a theorem.
In short: split the quadrilateral into two triangles!

9 **a.** This is a sketch, which belongs to it. However, sometimes it is possible that the



connection AC does not lie in the quadrilateral. Sketch such a case and divide that quadrilateral in two internal triangles with a connection line. As long as we do not use anything, but the properties of triangles, everything will work out just fine after this case distinction.

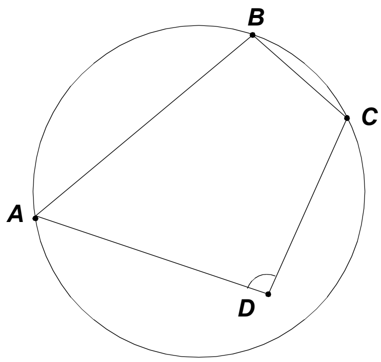
b. Now add the necessary numbers and arches in the figure and write down the proof using the format of exercise **4**.

The proof of theorem 6 is now complete. It is now safe to use the theorem to improve the temporary theorem of the cyclic quadrilateral to:

temporary theorem of the cyclic quadrilateral, improved version

For each cyclic quadrilateral *ABCD* holds $\angle A + \angle C = \angle B + \angle D = 180°$

Problem C



Now we will look at the situation where D lies within the circle through A, B and C.
The next assumption almost goes without saying: $\angle B + \angle D > 180°$. Imagine that you are standing somewhere on the circle, opposite B. If you starting walking forwards, you need to widen your view to left and right to be able to still see A and C. The proof of $\angle B + \angle D > 180°$ shall be based on this idea: we will compare the situation in point D to one with a point on the circle.

10 For that point we do not choose a completely new point, but a point, which has a relation to the other points.
- **a.** Choose a point on the extension of AD and call it E. Make sure that the quadrilateral $ABCE$ is fully drawn.
- **b.** Now show, using a familiar property of triangles that
  $$\angle ADC = \angle DCE + \angle CED.$$
- **c.** Which inequality does now apply to $\angle ADC$ and $\angle CED$?
- **d.** Complete the proof of $\angle B + \angle D > 180°$ by applying the temporary theorem of the cyclic quadrilateral to $ABCE$ and combining that with the result of **c**.

The proof has not been put in a strictly organized form, but this is not always necessary.

Now the following has been proven:

I. If in a quadrilateral $ABCD$ point D lies **INSIDE** the circumcircle of triangle ABC, then $\angle B + \angle D > 180°$ applies.

We already knew:

II. If in a quadrilateral $ABCD$ point D lies **ON** the circumcircle of triangle ABC, then $\angle B + \angle D = 180°$ applies.

And of course we also expect that:

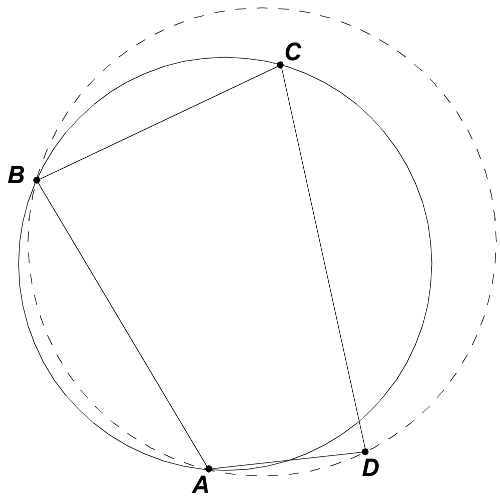
III. If in a quadrilateral $ABCD$ point D lies **OUTSIDE** the circumcircle of triangle ABC then $\angle B + \angle D < 180°$ applies.

You could prove III almost similar to exercise **10**, and surely you would be able to. Without doing this, you may also assume the correctness of III; it is no fun doing the same over and over again. But ....

On the other hand: it is better using the proven cases I and II in a smart way and deducting case III from there. We will do this in this extra-exercise.

extra

11 First of all the sketch. D lies outside the circle through A, B and C.
The dotted circle goes through A, B and D. It looks like the complete arch from A via D to B lies outside the circle through A, B and C.



a. Argue that using theorem 5, page 31.
Actually, you need to show that both circles cannot have a third point X in common, because what would then be the circumcircle of AXB?

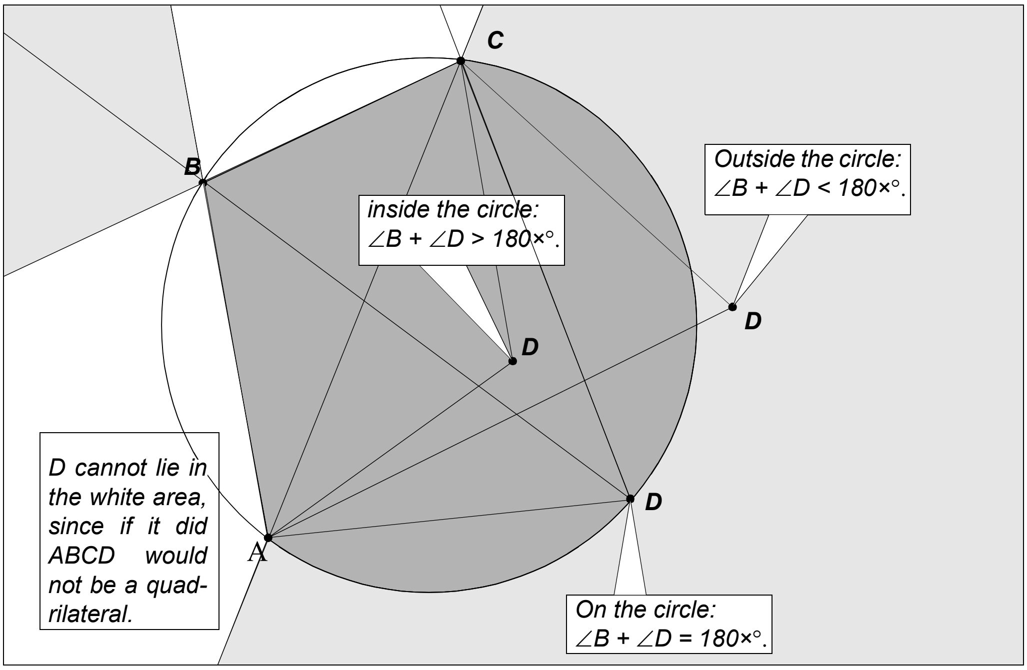
b. Thus C lies within the circle through A, B and D. The proven part I of above now leads to an equality in where $\angle C$ occurs. Write it down.

c. Now deduct, using theorem 6, the desired statement $\angle B + \angle D < 180°$.

Problem C is solved and we will summarize the results of this section in one theorem.

Theorem 7 **If *ABCD* is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.**
If *D* lies inside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D > 180°$.
If *D* lies outside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D < 180$.

This deserves a survey illustration.



Remark: Since the three cases of the theorem exclude each other, you can immediately draw conclusions like:

If in a quadrilateral $\angle B + \angle D = 180°$, then that quadrilateral is a cyclic quadrilateral.

After all, if this equality holds, then the last two statements of the theorem ensure the fact that *D* neither lies inside nor outside the circle. Remains: on the circle.

extra, finding a good definition

12 It is said that *D* cannot lie in the white area, since then *ABCD* would not be a quadrilateral.
This is rather vague as long as we not have agreed upon what a quadrilateral is.
Find out what is going on, and give a definition of ‘quadrilateral’, which exactly excludes these cases.

extra

Warning!
In mathematics it can, and will, happen that you find a proof which looked right, but later on somebody remarks on a small error in it. It does not always mean that the proof is totally wrong; it can be fixed in most cases. You are now in such a situation.

13 Check again the position of point *E* in 10a above. It could be on arc *BC*! In that case you cannot work with *ABCE* as a quadrilateral.
a. How to fix this hole in the proof?

19: Using cyclic quadrilaterals

In this section we return to constructing Voronoi-diagrams. We use what we know of cyclic quadrilaterals, so mostly theorem 7. Since that theorem talks about angles, you need to measure angles very precisely several times.

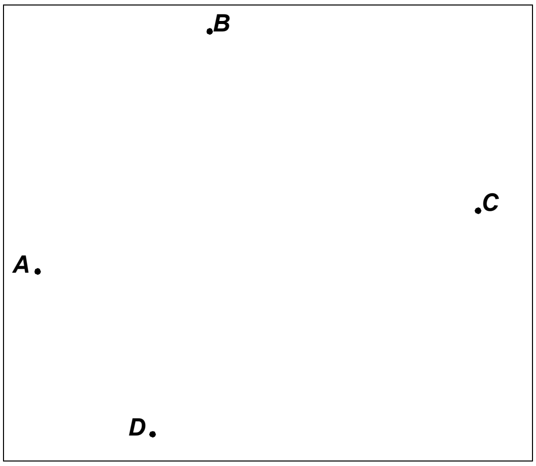
14 Given are four centers.

a. Find out whether in this Voronoi-diagram of these four points the cells round A and C, or the cells round B and D adjoin.

b. Sketch all connection lines of centers which have adjoining cells with a color.

c. Finish the Voronoi-diagram by sketching perpendicular bisectors.

d. Unlike exercise **4 a**, page 7, now you did not sketch too many perpendicular bisectors. How come ?



15 a. Is this also true?

If a Voronoi-cell is a quadrilateral, then that quadrilateral is a cyclic quadrilateral.

If necessary, give a counterexample.

16 Sketch a situation with six centers, where you have two four-countries-points in the Voronoi-diagram. (Here avoid the flat example that the centers which have a four-countries-point, form a square or a rectangle.)

Summary of chapter 4

reasoning

In this chapter we got results through reasoning.
The direction of concluding things was from former knowledge to new all the time.

Writing down proofs

You have learned that you can write down proofs in a neat way. Two aids were:

a. Indicating angles with indices: A_1, B_2, etcetera. In the sketch you can also indicate angles with symbols like $*$, $\circ$, $\times$, and $\bullet$. However, it looks kind of weird if you start your proof with: * = *. A good compromise is: indicate in the sketch equal angles with the same color or symbols and use unambiguous denominations.

b. Note the statement and motivation in this format:

a statement

with a motivation.

motivations

As motivations the following are allowed:
- references to definitions
- basic unproven known facts we agreed about (like the triangle inequality)
- statements that have been proven earlier.

theorems

Important things which we know to be true and which we will use again, are laid down in the form of a theorem.
Several theorems mentioned below have not been proven in this book. Those are the first two of the following survey.

survey of theorems of this chapter

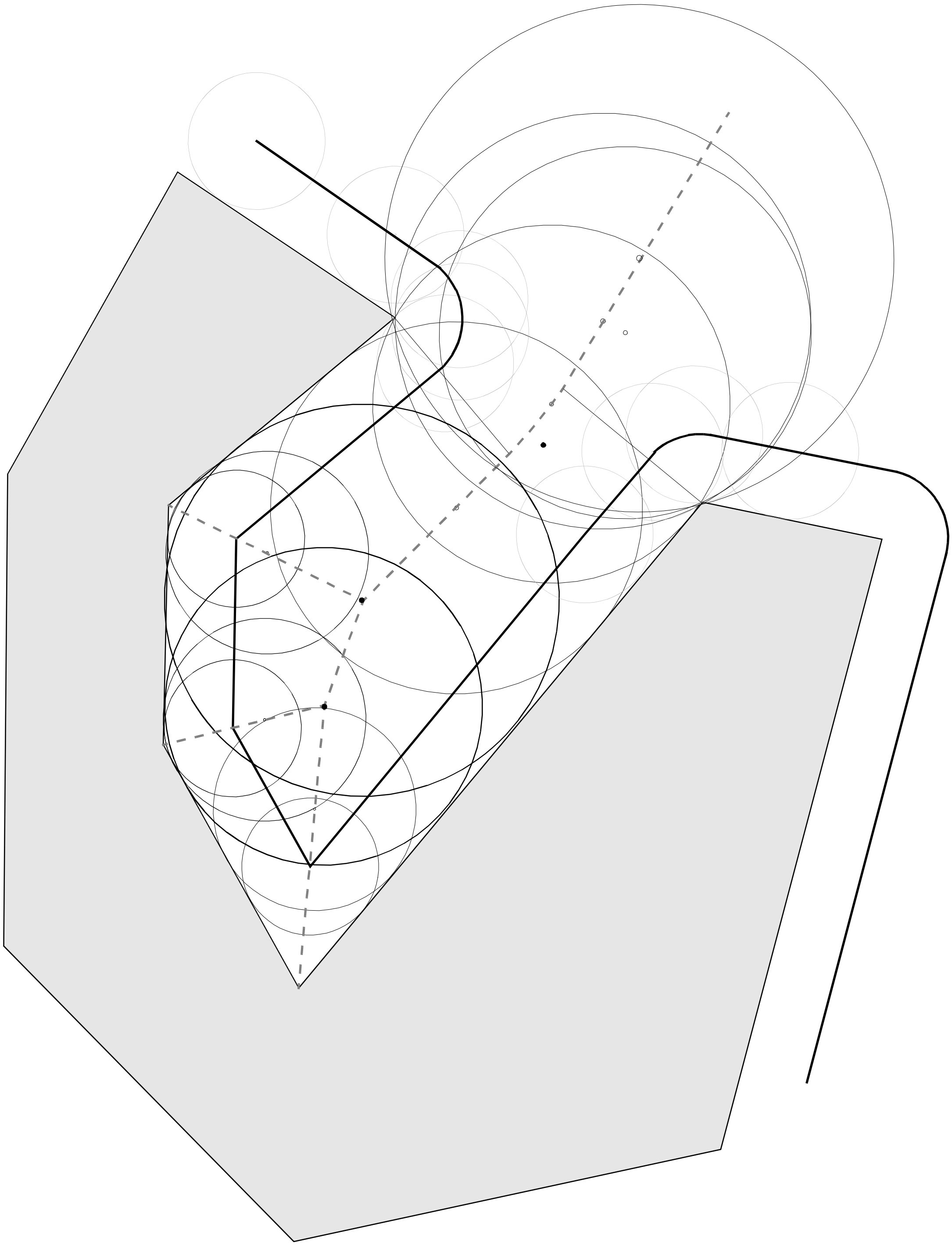
Theorem 1 **(Isosceles triangle)**
If in triangle ABC d(A, C) is equal to d(B, C), then ∠CAB is also equal to ∠CBA.
Also the reverse is true:
If in triangle ABC ∠CAB is equal to ∠CBA, then d(A, C) is equal to d(B, C).

Theorem 2 **(Sum of angles in a triangle)**
In each triangle the sum of angles is equal to 180°.

Theorem 3 **(Sum of angles in a quadrilateral)**
In each quadrilateral the sum of angles is equal to 360°.

Theorem 4 **(Properties of cyclic quadrilaterals)**
If ABCD is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.
If D lies inside the circumcircle of A, B and C,
then $\angle B + \angle D > 180°$.
If D lies outside the circumcircle of A, B and C,
then $\angle B + \angle D > 180°$.

Chapter 5
Exploring isodistance lines



When dividing fishing grounds between different countries the concept of the isodistance line plays an important role. This is a line which lies at sea at a fixed distance from the different shores.
Studying these lines puts us on the track of several geometric relations, where tangent circles and bisectors play a big role.

20: Isodistance lines, distance to areas

The small island Mururoa in the South Pacific was frequently in the news in 1995. An international fleet, lead by two Greenpeace ships, wanted to protest on the spot against nuclear activities which would take place on the island. When rubber boats and the helicopter from the Rainbow Warrior operated within the 12-mile-zone round Mururoa, the French government saw this as an occasion to board this ship. On the one hand this lead to dramatic TV, but on the other it was a sensitive blow to Greenpeace's further plans.

The 12-mile-zone is seen as part of a country's territory. Unauthorized trespassing of the imaginary edge of this zone means a violation of international rights.

isodistance line

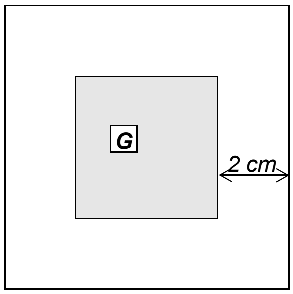
The edge of this 12-mile-zone is an example of an isodistance line: each point of this line has a distance of 12 miles to the coast. More exact: it is the iso-12 mile-line.
But how do you know where this imaginary line lies? To what point on the shore is the distance 12 miles? What determines the shape of such a line? We will deal with these questions in this chapter.

1 Mururoa is an atoll. The diameter of this small island is a couple of hundred meters, very little compared to 12 mile (1 mile = 1,61 km).

a. First consider Mururoa as a pointlike area.
Which shape does the edge of the 12-mile-zone have?

b. Consider Mururoa as a circular area with a radius of 680 meter. What then is the iso-12-mile-line?

2 On the right a square – reduced – with a side of 4 cm is sketched. That is the area G which we look at. Around the square a second square with a side of 8 cm is sketched.
The edge of this square is *not* the iso-2 cm-line of G.
You can also find a full-size sketch on the worksheet at page 125.



a. Indicate on the edge of the larger square five points that do belong to the iso-2 cm-line of G and also five that do not.

b. Now sketch the iso-2 cm-line of G.
Also sketch the iso-1 cm-line, the iso-3 cm-line and the iso-4 cm-line of G.

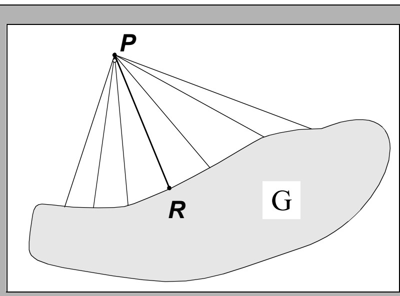
c. Is the iso-300 cm-line of G a circle?
Explain your answer.

Before we start sketching isodistance lines for more complicated areas, we will first describe what we mean by the distance from a point P to an area G.
The description uses the familiar distance concept for two points.

description

Determine for all points on the edge of an area *G* the distance to *P*.
For one (or more) of these points, say *R*, is *d(P, R)* the smallest.

We then call:
– ***d(P, R)* de *distance* from *P* to *G***
– ***R* a *footpoint* of *P***
We denote the distance from *P* to *G* by *d(P, G).*

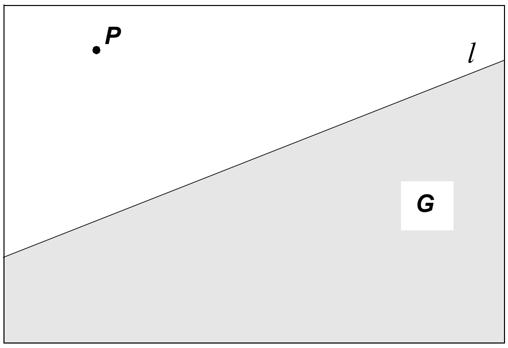


In the description a lot of words are used which are not defined: area, edge. Also, without further introduction it is simply assumed that such a smallest distance exists.
In this chapter we will use the terms *area* and *edge*, just as if they have a solid mathematical meaning, and we will accept also that the minimum distance exists. We will agree on one other thing: *the edge belongs to the area*.
Furthermore: if misunderstandings are not possible, from now on we will leave out the unit for distances.

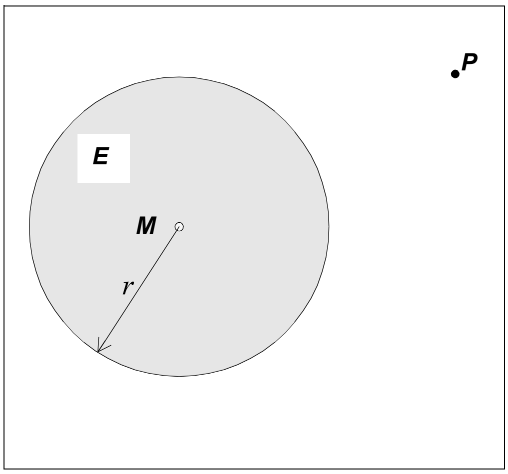
simple areas

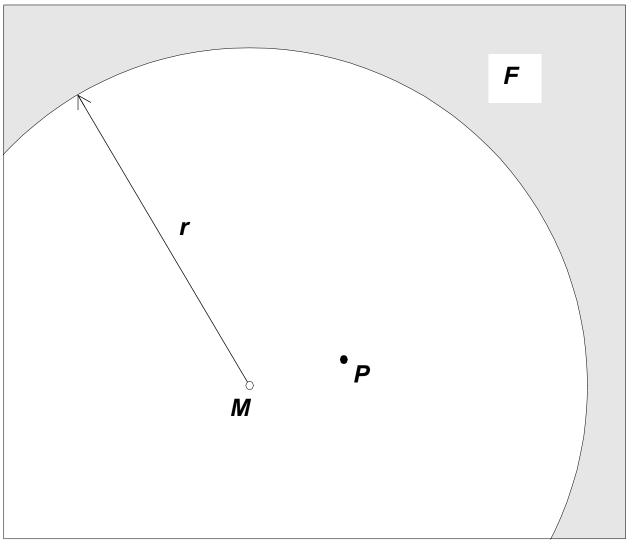
In some cases it is easy to determine *footpoint* and *distance*. We are also able to give a full proof in those cases. We will do so in the following exercises.

3 Here the area G is a half-plane; the edge of A is a line, say l.
  a. How can you determine the footpoint R of P and the distance $d(P, G)$ immediately?
  b. In what way(s) has been proven that this is true?



4 A circular island E with center M and radius r. Again a point P outside of E is indicated.
  a. Of course you expect the footpoint of P on E to be on the connection line PM. Sketch the line and name the point R.
  b. Take a point Q on the edge of E, not equal to R, and prove that $d(P, Q) > d(P, R)$ applies.
  Aside from the triangle inequality, you also need to use your knowledge of circles.
  c. Express, using the d-notation, the distance from P to E in the distance from P to M and the radius r.





5 Here the area F is the outside area of a circle. M again is the center of the circle. We need to find the footpoint R for the given point P.

a. Again sketch the points R and a point Q on the circle, (almost) like in the previous exercise.

b. Also here the triangle inequality helps while proving that $d(P, Q) > d(P, R)$. Now you definitely need $d(M, Q) = d(M, R)$.

c. Again express, using the d-notation, the distance from P to F in the distance from P to M and the radius r.

Now we know how to find footpoints of given points for line-shaped and circular edges of areas, it is also clear that footpoints can be found for areas which are restricted by several pieces of lines and circles. With that, for these types of areas, the existence of a minimum distance, which is assumed in the description on page 60, is made secure.

Now the concept distance to an area is known, you can also define what an isodistance line is.

definition iso distance line

For a given positive distance *a*, the isodistance line at distance *a* from an area *G* is the set of all point to which apply: $d(G,P) = a$.
The isodistance line at distance *a* is indicated by iso-*a*-line.

6 **a.** Sketch for exercises **3**, **4** and **5** the isodistance line at the distances 1, 2 and 3 and also sketch the isodistance line which crosses point P.

b. From what does it follow immediately that these are indeed circles in exercises **4** and **5** ?

7 On the next page a triangular area G is sketched.

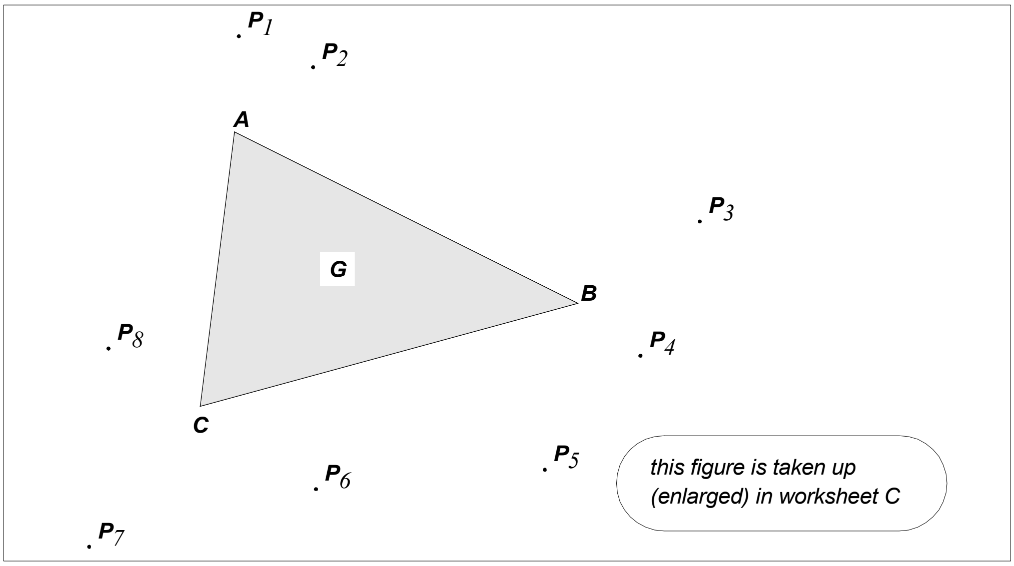
a. Sketch for each of the points P_i the accompanying footpoint R_i on the edge of G.

b. The vertices A, B and C of the area play an important role.
Sketch in the figure the ‘zone’ for which point A is the closest point for points in the area G. For instance P_1 is in this zone.
Also sketch these zones for the points B and C.

c. The outside area of G is now divided into six zones.
Describe for each zone the shape of the isodistance lines.

d. Sketch the isodistance lines for d=2 cm, d = 4 cm and d = 6 cm. What stands out?

e. The isodistance lines are longer than the circumference of area G. How much longer?

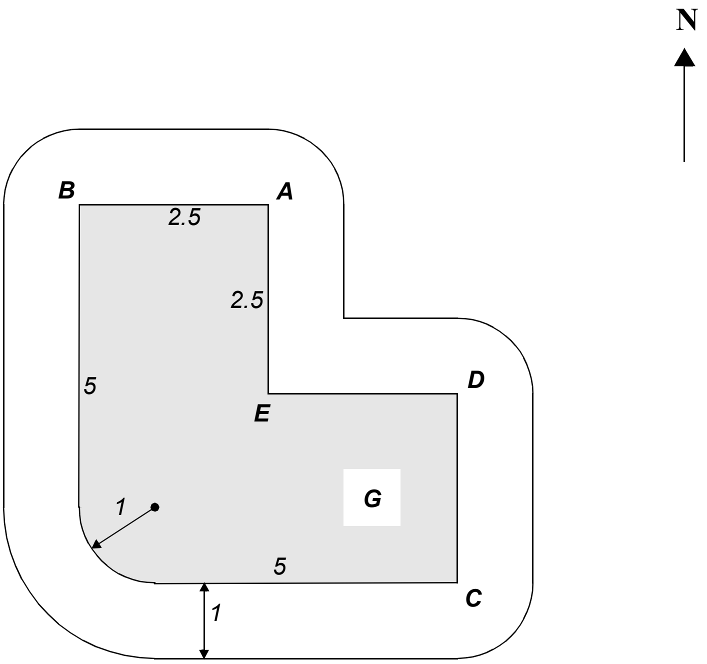


cape

The points *A*, *B* and *C* are a kind of *cape*. According to the dictionary a cape is *a strip of land projecting into a body of water*. Such a projecting point is the footpoint for a large number of points: all points, which lie in a certain sector.

8 **a.** Explain why all the isodistance lines in the neighborhood of a *cape* are parts of concentric circles.

b. How do you find the lines which confine the sector of such a cape?



9 Up here you see an L-shaped area *G* with iso-1-line. *A*, *B*, *C*, and *D* are capes.

a. Now sketch the northeast part of the isodistance lines for the distances 0.5, 2, 3 and 4 very exact.

Near the capes the arcs of the circle smoothly run into the straight pieces, but this is somewhat different for *E*. There, the isodistance lines show a kink.

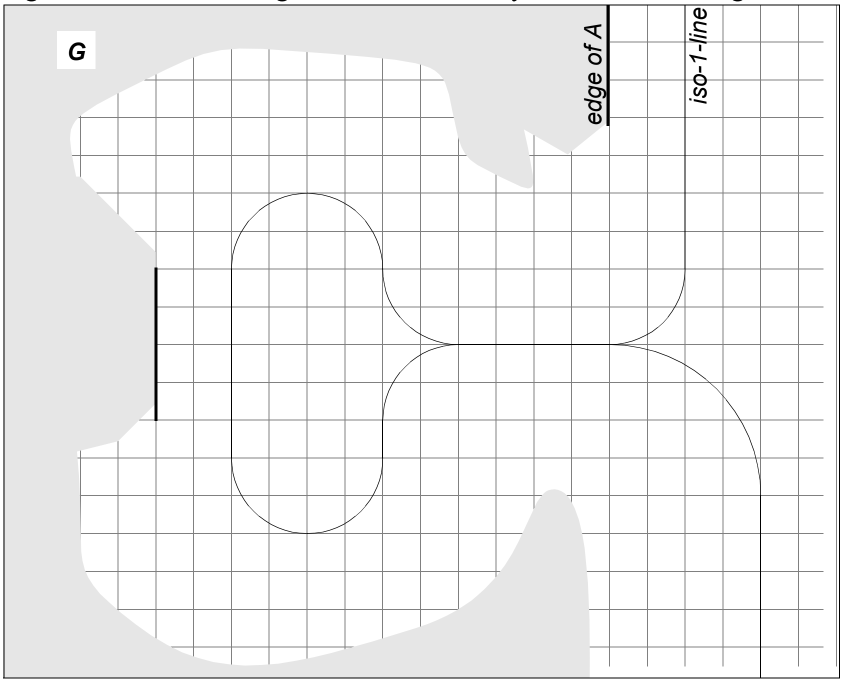
b. On which line do these kinks lie?
c. The iso-1-line makes a kink of 90^o. How large is the kink for the other four iso-distance lines?
d. Up to which value of *d* does the de iso-*d*-line make a kink of 90^o? Explain.
e. As the distance increases, the angle between the two arcs decreases. Will the kink completely disappear at a great distance?
Give a clear reasoning.

Areas with bays

For the simple areas which have come up till now, the isodistance lines were built of arcs of circles and straight line segments and could be sketched precisely.
You can say from the L-shaped area of exercise **9** that the northeast has a bay. Near this bay the isodistance lines showed kinks. In the next section we will go further into bays with straight-lined shores.
The next two exercises deal with bays with a small passage to the open sea.

10 The bay above has a passage to the open sea with a minimal width of 2 kilometer. The scale of the map is therefore determined by:
1 cm on the map is in reality 1 kilometer.
a. Successively sketch in different colors the iso-0.2-, iso-0.5-, iso-1-, iso-1.50- and iso-2-lines. You do not need to measure very precisely, but the important characteristics of the isodistance lines should clearly come out.
b. What, according to you, is the most important difference between the shapes of the iso-0.2-line and the iso-1.5-line?
c. Indicate all points on the iso-1-line, which have two footpoints.
d. Indicate with an extra color on the iso-2-line those pieces of which the footpoints lie on the coast of offshoot *B*.

11 Below, an iso-1-line is shown, as well as two short pieces of the edge of the involved area. The isodistance line consists entirely of quarter circles, half circles and straight line segments. Finish the edge of the area exactly. You can use the grid lines.



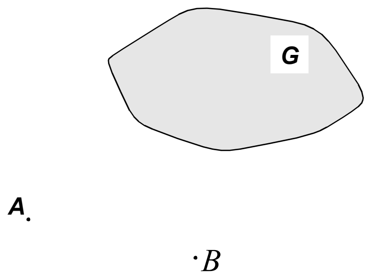
You can see from this example that the name isodistance *line* is not perfect: a line does not split up in two pieces halfway through. No problems occur if you know what you mean by the total concept *isodistance line*, namely the set of points with ... etcetera.

extra

12 In the proofs at the beginning of this section (the half plane and the circular island) the triangle inequality played an important role. The triangle inequality applies to three arbitrary *points*. Now the concept of distance is extended with distances to areas. Would the triangle inequality still be true if you were to replace one or more points by areas? In short:

a. Does $d(A,B) \leq d(A,G) + d(B,G)$ apply to the sketched situation?

b. Move B in such a way that this 'triangle inequality' is no longer true.



21: Angle bisectors

In this section we investigate isodistance lines in bays which are bounded by straight-lined shores. In here bisectors play an important role.

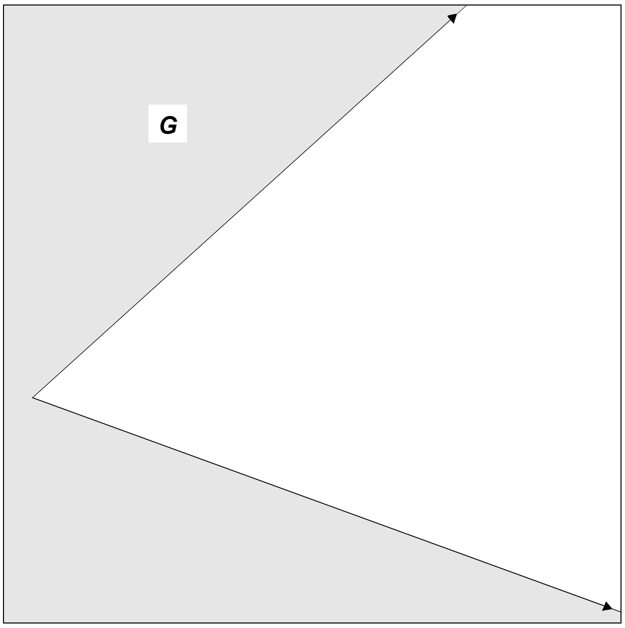
13 You only see a part of the area *G*. Suppose that the straight edges of the bay continue indefinitely. The arrows should represent this.

a. Sketch accurately in the bay
 the iso-1-line
 the iso-2-line
 the iso-3-line
 the iso-4-line.

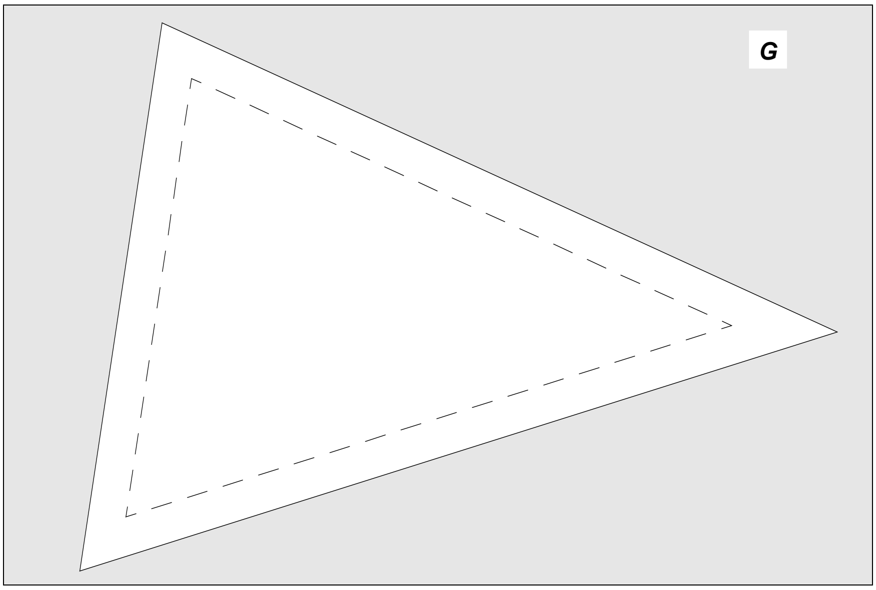
b. All these lines have a kink. Which figure do these kinks form?

c. How many footpoints does the 'kink point' of an isodistance line have?

d. Sketch the footpoints of the 'kink point' of the iso-3-line.



14 Here you also see part of an area. The white triangle does not belong to the area and the isodistance lines lie inside the triangle.



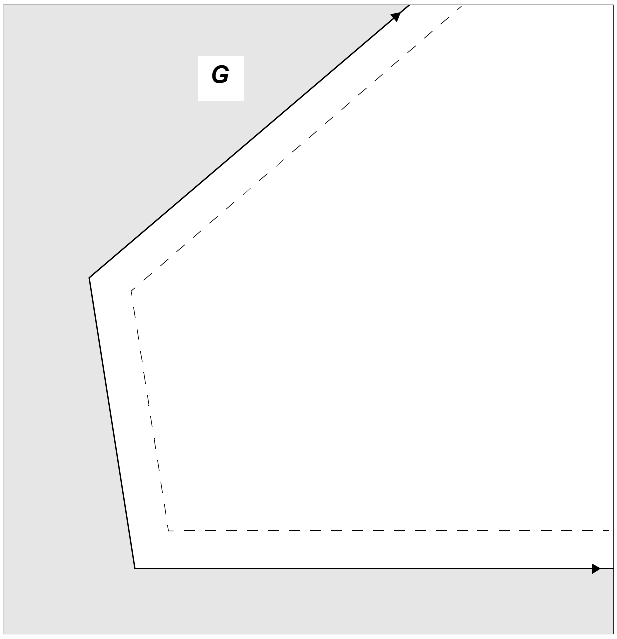
a. The iso-0.5-line is sketched. Also sketch the distance lines at the distances 1, 1.5, 2, 2.5, etcetera.

b. Describe what you notice.

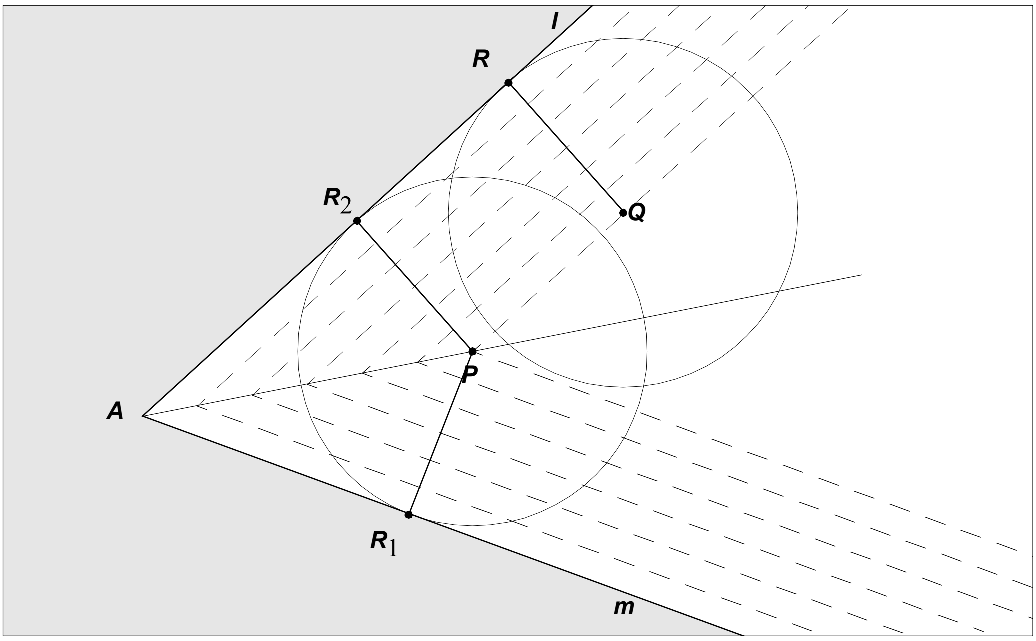
15 This area has a bay with two kinks. Also the iso-0.5-line shows two kinks.

a. Do all isodistance lines have two kinks?

b. Find a point *P* in this bay that has three footpoints on the area *G*. Demonstrate how you found *P*
Also sketch the circle with center *P*, which runs through the three foot-points.



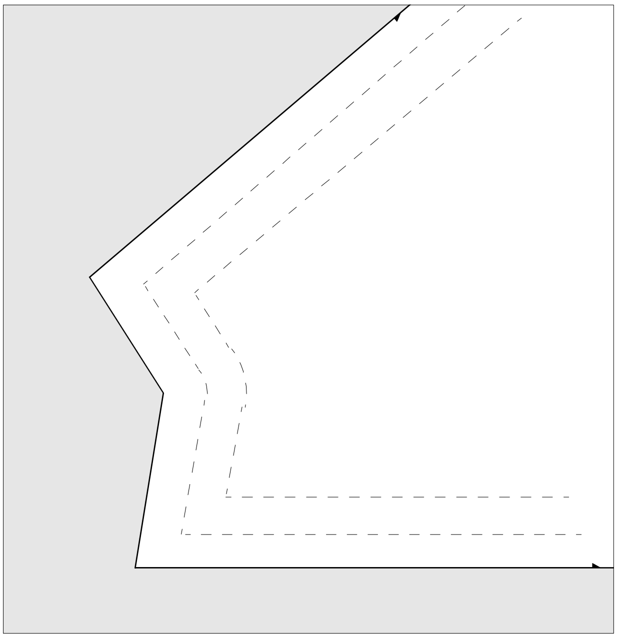
In a bay which is bounded by straight line segments the isodistance lines show kinks. Those kinks lie on a straight line. Each kink has a footpoint on both edges of the bay, in contrast to all the other points on an isodistance line.



For the two lines which bound the bay, a 'kink line' plays a similar role as the Voronoi-edge for two points.
To the points on the Voronoi-edge of centers *A* and *B* applies: $d(P, A) = d(P,B)$
To the points on the 'kink line' for *l* and *m* applies: $d(P, l) = d(P,m)$.
In the next section this analogy will be looked into further.

extra

16 Also, for the bay on the right we have a lot of isodistance lines with two kinks, but not all.
Sketch an area which has bay(s) with straight-lined shores and for which *all* isodistance lines have two kinks.



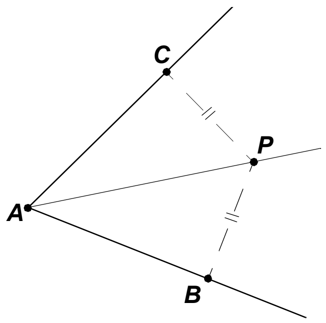
22: Theorems about angle bisectors

Here is a definition of *bisector of an angle.*

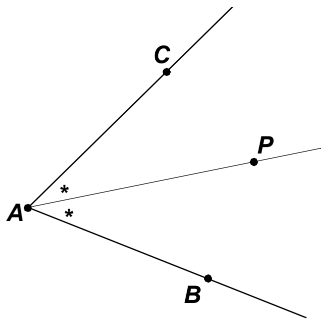
definition angle bisector

> **The angle bisector of an angle *CAB* is the line which makes equal angles with the legs *CA* and *BA* of that angle.**

To a point P on the bisector applies $\angle PAC = \angle PAB$.
In the previous section we found that to points on the 'kink line' of a bay which is bounded by straight line segments, applies: $d(P, C) = d(P, B)$.
There, B and C are the footpoints of P on the legs of the angle.
The figure strengthens the impression that the 'kink line' and the angle bisector coincide.

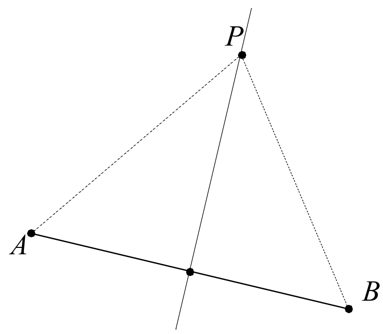


'kink line': d(P,C)=d(P,B)

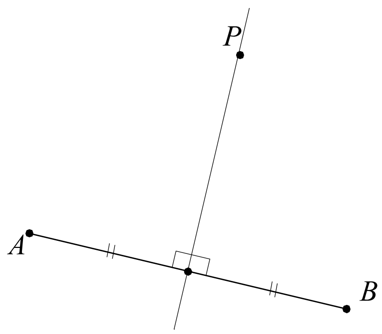


angle bisector:
$\angle PAC = \angle PAB$

It all looks a lot like the equality of the Voronoi-edge and the perpendicular bisector.



Voronoi-edge: d(P, A) = d(P, B)



perpendicular bisector:
through the middle of AB
and perpendicular to AB

In section 11 of chapter 2 (page 26) we have proven this equality. The proof consisted of two parts.

17 Now prove only that each point on the 'kink line' also lies on the bisector.

On page 26 we have formulated the following theorem about the properties of the perpendicular bisector.

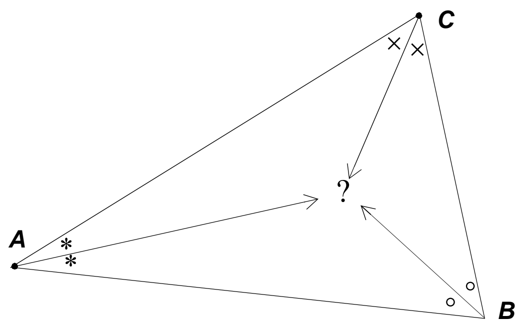
Theorem 3 **The perpendicular bisector of line segment *AB* is the set of points *P* for which holds *d(P, A) = d(P, B)*.**
For points *P* outside of the perpendicular bisector the following holds:
If *d(P, A) < d(P, B)*, then *P* lies on the *A*-side of *pbs(A, B)*.
If *d(P, A) > d(P, B)*, then *P* lies on the *B*-side of *pbs(A, B)*.

18 Formulate an analogue theorem about the angle bisector.

Theorem 8 **Angle bisector theorem**
....
....

This theorem can be proven just as fundamentally as theorem 6; we will not do this in detail. In exercise **17** you have already done a part.

There also is an analogue for theorem 4, page 30.
The figure below already suggests it.



Theorem 9 **The three angle bisectors of triangle *ABC* meet in one point.**

In the proof of the theorem of the perpendicular bisector we used the concept of distance. Since theorem 8 associates bisectors with distances, we can do the same here.

19 Give a proof of theorem 9; use the structure of page 30.

We will carry on with the analogy in a minute.The *circum*scribed circle of the triangle belongs to the intersection of the perpendicular bisectors. Here we have an *in*scribed circle.

definition inscribed circle

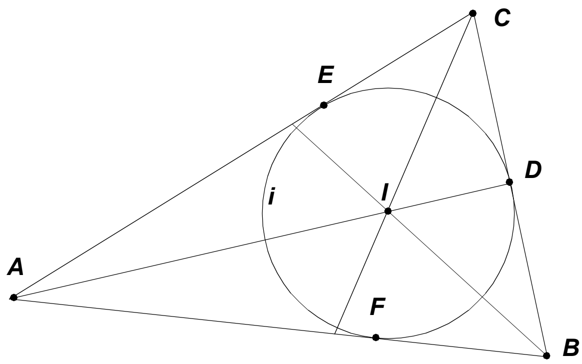
An inscribed circle of a triangle is a circle which lies in the triangle and which has one, and only one, point in common with each of the three sides.

This could be the theorem that looks like theorem 5.

Theorem 10 **Each triangle has an inscribed circle (=incircle). The center (=incenter) of the incircle is the point of concurrence of the triangle's angle bisectors.**

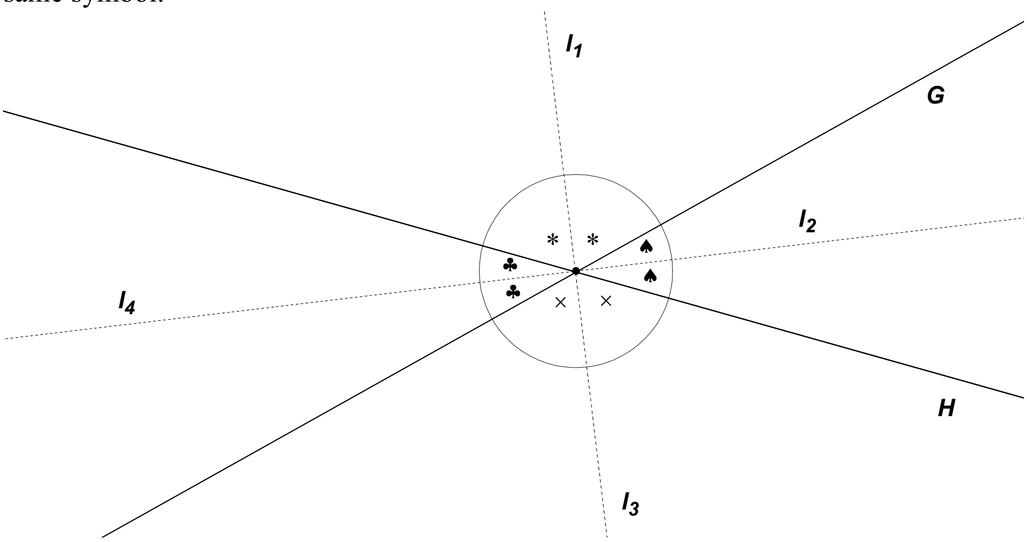
glance forward

The figure on the right shows a triangle with its incircle.
It is tempting to say that the sides are tangents to the circle. This is not wrong, but then we need to underpin the word 'tangent'. We will elaborate on this later.



As the theorem was formulated here, everything is fine; the circle lies completely on the right side of all three lines, so within the triangle.

For preparation purposes on a theorem about angle bisectors, which does *'not'* have an analogue in perpendicular bisectors, we investigate converging angles and their bisectors.
Here two line shaped areas G and H are indicated. (Therefore, the intersection of the lines belongs to both areas, but this is not very important). The two areas enclose four bays. For each of the bays the bisector is sketched. Equal angles are indicated with the same symbol.



According to the definition on page 68 we should talk about *four* angle bisectors. Up to now we only had *half* angle bisectors. We will rectify this in a moment.
Of course you know that these four half bisectors complete each other into two lines. You can also prove this.

20 Prove that l_2 and l_4 indeed form a straight line.
First prove that l_1 is perpendicular to l_4 and also to l_2.
Use the facts that H and G are 'whole' lines and that a straight angle is 180°.

agreement

From now on we mean by bisector a *whole* line, which passes through the intersection of two lines and makes equal angles with those lines.

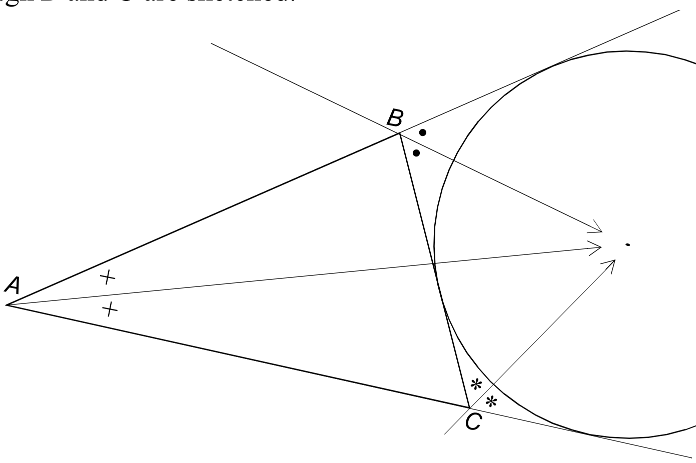
The result of exercise **20** can now be formulated as the following theorem.

Theorem 11 **The two angle bisectors of two crossing lines are perpendicular to each other.**

internal and external angle bisectors

In a triangle *ABC* we now have six bisectors:
- three, which coincide in one point inside the triangle; these are called the *internal angle bisectors*;
- three others, which, besides a vertex they cross, lie outside the triangle; they are called the *external angle bisectors*.

21 In this triangle the internal angle bisector through *A* and the external angle bisector through *B* and *C* are sketched.

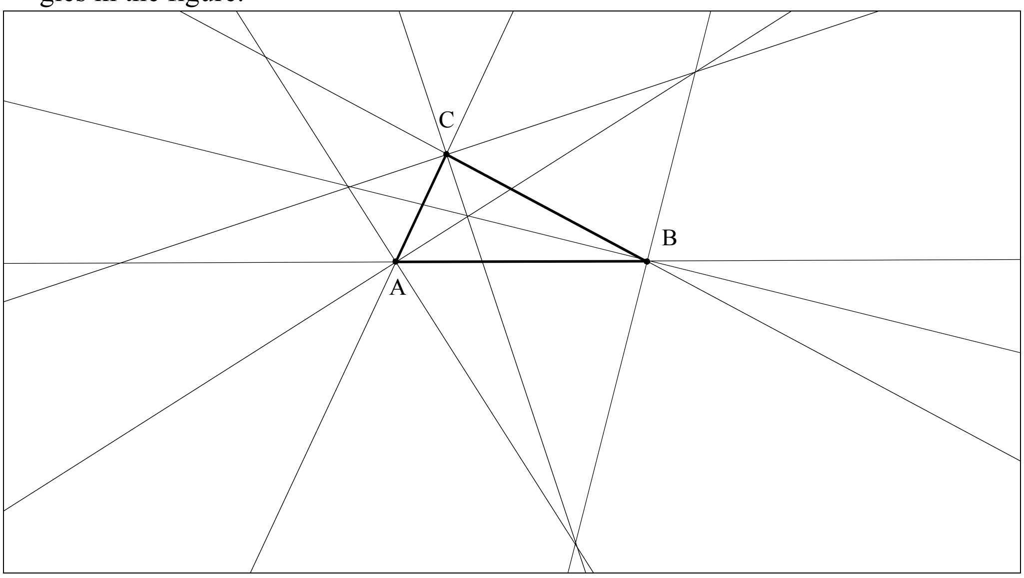


Prove that these three bisectors also coincide in one point and that this point is the center of a circle, which only has one point in common with each of the three lines.

escribed circle

Such a circle is called an *escribed* circle or *excircle* of triangle *ABC.* Each triangle has three excircles.

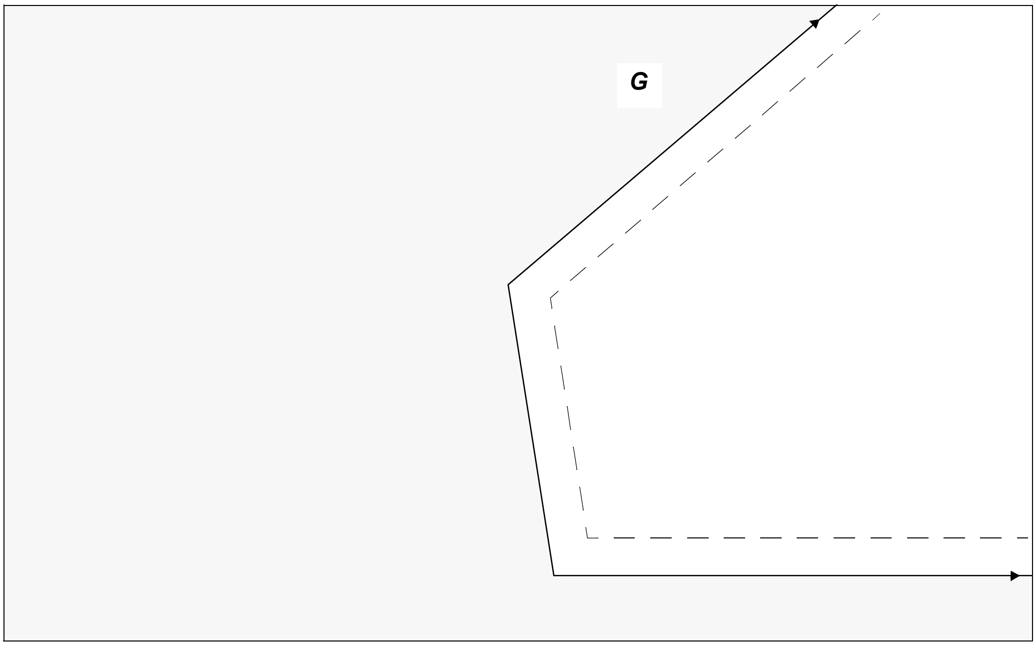
22 The following figure shows a triangle *ABC* and all its internal and external angle bisectors. Sketch the *incircle* and the three *excircles*. Also indicate equal and right angles in the figure.



The following theorem does not state anything new, it is a summary of the preceding. It even repeats the previous theorem. There is nothing against that in such a summary. We will leave the undefined concept of '*tangent*' in there. For now you can think about 'tangent' as just having one point in common and for the other points as lying on one side of the lines.

Theorem 12 **In a triangle the three internal angle bisectors coincide in one point. That point is the incenter of the incircle.**
If one chooses the external angle bisector at two vertices and the internal angle bisector at the third vertex, then those three lines also concur in one point. This point is the excenter of an excircle of the triangle.

23 In exercise **15** you have sketched a point *P* in a bay, which has a footpoint on each of the three shores. You can now think of that point as the center of an excircle of some triangle.
But then *P* needs to be the intersection of two external angle bisectors and one internal angle bisector of this triangle.
Sketch those three angle bisectors of the triangle.



extra

24 Do the incircle and excircle have the same point in common with a side in a triangle?
Find out for which triangles this is true.
You do not have to prove your statement.

23: Tangent circles

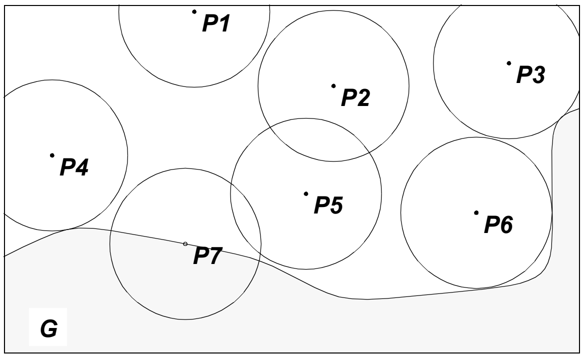
In this section we discuss a relatively easy way to find isodistance lines of capricious areas.

bumping circle

We call a circle round a point P outside G a *bumping circle* for G if there is one (or more) point of G **on** the circle and there are no points of G **inside** the circle.

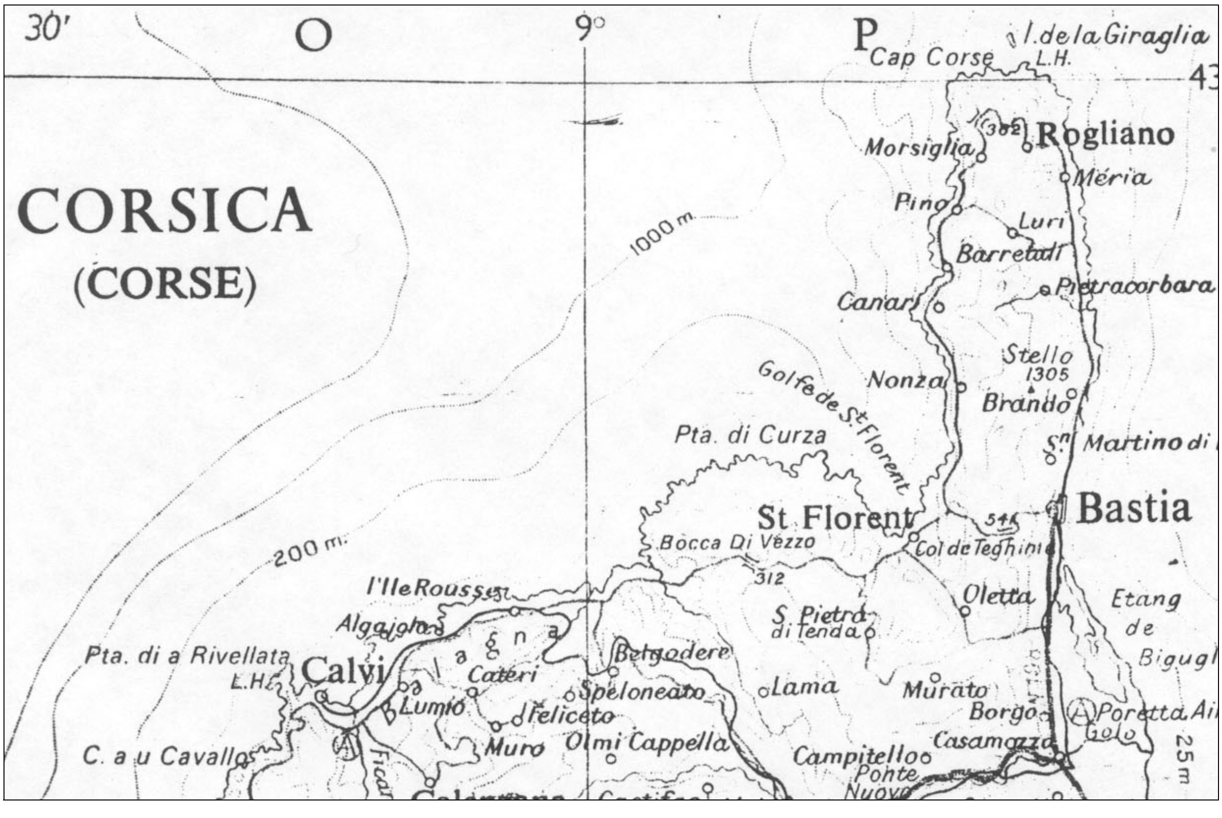
25 a. Here an area G and a number of circles with radius 1 with centers $P_1, \ldots, P_7$ is sketched. Indicate which points P_i belong to the iso-1-line.

b. If a point Q lies on the iso-1-line, is it certain that there is a bumping circle with radius 1 and center Q?



With one bumping circle, moving around the area, and little hole in the middle, you could quickly sketch an isodistance line. Let's try!

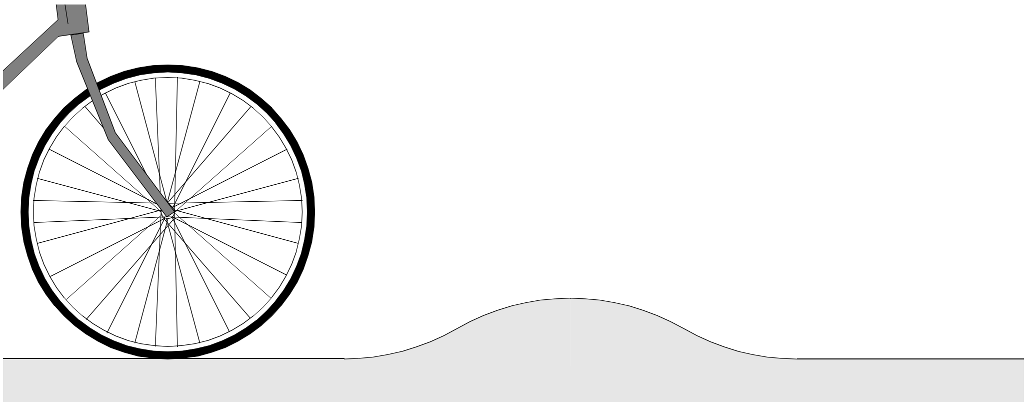
26 Below you see part of a map of Corsica; the coast of Corsica is very varied. Make a cardboard circle with radius 2 and a hole in the middle. With a pencil, which sticks through the hole, you can very quickly sketch the iso-2-line. The scale of the map is 1 : 600000; so you are actually sketching the iso-12 km-line.



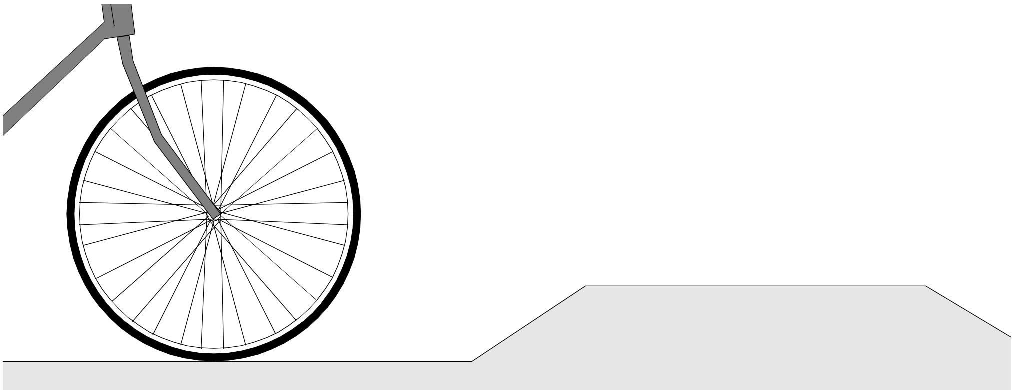
27 Consider a turning cycle wheel as a moving bumping circle. In the following three figures you see different obstacles.

a. For each case sketch the path that the middle of the wheel follows.

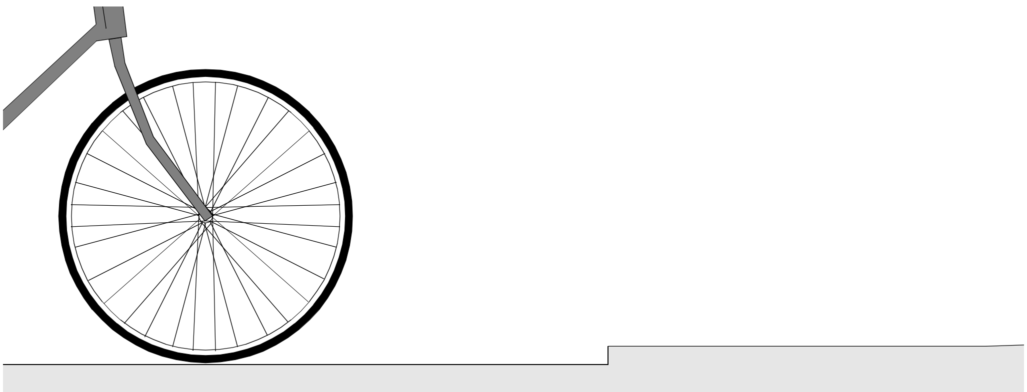
1. round bump



2. speed ramp



3. low pavement



If the path of the middle of the wheel makes a kink, the cyclist will feel a blow. The blow is heavier when the angle between the directions before and after the kink is larger.

b. How many blows does the cyclist feel at the speed ramp?

c. In which case does the cyclist feel the severest blow: at the ramp or at the low pavement?

The theorem of the bumping circle

Theorem 13 **The centers of all bumping circles with radius *a* of an area *A* form the iso-*a*-line of that area.**

28 The statement has two directions:
- each center of a tangent circle with radius a lies on the iso-a-line
- each point of the iso-a-line is the center of a tangent circle with radius a.

Did we prove this theorem? Where and how?

29 The next statement is *not* true in general:

If a bumping circle with radius a touches two or points of an area, then the iso-a-line has a kink.

Find in section 20 an example and counterexample for this statement.

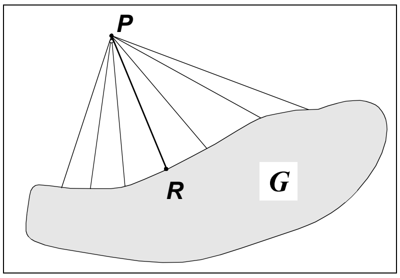
Summary of chapter 5

area, edge

We did not say what an area and an edge are in general. We could do this, but it would go too far.
No difficulties occur if we stick to the following:
– An area in the plane is nothing else but a piece of the plane we work in. It could be bounded or unbounded. A piece of straight line is also an area.
– The edge of an area always belongs to the area.

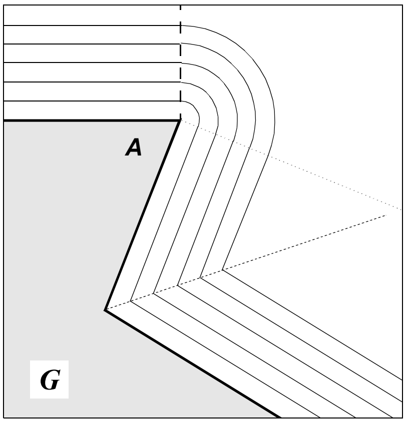
distance from a point to an area, footpoint

The concept distance between points has in this chapter been expanded to distances from points to areas. The distance from a point P to an area is the smallest possible distance from P to the edge points of the area. Such an edge point is a footpoint of P.
P can have more footpoints.
When we deal with areas, bounded by several arcs and line segments, we can be sure of the fact that each point outside the area has one or more well defined footpoints on the edge of the area.



isodistance line

The iso-a-line of an area G consists of all points which have distance a to the area G.
In the sector which belongs to cape A, the isodistance lines are pieces of concentric circles with center A.
If the footpoints lie on a straight line, then the isodistance lines will also be straight-lined.
For bays, which are bounded by straight line segments, the isodistance lines consist of parallel line segments, which are connected with a kink. The 'kink points' have two footpoints. They lie on the bisector of the angle the two straight edge segments make.



angle bisector

An angle bisector is a line which makes equal angles with the legs of the angle.

similarities between angle bisectors and perpendicular bisectors

Angle bisectors divide angles in half, perpendicular bisectors divide line segments in half. We have found several theorems about angle bisectors, partly analogous to the theorems of the perpendicular bisector.
Each triangle has three perpendicular bisectors, their intersection is the center of the circumcircle. Each triangle has three internal angle bisectors and three external angle bisectors. There are four points where three of those lines intersect. Those are the centers of the incircle and the three excircles.

theorems about angle bisectors

Since we state here the complete formulations of 8 and 12, theorems 9 and 10 do not have to be repeated here.

Theorem 8 **(Properties of the angle bisector)**

The two angle bisectors of an intersecting pair of lines l and m form the set of points P to which applies d(P, l) = d(P, m).
To points P within triangle ASB applies: (Here A and B are points on a and b respectively, not equal to S)
If d(P, a) < d(P, b), then ∠ASP < ∠BSP
If d(P, a) = d(P, b), then ∠ASP = ∠BSP
If d(P, a) > d(P, b), then ∠ASP > ∠BSP.

Theorem 11 **(Perpendicular position of two angle bisectors)**

The two angle bisectors of two intersecting lines are perpendicular to each other.

Theorem 12 **(Angle bisectors inside and outside of the triangle)**

In a triangle the three internal angle bisectors intersect in one point. That point is the center of the incircle of the triangle.
If one chooses two external angle bisectors at two vertices and the internal angle bisector at the third vertex, then those three lines also intersect in one point. That point is the center of an excircle of the triangle.

bumping circles

A *bumping circle* to an area *G* has one or more points of G on the circle and no points of G inside the circle.
Two theorems describe the role of these circles.
When sketching isodistance lines of complicated areas you can use bumping circles.

Theorem 13 **(Theorem of the tangent circles)**

The centers of all bumping circles with radius a of an area G form the iso-a-line of that area.

Chapter 6
Shortest paths

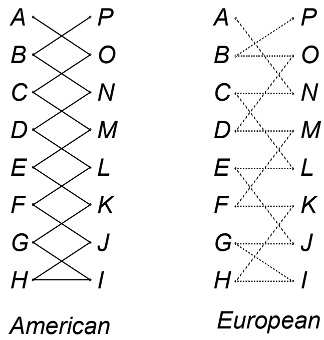
In this chapter a different type of geometry problems comes up: finding shortest paths.
We start off with finding economic ways to tie your shoelaces.
Reflection and working with angles will be used many times.
Thus you will often need your protractor.

The character of this chapter is mostly very practical. In many exercises sketching is required.

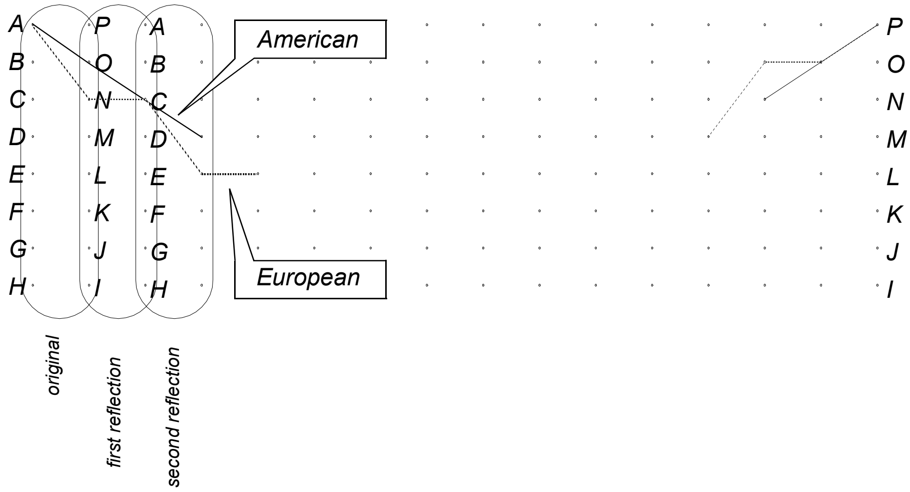
24: From shoelaces to shortest paths

The pictures on the first page of this chapter show two different ways to tie your shoelaces. The dressy shoes are tied European style, the cowboy-boots American style.
We will compare the styles on efficiency of use of shoelace and work with a method which gives rise to an interesting geometrical investigation: optimizing through reflection in mirrors.

1 The two methods are here applied to the same training shoe. Alongside you see the schemes. Which method uses, according to you, the shortest length of shoelace between *A* and *P*? (Base your answer on the schemes and restrict yourself to a first estimate.)



2 Below the start and finish of the shoelace patterns are represented transformed; reflections were used, so that the crossings no longer appear See the shoelace going from A to O, to C: the American style.



a. First finish both patterns in this fold-out representation.

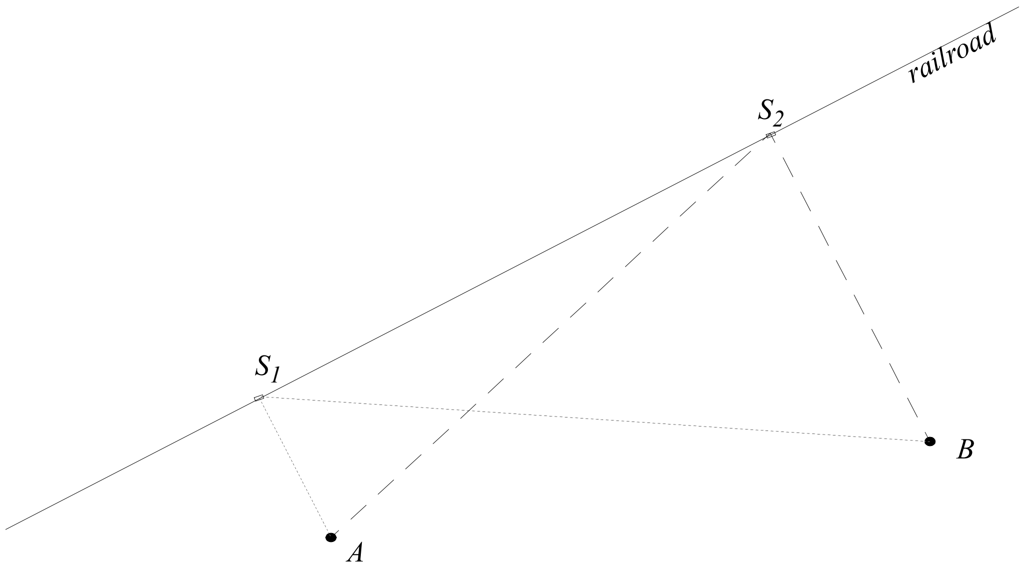
b. lengths are not changed in this new representation, compared to the earlier scheme. Why is that so?

c. Now decide again which pattern uses the shortest length of shoelace. Explain your choice.

shortest road length

In the shoelace-problem you saw that it is sometimes easier to determine a shortest path by reflection if the path needs to meet certain conditions of reflection.
We have seen an earlier situation in which this method was applicable.

old problem The straight line below represents a railroad line; A and B are cities, which lie at some distance from the railroad. One station needs to be built and of course roads from both cities to the station need to be constructed.



The separate city councils of the cities A and B suggested S_1 and S_2. The roads necessary for both propositions are indicated with dotted lines.

3 After deliberation the cities decide in favor of the overall cheapest solution:

position S of the station needs to be located in such a way that the total length of constructed roads $|AS| + |SB|$ is minimal.[*)]

a. Determine the exact point S, which meets this condition.
(You have done this before! But again a hint: if the cities lie on different sides of the railway, it would not be difficult to decide where S needs to be located.)

b. Now also show that for the point S holds: $\angle S_1SA = \angle S_2SB$.

think ahead for a moment Strictly speaking, you have not learned why that last thing is true in this book! You need some knowledge of angles, but not all that much. In the next geometry book you will work more with angles and then proving something like this would be a piece of cake.
In this chapter you will gain some experience in using angles.
Simply use what you already know:
– opposite angles of intersecting lines (as was used here)
– F- and Z-angles
– the total sum of angles in a triangle is 180 degrees.

We summarize what we found in the form of a theorem.

[*] Here the notation $|XY|$ is used for the length of the line segment XY. See, if necessary, page 23.

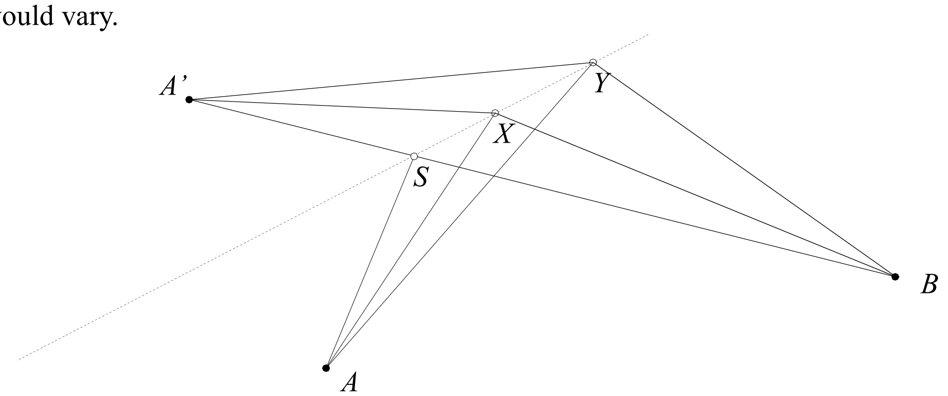
Stelling 14 **If a line *l* is given and two points *A* and *B* on the same side of the line, then there is exactly one point *S* on the line *l* for which |*AS*| + |*SB*| is minimal. For that point *S* holds:**
- ***S* is on the connection line of *B* and the *reflection of A* in *l*,**
- ***AS* and *BS* make equal angles with *l*.**

extra

If you were to move a point X from S_1 to S_2, then the total length

$$|XA| + |XB|$$

would vary.



You only know that this quantity would be minimal if $X = S$.
It is also true that the length decreases continuously if you move from X to S_1 to S and after that increases continuously. Said plainly: if you make a bigger detour, it will be longer. However, the mathematical proof is very subtle.
Thus you could add to the theorem:

plain, but subtle

Furthermore: if P is point which moves away from S along l,
then |AP| + |PB| constantly increases.

The proof of this statement does fit within the framework of the proving with the *triangle inequality*. But it is not easy!
With this figure you should show that:

$$|A'Y| + |YB| > |A'X| + |XB|.$$

Unpleasant as it is, it does seem that:

$$|A'Y| > |A'X|$$

but also:

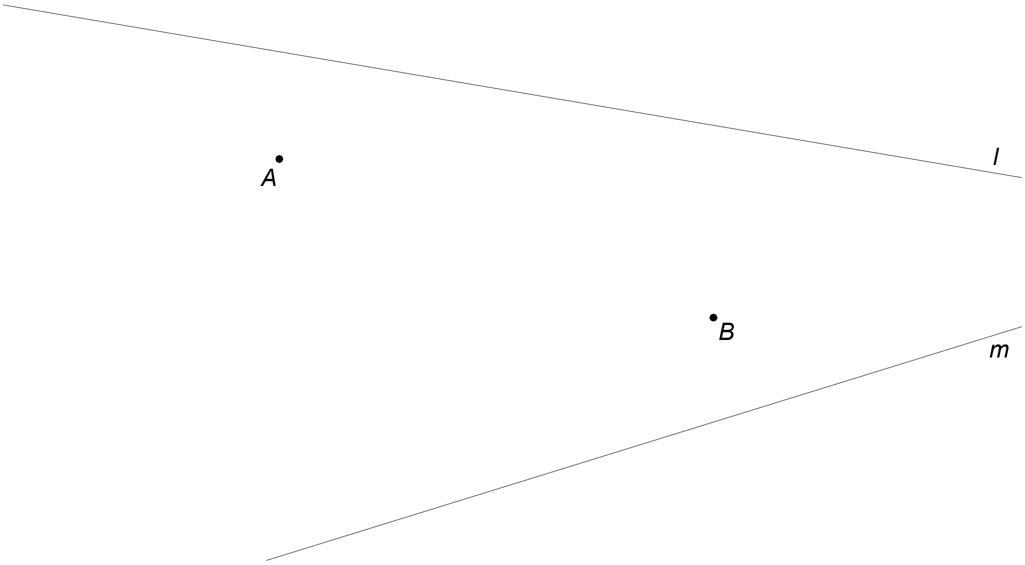
$$|YB| < |XB|.$$

4 You still need to find a proof!

Hint :
Bring a third route from *A'* to *B* into the game! First, walk partof the way from *A'* to *X* and go straight on until you hit line *YB*. Then follow the rest of the *Y*-route to *B*. Compare that route with both other routes.

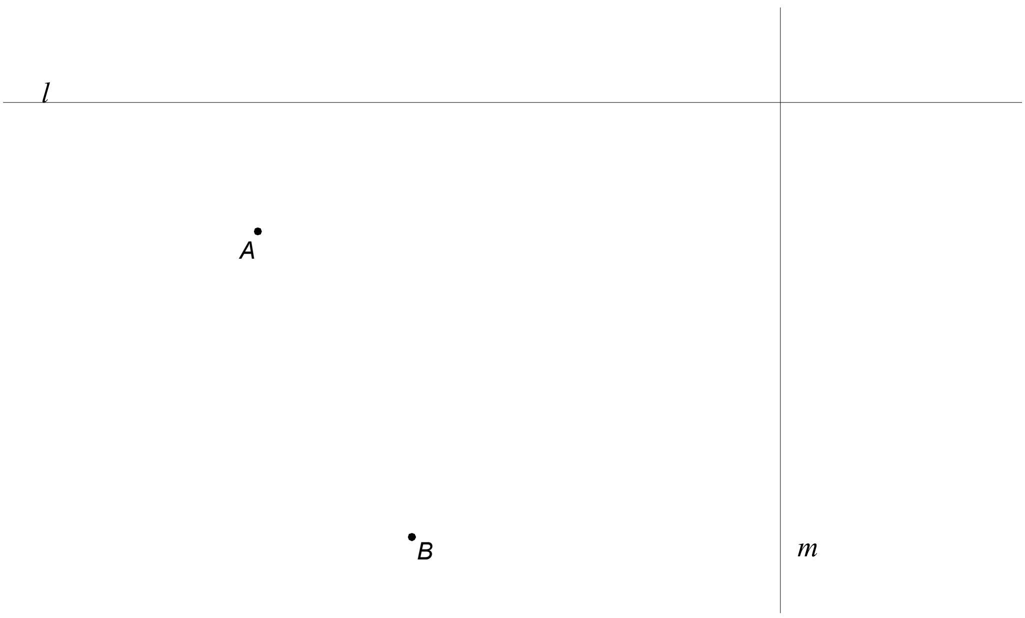
more lines

5 Here you see two lines *l* and *m* and two points *A* and *B*.



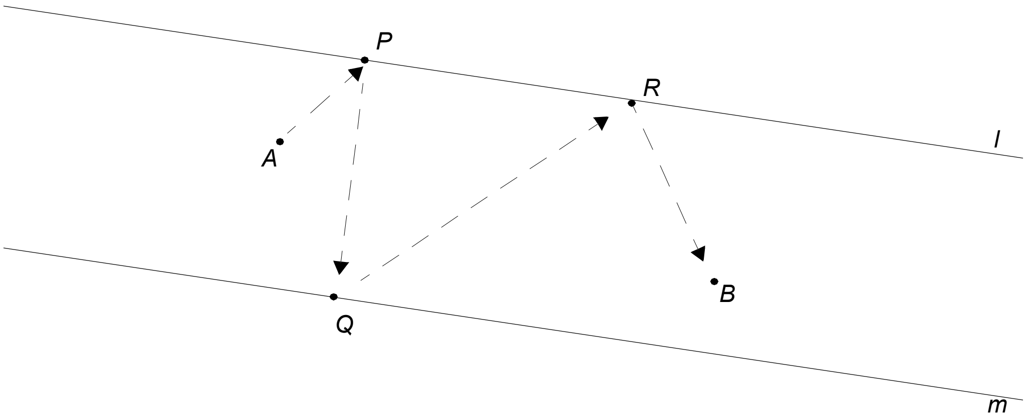
a. Determine the shortest path from *A* via *l* and *m* to *B*, in this order; first make a sketch and then think about how you can make use of reflection(s).
b. Also construct the road from *A* via *m* and then via *l* to *B*.
c. Finally: the shortest path from *A* via *l* and *m* back to *A*.

6 In this next situation the lines *l* and *m* are perpendicular.



a. Again construct the minimal route *APQB* with *P* on *l* and *Q* on *m*.
b. What do you notice about the directions of *AP* and *QB*?

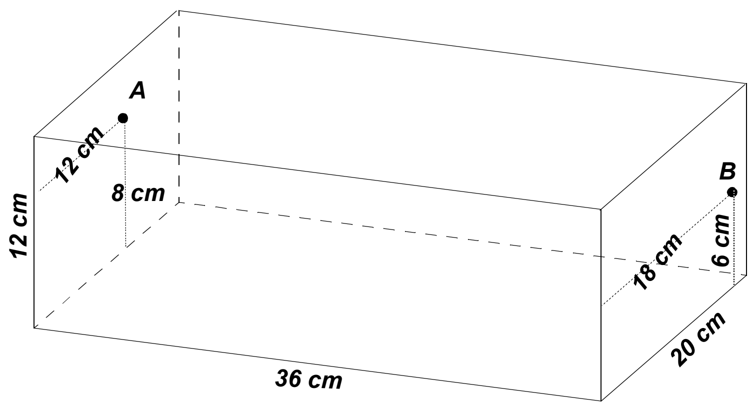
7 In the situation below l and m are parallel.
A route has been sketched from A to l, to m, again to l and only then to B. The consecutive contact points with l and m are P, Q and R, but this is not the shortest route.



a. Suppose Q has already been found. How to find R to make Q-R-B as short as possible?
b. What about the angles of PQ and QR with the line m?
c. Think harder, and think back to the shoelaces; now construct the shortest path precisely.
d. Also indicate why, in the ultimate solution, AP and QR run parallel and so do PQ and BR.

shortest route on a shoebox

8 An ant is sitting on the back of this (shoe)box of 12 by 20 by 36 cm at point A.



Determine the fastest route from A to B for this ant along the outside of the box.

First try to find a way yourself, before you use the following hint.

hint *On worksheet E (page 129) you find a possible net for part of the box. Now it looks like earlier problems. It is just an idea, but this net might not help you to find the shortest path...*

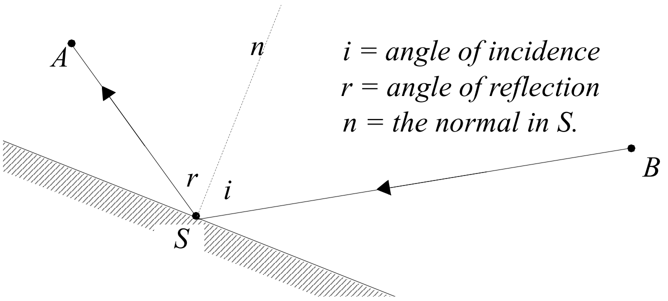
The principle of Fermat

The reflection principle applied to shortest paths via a straight line, and then you found equal angles.
There are also situations where you know that there are equal angle,s and thus you could work in the opposite direction.

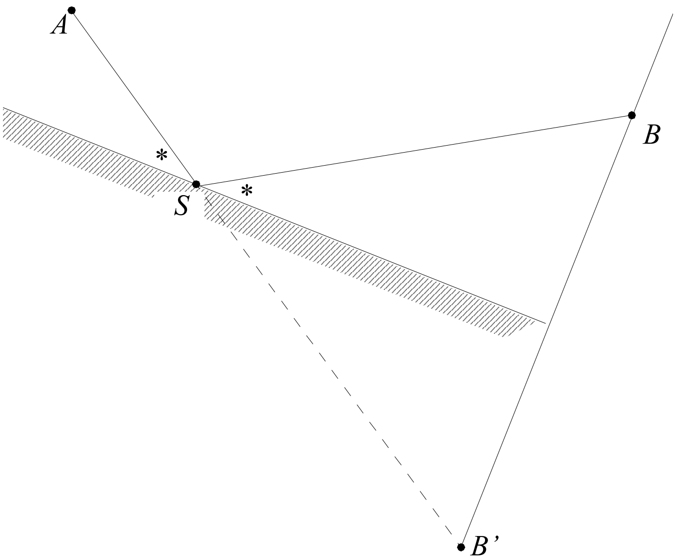
You know that for the reflection of a ray of light the principle of

angle of incidence
is angle of reflection

holds.
See the illustration.
The *normal n* is the line perpendicular to the mirror.



Our figures looked a lot more like the figure on the right, where it is also indicated how you can find the reflection line from *A* to *B*.
We looked at the angles which *AS* and *SB* make with the *mirror*, not at the ones with the *normal* on the *mirror*.



You see that also that

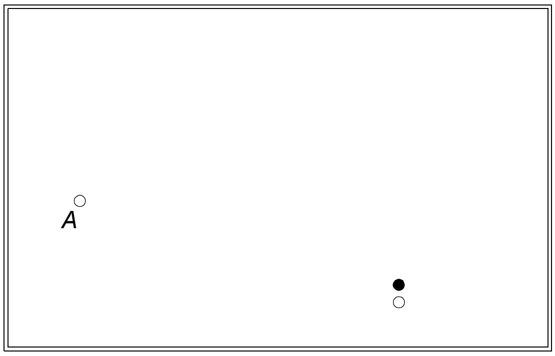
the lightray chooses the shortest path to go from B to A via the mirror.

This is the so-called principle of Fermat.

billiards problems

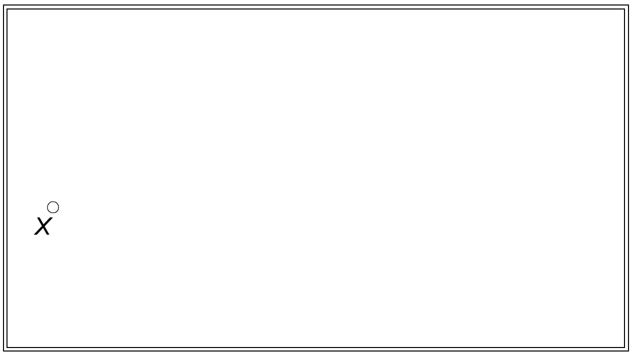
In billiards the balls bump against the so-called *cushions* (the edges round the green cloth) and go on from there.
In this part we assume that this happens according to the reflection principle. This is not true, say experienced billiards players, especially if some *effect* (spinning of the ball) is added to the shot or effect arises when touching the cushion.
If you think this is becoming a bit theoretical, you could also think of a room with mirror walls instead of billiards, and then you'd change from ball routes to light paths.

9 Here a billiard table is sketched in top view. Your cue-ball is *A*. It is a three-cushions billiards game, i.e. the shooter's cue ball must contact the cushions of the table at least three times before first touching the *third* object ball of his shot. You can also hit the same cushion twice, as long as the cue-ball touches a cushion three times. When you hit the *second* ball is irrelevant; in this case ball two and three lie so close together that it is very likely you will hit the three cushions first before touching the other balls practically at the same time.
Now construct *two* different possible routes for ball *A*.



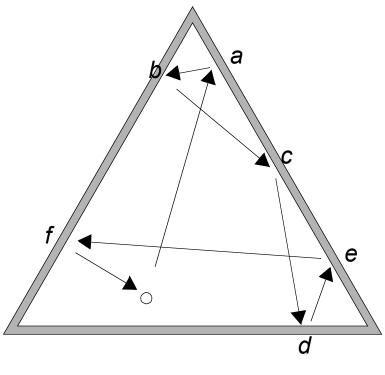
10 Play billiards with one ball! Show that you can shoot the ball in such a way that after contacting all four cushions, it returns to the same spot and continues in the same direction.

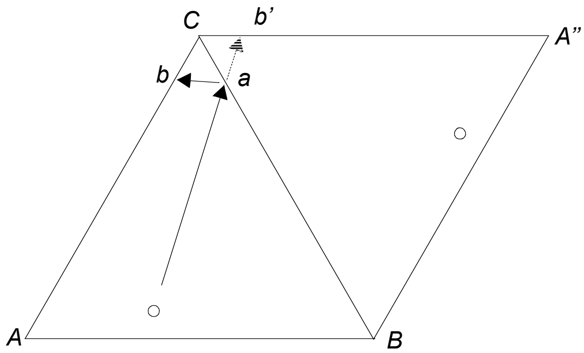
Hint: reflect the billiard table repeatedly to all sides. Set *A, B, C, D* on the vertices and reflect these letters at the same time. Then it is easier to indicate the reflected *X*-positions.



extra

11 We want to draw the exact six-cushion path in this triangular billiards. On the right, part of a solution is suggested. Complete it, if necessary use worksheet F (page 131).





summary of chapter 6

shortest connection

In this chapter you used the fact that the shortest connection between two points is a straight line.

principle of Fermat

You noticed that you could find the shortest path via one or more lines by applying the reflection principle. This principle is known as the *principle of Fermat.*

reasoning with angles

Therefore you needed to reason with angles in various situations.
It is time we pick up the how and why of that reasoning.
This will happen in part two of Advanced Geometry: *Thinking in circles ans lines*.

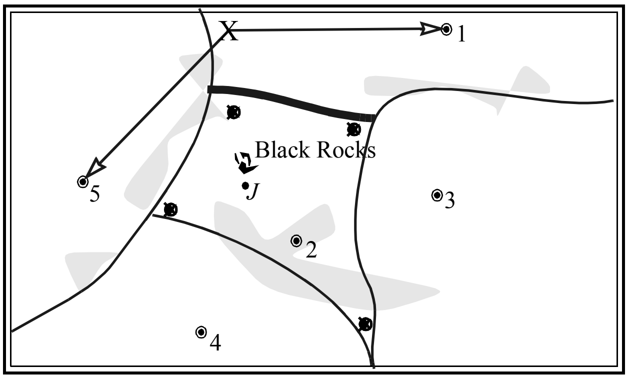
Example Solutions

General remarks

- Different solutions are possible for many exercises. Thus your answer will often differ from the solution given here, without it being wrong. In most cases, the example solutions will be sufficient to determine whether you answer is acceptable.
- Some comments with these solutions will give a deeper explanation. You will see this when you encounter them.
- The original drawings from the books were used in these solutions. That is why the solutions look prettier than is expected of you. You yourself will have to copy figures and then only sketch the necessary things. Do maintain clarity in your sketches.
- A few times you will not find an answer here, for example, when you were asked to make a figure with the computer. Sometimes you will find a printed screen. It will help you to check whether you are heading in the right direction, but of course you do not need to reproduce such pictures in your notebook, unlessabsolutely necessary.

Chapter 1: Voronoi-diagrams

1: In the desert

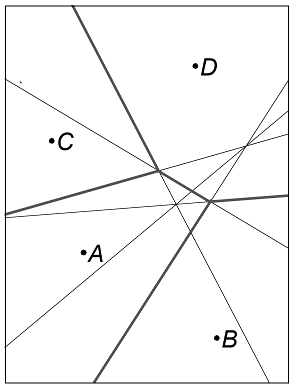


1 **a.** 2
b.
c. See figure.
(It does not need to be exact, that is for later.)
d. Choose to which well you go.
e. Yes. See point *X*. The arrows are of equal length and the distances to well 2,3 and 4 are larger.
f. Yes. Very far to the southeast.
g. No. Not for 3 and 4. (It does hold for the extensions of the edge.)
h. Straight. (This will be investigated thoroughly later.)

2: The edge between two areas

2 That is a good name. If we were dealing with the partitioning of for example oil fields, you definitely would get conflicts on the edges, because the points on the edges belong to both fields.
By the way: Geologist can partly explain why most of the oil under the Nord Sea is in the middle, at equal distances from the coasts.

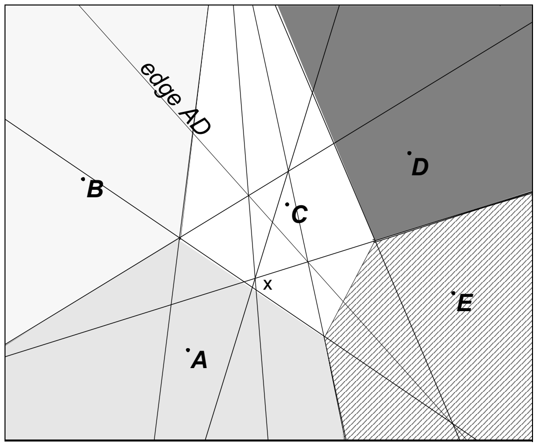
3: More points, more edges



3
a. 6. For every pair one. Thus 3 + 2 + 1 = 6.
b. Sometimes 3 folding lines intersect. For the other intersections 2.
c. The one between *B* and *C*.

4 **a.** 4 + 3 + 2 + 1 = 10.
b. It is closer to *D*; you can tell by the edge *BD*.
c. It does not belong to *A* (edge *AC*).
Also not to *D* (edge *AD*) and not to *E* (edge *AE*). Remains *C*.
Be careful: Not belonging to *B* because of edge *BD* only means: the distance to *D* is smaller than to *B*. It does not mean that the distance to *D* is the smallest.

d.



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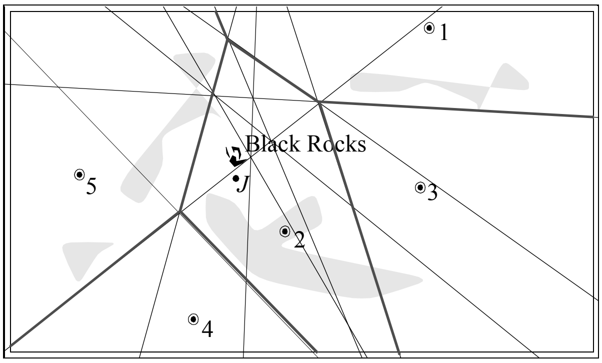
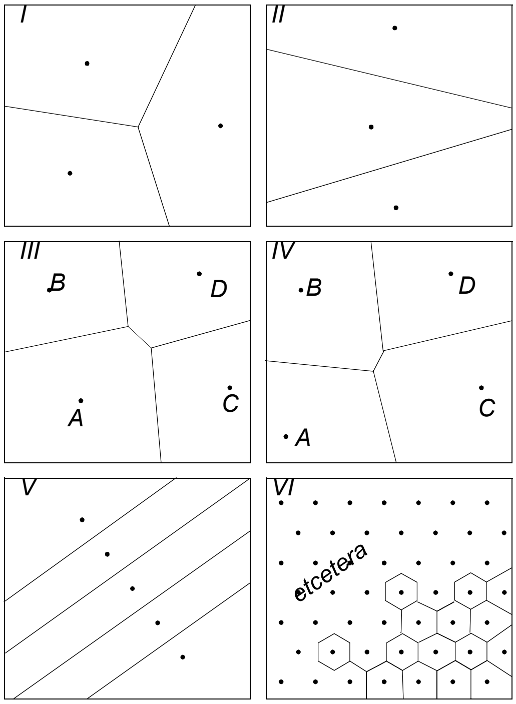


4: Voronoi-diagrams, centers, edges

6 **a.**



b. That point lies on equal distances from the centers.

c. Yes, but that lies outside the figure.

d. For IV point *A* is further away. Therefore the cells round *B* and *C* adjoin instead of those round *A* and *D*.

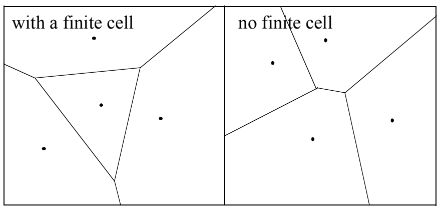
e. The edges are parallel. Ribbon villages in Noord-Holland have such fields.

f. The honeycomb.
Basalt (solidified lava) often shows these hexagons. Famous examples are the Giants Causeway in Northern Ireland and the island Staffa in Scotland.

7

a. 4.

b.



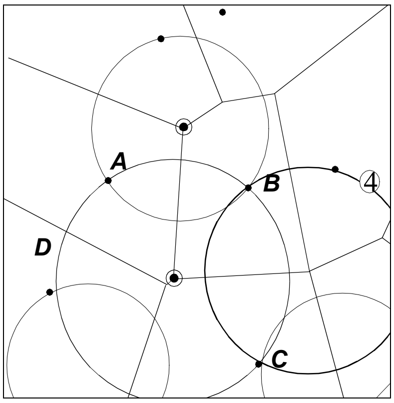
c. 20 points on a circle or on a straight line. Or on a different figure, which does not have bays, for example an oval. It does not matter how they lie on that circle, line or curved figure.

d. On the edges. (We will investigate this later on.)

5: Three-countries-points, empty circles

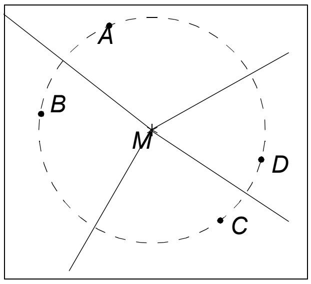
8 **a.** It has equal distances to the three cities.
b.
c.

9 **a.** See ⦿ ⦿

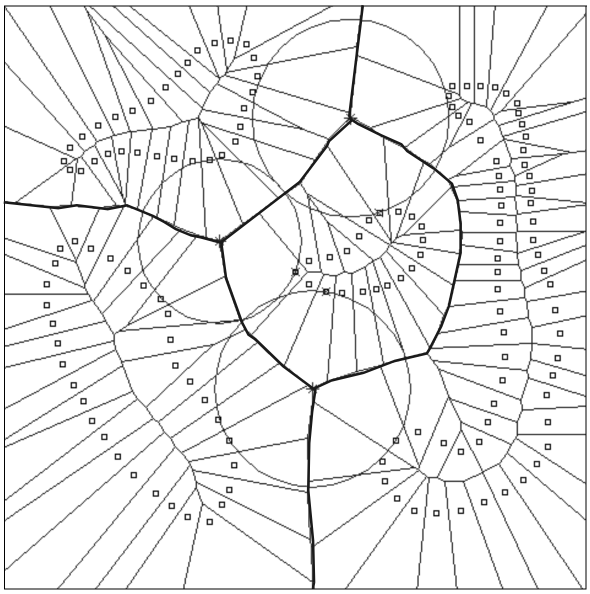


b. See figure.
c. There are always three.

10 **a.** *D* lies closer to *M*. Thus the circle through *A*, *B* and *C* is not a largest empty circle.
b.
c.
d. If there are three centers, there always is a three-countries-point (except if the centers lie on one line).
If a four-countries-point needs to occur, the fourth center needs to lie on the circle through the three other points. This happens by coincidence. Thus it is something special.



11
a.
b. Where the three colored edges converge.



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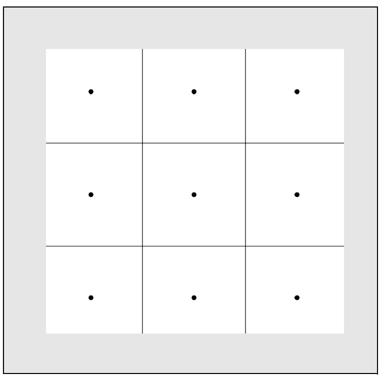


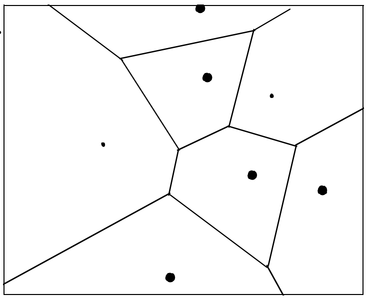
6: Chambered tombs in Drente; reflecting

13 a. The centers all lie at the same distance from the edge and the edge is perpendicular to the connection line.

b. It is not completely accurate, but it should be so.

c. The round dot closest to the east edge.

14 .



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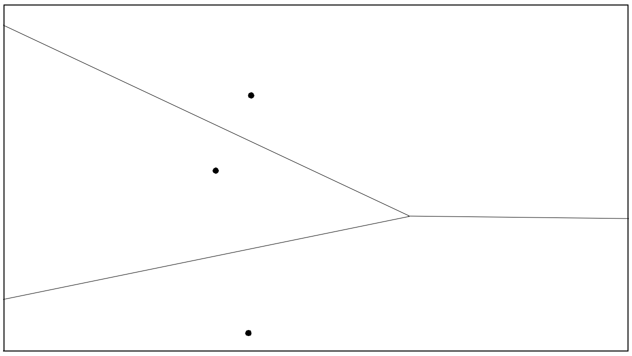

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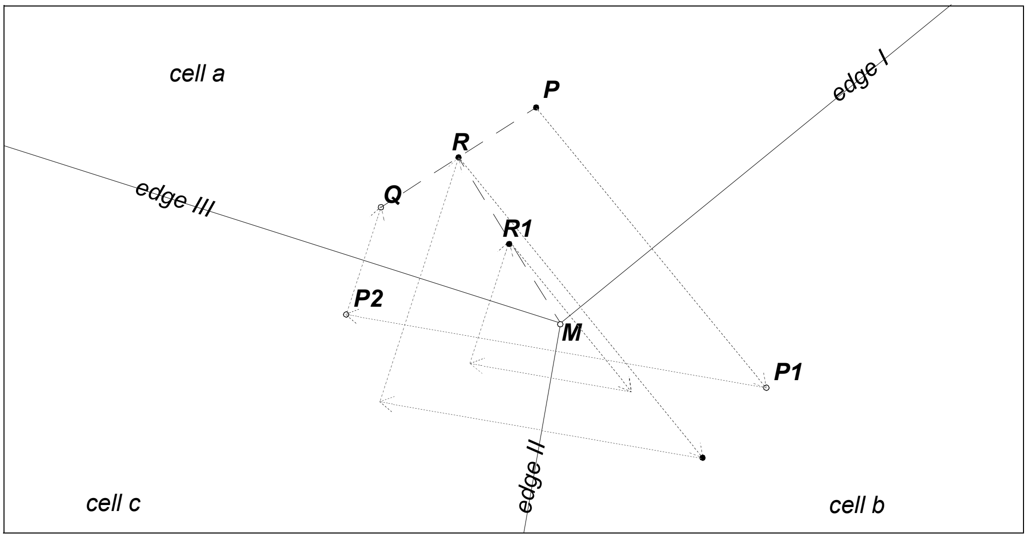


a. *P* and *Q* are different. If *P* was the center of cell *a*, then P_2 should be the center of cell *c*. This is not possible.

b. *S* is equal to *R*.

c. See R_1.

Chapter 2: Reasoning with distances and angles

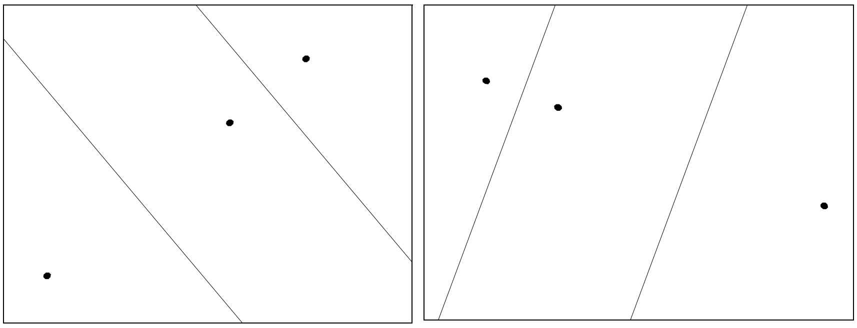
7: Introduction: reasoning in geometry

1 a. That it is a line and not part of an area, that it is a straight line, that the points lie symmetrical in relation to the line (There probably is more!)

b. The symmetry was used to find the missing centers.

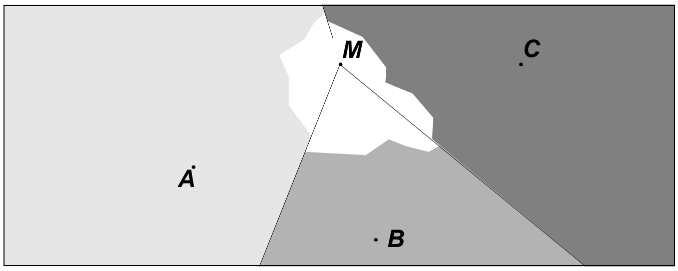
8: An argumentation with three-countries-points and circles

2 a.



b. The three centers lie on one line.

3



a.

b. (distance M to A) = (distance M to B)
(distance M to B) = (distance M to C)
From this directly follows:
(distance M to A) = (distance M to C).

c. Thus M lies on the Voronoi-edge of A and C.

d. Yes, all of the three edges pass through M.

4

a. The distances from M to A, B and C are all three equal. Thus the circle with center M, which passes through A, also passes through B and C.

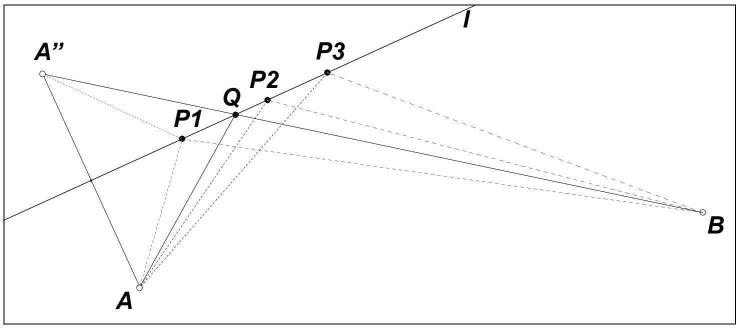
b. That was the largest empty circle round the three-countries-point M.

5

a. That the two edges indeed have a point of intersection.

b. No, apparently not. If the intersection of the AB-edge and the BC-edge *exists*, then it also lies on the AC-edge. The shape of the edge is irrelevant for the argumentation, only the distance equalities matter. (That the intersection exists, follows from rectilinearity and non-parallelism.)

9: Shortest paths and triangle inequality



6

a. Via P_2. But it differs very little.

b. Since $d(A, P_1) = d(A', P_1)$.

c. Draw the straight line A’B. The intersection with l is Q.

d. *Example one*: Getting water from river l and bringing it to village B.
Example two: l is a railroad line: the cities A and B get one station together.

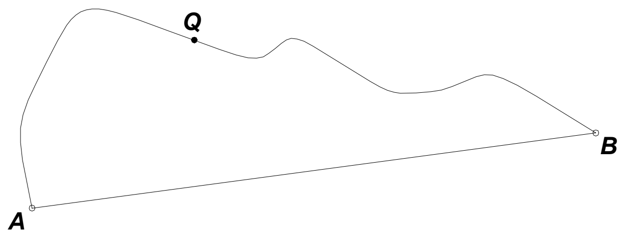
7

a. From P to Q is as far as from Q to P.

8

a. For example on $A'P_1B$.

b. $d(A', Q) + d(Q, B) = d(A', B)$.



9

a. Something with a route that does not consist of straight pieces.

b. It only deals with distances.

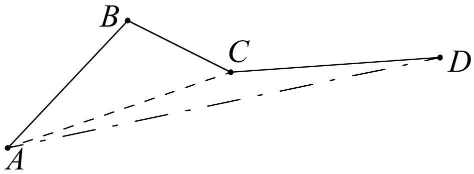
10

a. You could think of the Pythagorean theorem.

b. What is this Pythagorean theorem based on?
But also: how does the triangle inequality follow from the Pythagorean theorem?

11 Draw AC and apply the triangle inequality twice: to AD with a detour via C and to AC with a detour via B:

$$|AD| \le |AC| + |CD| \le |AB| + |BC| + |CD|$$



10: The concept of distance, Pythagorean theorem

12 If in triangle ABC angle B is right, then:

$$d(A, C)^2 = d(A, B)^2 + d(B, C)^2.$$

13 a. $d(A, Q)^2 = d(A, P)^2 + d(P, Q)^2$.

b. Since $d(P, Q)^2 > 0$, $d(A, Q)^2 > d(A, P)^2$ and since distances are non-negative: $d(A, Q) > d(A, P)$.

14 a. There are many ways to do this. Also put letters in the areas of the squares and the triangular areas.

Area(AEBF) = Area(CHDG)

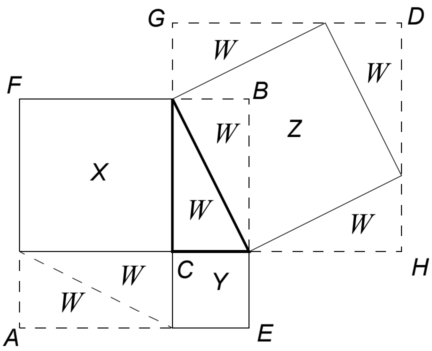
Thus*:*

$X + Y + 4 \times W = Z + 4 \times W$

Thus

$X + Y = Z$

b. That the sides are equal and perpendicular to each other.



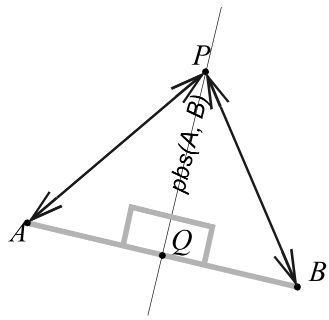
15 You do the same as in exercise **6**, for the case that *A* and *B* coincide.

11: Properties of the perpendicular bisector

16

a. = green. Equal line segments and angles of 90°.

b.  = red. Equality of the two line segments needs to be proven.



c. Since $\angle PQA = 90°$:

$d(P, A)^2 = d(P, Q)^2 + d(Q, A)^2$

Since $\angle PQB = 90°$:

$d(P, B)^2 = d(P, Q)^2 + d(Q, B)^2$

Since we can use that $d(Q, A) = d(Q, B)$, we conclude:

$d(P, A) = d(P, B)$.

d. Yes, perpendicularity is used in Pythagoras, and the equality of the lengths *AQ* and *BQ* is used in the deduction.

17

a. To prove:

$d(Q, A) \neq d(Q, B)$.

But we actually prove:

$d(Q, A) < d(Q, B)$.

b. *R* is on the perpendicular bisector, thus:

$d(R, A) = d(R, B)$.

c. Triangle inequality for *Q*, *R*, *A*. *R* is not on *QA*, thus we have a true inequality:

$d(Q, A) < d(Q, R) + d(R, A)$.

d. $d(Q, A) < d(Q, R) + d(R, A) =$

$d(Q, R) + d(R, B) = d(Q, B)$.

the last equality holds, since *R* does lie on *QB*.

18 Finding the shortest route via a line. Here you could go from *A* to *Q* via the perpendicular bisector of *AB*.

12: From research to logical structure

19

a. *To prove*

$d(P, A) + d(P, B) + d(P, C) + d(P, D) < d(Q, A) + d(Q, B) + d(Q, C) + d(Q, D)$

b. *Proof*:

Use the triangle inequality in triangles *ACQ* and *DBQ*.

$d(P, A) + d(P, C) = d(A, C) \leq d(Q, A) + d(Q, C)$.

and

$d(P, B) + d(P, D) = d(B, D) \leq d(Q, B) + d(Q, D)$.

In at least one of the cases we have an inequality, because else Q would be equal to P. By adding, what needed to be proven follows immediately.

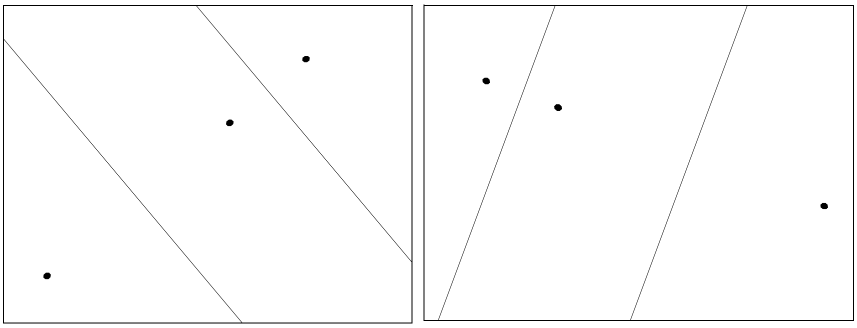
20 *Idea*: Use that B and D lie ***on*** the ‘Voronoi-edge’ of A and C.

Proof:

$d(A, B) = d(B, C)$ (this is given). Thus B lies on *pbs(A, C)* according to theorem 3.
$d(A, D) = d(D, C)$ (this is given). Thus D lies on *pbs(A, C)* according to theorem 3.
Combine: the line through B and D *is* the *pbs(A, C)*. Thus BD is perpendicular to AC.

21 **a.** By demanding that A, B and C form a triangle.

b. The perpendicular bisectors are perpendicular to that line and are thus parallel. Re-using an old figure.



c. Then the perpendicular bisectors are not parallel and thus do intersect.

22

a.

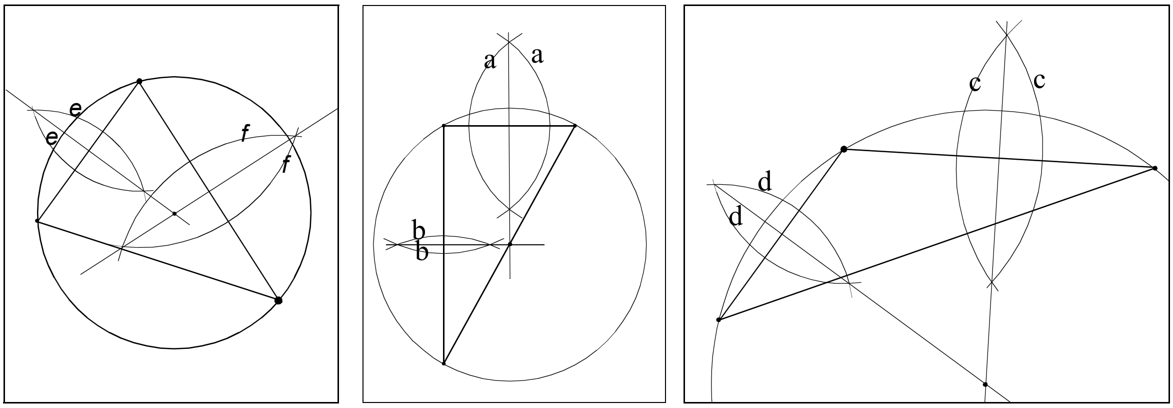
3b, writing down the equalities in distance	<>	step 1 and step 1bis
3b, conclude third equality	<>	connection step
3c conclusion that M lies on the third Voronoi-edge	<>	conclusion step.

b. step 1 and 1bis: from *on perpendicular bisector* to *equality of distances*,
The conclusion step is the other way around: from *equality of distances* to *on perpendicular bisector*.

23 P and Q lie at equal distances from A and B and thus on the perpendicular bisector of A and B. Theorem 3.

24 With the techniques of the two circles, Dürer constructs the perpendicular bisectors of a and b, and also those of b and c. He calls the intersection d. That intersection is the center of the circle through a, b and c.

25 Three examples of the construction: (circles with equal letters have the same size.)

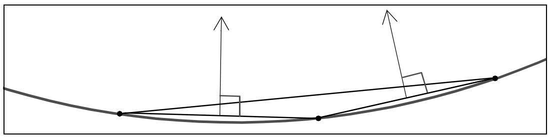


Acute triangle:
center circumcircle lies inside the triangle.

Right triangle: center circumcircle lies on the triangle

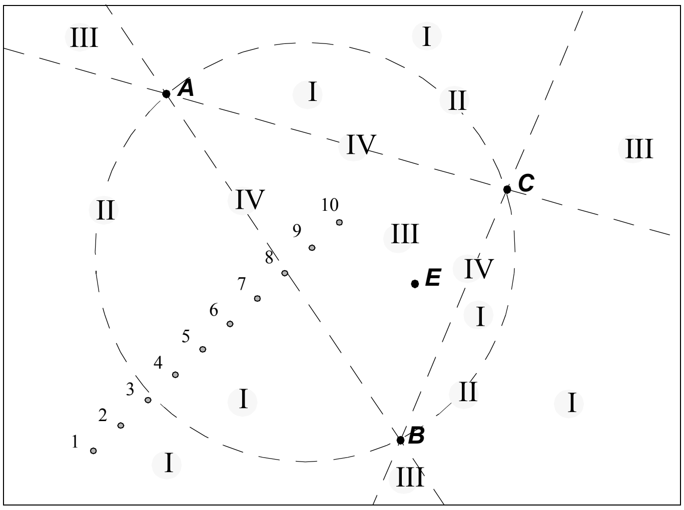
Obtuse triangle:
center circumcircle lies outside the triangle

26 Choose three points on the circle. Construct the circumcircle of that triangle. You already have the circle, but the construction also gives the center!



Chapter 3: Computer practical Voronoi-diagrams

13: Introduction



1 **a.**
b.

2 **a.**
b. the angle bisector of the angle of the two lines through the points.
c.

3

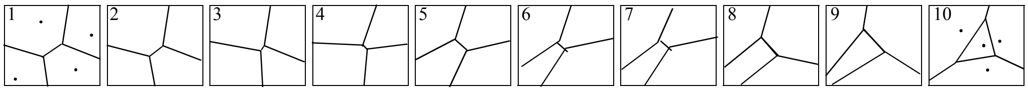
14: The influence of the fourth point

4

a. For example point *E* shown above on the right.
b. The button `circle(A, B, C)` does exactly that.
c. Then there are two perpendicular edges. Thus three points need to lie on one line. Thus the three lines *AB*, *CB* and *AC*. All of them!
d. Type III.

5 **a.**

Voronoi:



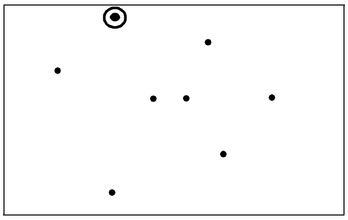
	1	2	3	4	5	6	7	8	9	10
Type:	I	I	I	I	I	I	I	IV	III	III

b. When *D* passes through the circle.

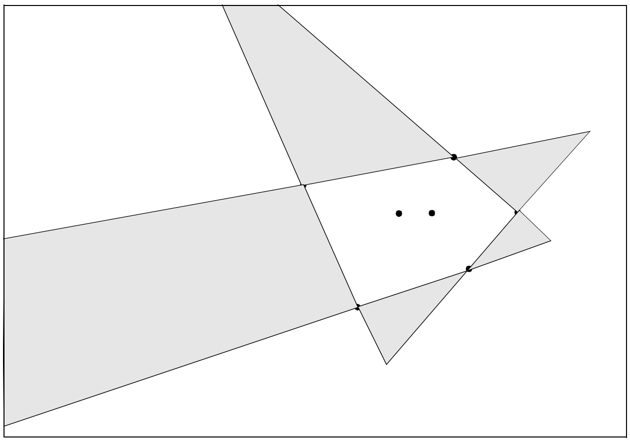
6

a. See above.
b. Type II and IV only occur when *D* lies on certain lines lies. This is rare.

15: Infinitely large cells



7 **a.**
b. Five rays from one point.
c. A rubber band.
d. No new infinitely large cells.
e. For example:
f.



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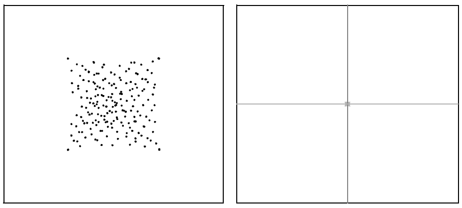
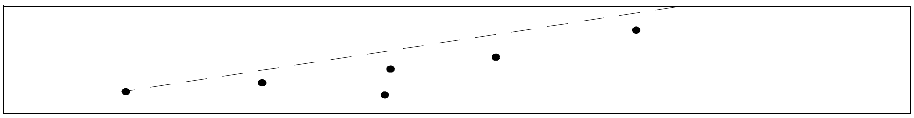


9 **a.** They adjoin.
b. Those centers lie at equal distances from *A* and of *B*.
c. Those cells still adjoin. The perpendicular bisector of AB continues past the center of the circle through *A*, *B* and *E*. There it is the Voronoi-edge of *A* and *B*. If you take more points in the half-plane of the line *AB* on the side of *C* and *D*, it still holds that far on the perpendicular bisector of *AB* lie points for which the closest centers are *A* and *B*.

10

11



16: Extra assignments Voronoi-diagrams

12 No, the computing time increases faster.

13

a. The Voronoi-cells round Maastricht and Assen adjoin.
They lie next to each other on the hull. You can also see this by drawing the line through those two centers.

14 **a.**

b.

15 a.

b. In certain places new railroad tracks are built which do not give much improvement. For example Assen-Maastricht. Leeuwarden-Haarlem via Alkmaar is not such a bad idea; it must be possible to go over the Afsluitdijk. Assen-Leeuwarden: do not do it, it is a nice quiet area and it does not yield a lot of profit from the detour via Groningen. Voronoi diagrams also do not ask for a high speed train through the Green Heart of Holland!

16

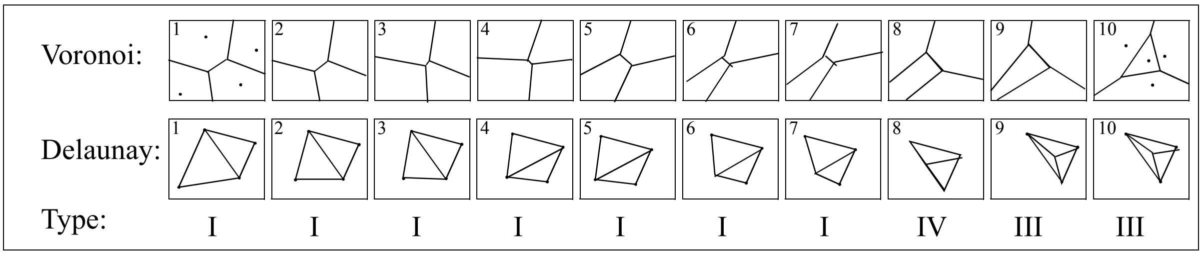
a. A connection line between two centers in the Delaunay-triangulation says per definition: these centers have adjoining cells. Thus there is a Voronoi-edge. Also the other way around.

b. The Voronoi-edge is a segment of the perpendicular bisector of the connection line of centers, which is exactly the connecting line segment in the Delaunay-triangulation. Since the edge can be more like a segment of the perpendicular bisector, the intersection does not have to exist. In the example of the map: Arnhem-Maastricht, Den Bosch-Den Haag.

c. Round a vertex you find three cells which are neighbors. Thus there you have a triangle of Delaunay-connections, and the other way around. Thus there are as many triangles as vertices.

d. Adjoining centers on the hull have adjoining cells around them. This has been established before. Thus the line segments of the hull are certainly part of the Delaunay-triangulation.

17



18 a.

b. You see the line which approximately runs the same distance from both banks. This is a smart way to find it.

c. The points, which lie at equal distance to both banks.

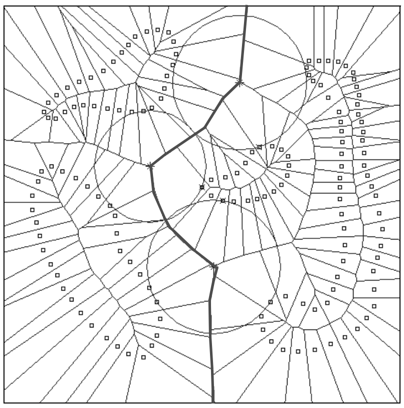
19

a.

b.

- This matters a whole lot. You could deduct from this that such small islands play a much too important role. One abandoned rock changes the whole partitioning. Furthermore, those remote islands are of great military significance.
- This will differ less, but still quite a lot in the far north.

20



a.
b. Along the lines which lie as far as possible from the islands: follow the edges of cells that belong to two islands. You have a choice, but you choose west for the middle island, since east puts you along the elongated island for a long period of time.

21

a.
b. infinitely large. Bend on the left side.
c. the cell gets smaller.
d. It looks like a *parabola*.
e. Till the middle of those line segments. Since the line is then perpendicular to the connection of such a middle with the center left on the line.
f. This results from **e**. Because the tangent line is perpendicular to the radius. (We will return to this later!)
g. It is perpendicular to the connection line of centers.

22

a. An oval form arises, or something that looks like a parabola, but much wider. (the correct names are ellipse and hyperbola; we will return to this later).
b. If F comes close to the circle, the figures are narrow.

23

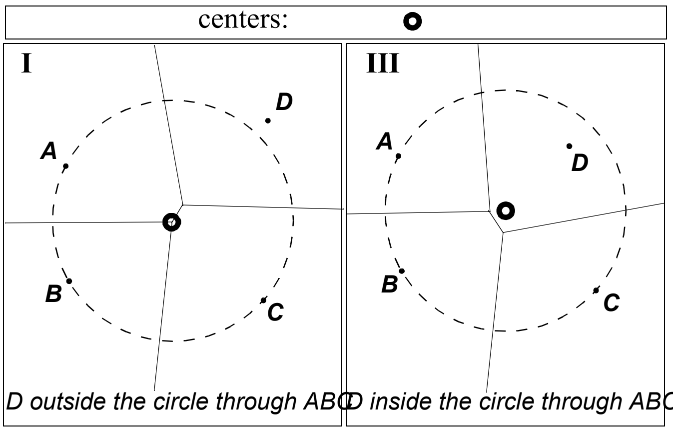
a. You get a more delicate honeycomb.
b.

24

Chapter 4: A special quadrilateral

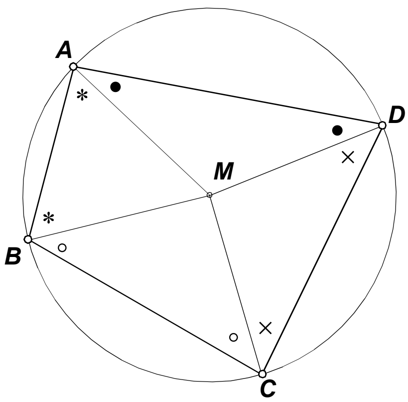
17: cyclic quadrilaterals

1 a. The center of the circle lies on equal distances from A, B, C and D. That point belongs to all four cells.
b. ○
c. D lies inside the circle through A, B and C; therefore the center of that circle lies in the cell of D.



2

a.



b. the four triangles are isosceles.

c. The same symbols can be found at A and C together as at B and D together. Thus ∠A + ∠C = ∠B + ∠D**.**

18: Scrutinize proving

3 **a.** Here M does not lie inside the quadrilateral.

b. When comparing the symbols in the angles. This is simply impossible.

4

a. $\angle ADC$. For the size of the angles holds: $\angle ADC = \angle D_1 - \angle D_2$.

$\angle D_1 = \angle A_1$, $\angle A_2 = \angle B_1$, $\angle B_2 = \angle C_1$, $\angle C_2 = \angle D_2$
(since the triangles DMA, AMB, BMC and CMD are isosceles

$\angle A + \angle C$
= (refinement angles)
$\angle BAD + \angle DCB$
= (subdividing angles)
$(\angle A_1 + \angle A_2) + (\angle C_1 - \angle C_2)$
= (use equal angles, working towards either A and C or B and D)
$(\angle D_1 + \angle B_1) + (\angle B_2 - \angle D_2)$
= (rearrange to lead to B and D)
$(\angle D_1 - \angle D_2) + (\angle B_2 + \angle B_1)$
= (use subdivision)
$\angle ADC + \angle ABC$
= (back to the needed angles)
$\angle D + \angle B$
= (calculation step*: $x+y = y+x$*)
$\angle B + \angle D$.

(Such motivations as in the last two steps need not to be indicated all the time!)

5 Replacing the minus signs by plus signs is enough.

6 **a.** M can still be on the quadrilateral.

b. Take as example M on DC.
In that case both proofs work.
(That is actually because then $\angle C_2$ and $\angle D_2$ are equal to 0. Then is does not matter whether it has plus or minus signs.)

7

a. Results are 180° or lie real close to that.

b. A reasonable statement is: the sum of two opposite angles in a quadrilateral is 180°.

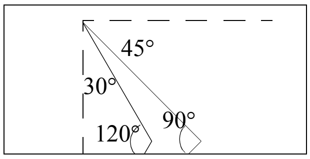
8 All four 360°:

a. 4 × 90° = 360°

b. 60° + 120° + 60° + 120° = 360°

c. 2 × 90° + 45° + 135° = 360°

d. the acute angles are (see figure) 90° – 30° – 45° = 15°.
90° + 2 × 15° + (360° – 120°) = 360°

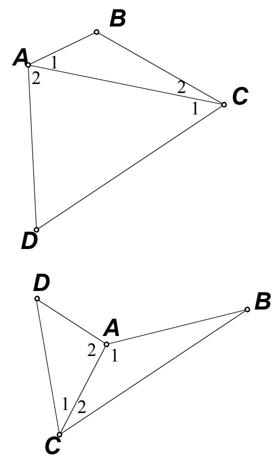


9 **a.**
b.
c.

Proof ONE

$\angle A + \angle B + \angle C + \angle D$
= (subdivision of the angles)
$\angle A_1 + \angle A_2 + \angle B + \angle C_1 + \angle C_2 + \angle D$
= (rearranging to triangles)
$\angle A_1 + \angle B + \angle C_2 + \angle C_1 + \angle D + \angle A_2$
= (sum of angles in triangle is 180×, twice)
180° + 180°
= 360°

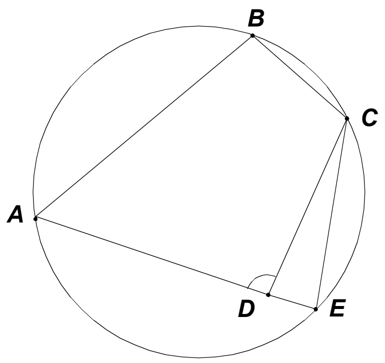
for 9 a and b



Proof TWO

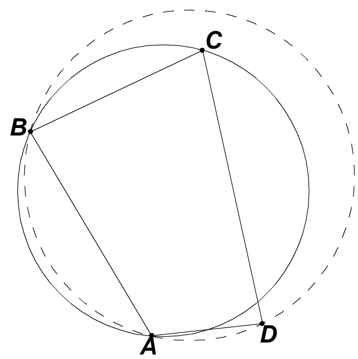
$\angle A_1 + \angle B + \angle C_2 = 180°$(sum of angles in triangle *ABC* is 180°)
$\angle C_1 + \angle D + \angle A_2 = 180°$(sum of angles in triangle *ADC* is 180°)
$\angle A_1 + \angle B + \angle C_2 + \angle C_1 + \angle D + \angle A_2 = 180° + 180°$ (combine)
$\angle A + \angle B + \angle C + \angle D = 360°$(rearrange)

10



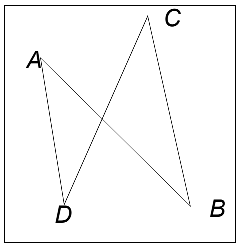
a.
b. $\angle DCE + \angle CED + \angle EDC = 180°$.
But also: $\angle ADC + \angle EDC = 180°$.
Thus: $\angle ADC = \angle DCE + \angle CED$
c. $\angle ADC > \angle CED$.
d. Since $\angle B + \angle E = 180°$ according to the temporary theorem of the cyclic quadrilateral, follows from the inequality in **b**:
$\angle B + \angle D > 180°$.

11



a. No other points but *A* and *B* can lie on the dotted circle as well as on the sketched circle, since then the circles would coincide according to theorem 5. Thus if *D* lies outside circle *ABC*, the whole arch *BAD* lies outside circle *ABC* and *C* lies inside circle *ABD*.
b. $\angle A + \angle C > 180°$.
c. Since the four angles together are 360°, this needs to be true: $\angle B + \angle D < 180°$.

12 *CD* then passes through *AB*; for example like this:

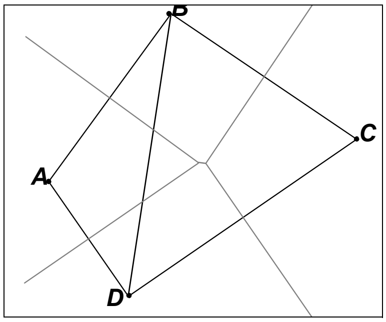


A **definition** of a useful kind of quadrilateral here could be:
A quadrilateral ABCD consists of four points, and the four line segments AB, BC, CD and DA, for which
no three points lie on one line,
the line segments AB and CD do not intersect
and the line segments AD and BC do not intersect.

13 a. In this case the angle at $\angle ADC$ on its own is already over 180°.

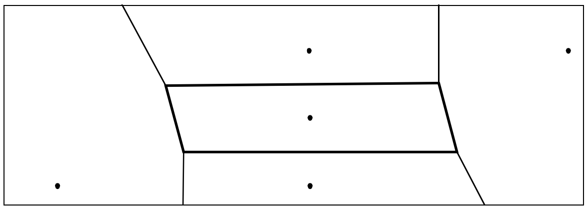
19: Using cyclic quadrilaterals

14



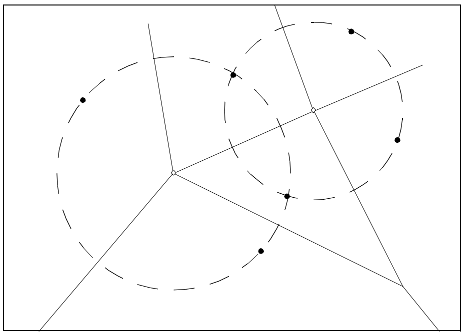
- **a.** according to the protractor is $\angle B + \angle D = 183°$. Thus *D* lies inside the circle through *A*, *B* and *C*. Cells *B* and *D* adjoin.
- **b.**
- **c.**
- **d.** You already knew which cells adjoined.

15 That is a mistake and here is a counter-example.



16

17



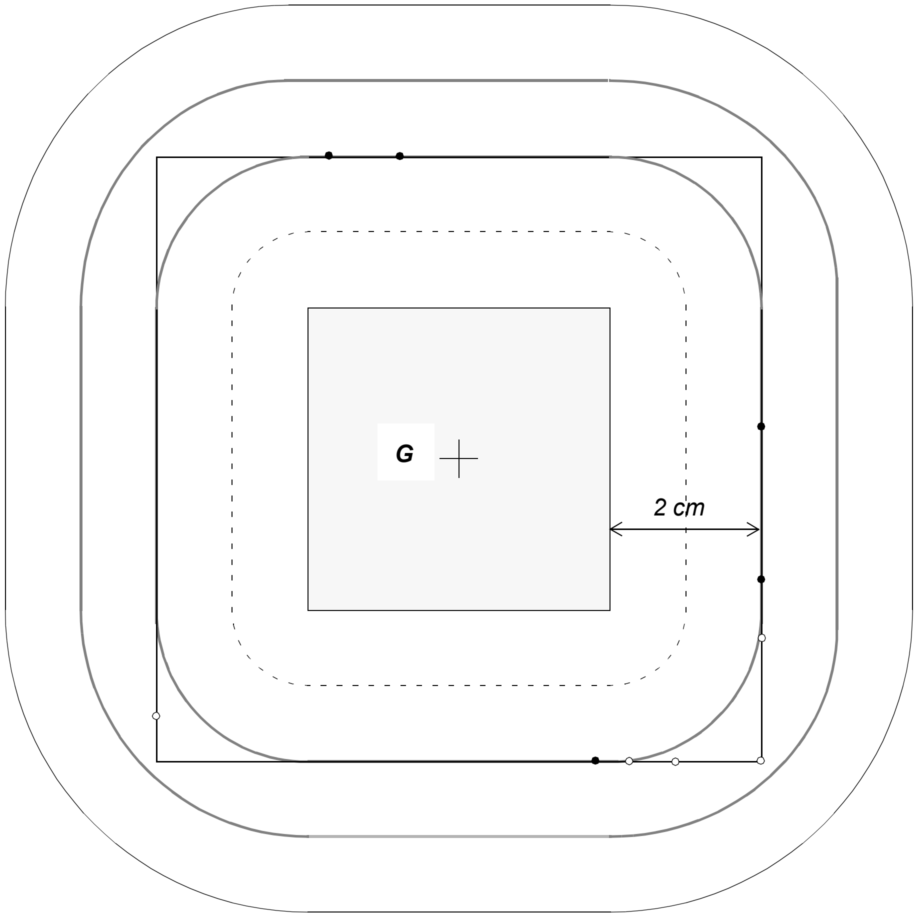
Chapter 5: Exploring isodistance lines

20: isodistance lines, distance to areas

1

- **a.** A circle with a radius of 12 miles.
- **b.** A circle with a radius of 12 miles + 680 meter.

2

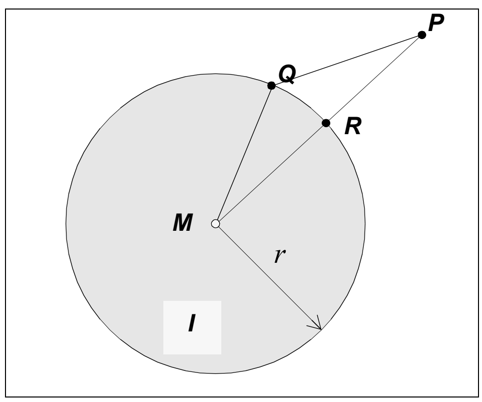


a. does: • does not: ○
b. iso-1 cm-line ; iso-2 cm-line ; iso-3 cm-line ; iso-4 cm-line .
c. *the iso-300-line is not a circle since it has four straight segments, each of 4 cm.*

3

a. Sketch the line through *P* perpendicular to *l*. In intersects *l* in *R*. $d(P, G) = d(P, R)$.
b. With the triangle inequality and with Pythagoras.

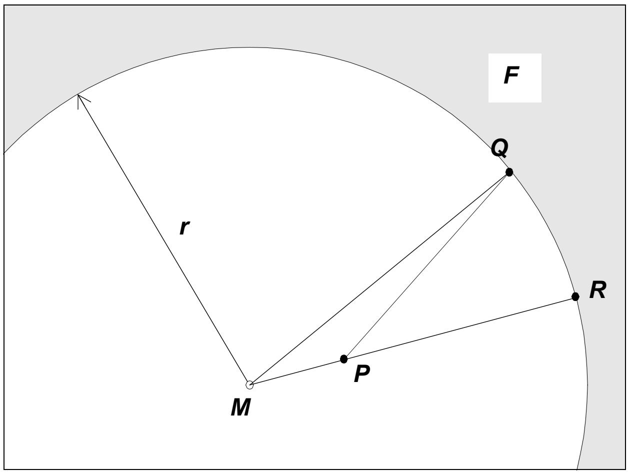
4



a. on the intersection of *MP* with the circle.
b. $d(P, Q) + d(M, Q) > d(P, M)$.
Since *Q* and *R* lie on the circle, holds:
$d(R, M) = d(M, Q)$.
$d(P, M) = d(P, R) + d(M, R)$ since *R* lies on *PM*.
Filling out gives $d(P, Q) + d(M, R) > d(P, R) + d(M, R)$.
Thus: $d(P, Q) > d(P, R)$.
c. Since $d(P, I) = d(P, R)$, holds:
$d(P, I) = d(P, M) - r$.

5

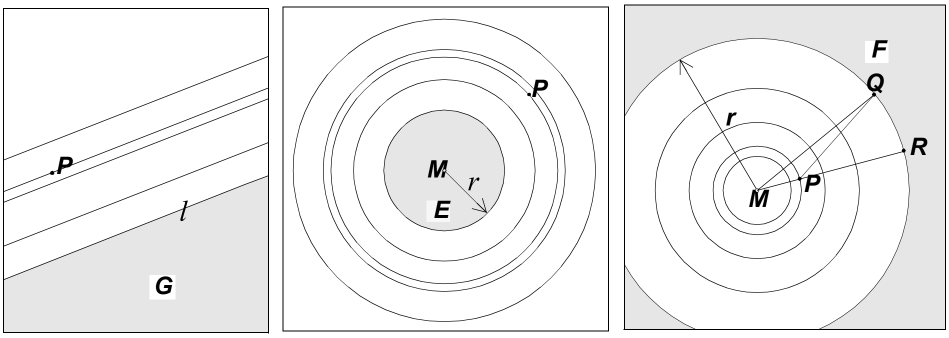
a.



b. the triangle inequality in triangle *MPQ* says:
d(M, R) = d(M, Q) < d(M, P) + d(P, Q).
Since also holds:
d(M, R) = d(M, P) + d(P, R)
needs to hold:
d(P, Q) > d(P, R).

c. *d(P, F) =r – d(P, M)*.

6 **a.** Reduced, scale 1 : 2.



b. From the expressions for the distance in the **c**-questions.
From those formulas, it follows that the points lie on a fixed distance to the circle and also lie on a fixed distance to the center *M*.
Elaborating on, for example, exercise 4:

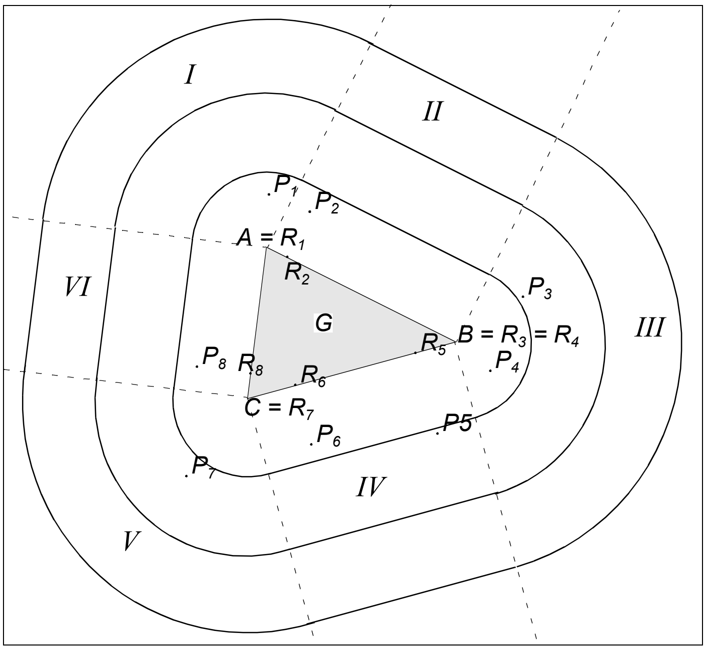
(1)	*d(P, E) = d(P, M) – r*	*(exercise 4 c)*
(2) **If** *P* lies on the iso-*a*-line:		
	d(P, E) = a	*(definition iso-a-line)*
(3)	*d(P, E) = d(P, R)*	*(according to exercise 4a)*

From 1, 2 and 3 follow:
d(P, M) = d(P, R) + d(R, M) = a + r
a + r is constant, Thus:
then *P* lies on a circle with radius *a + r* and center *M.*

Now we also need to show: *each point on the circle with radius a + r and center M, lies on the iso-a-line of E.*
That goes like this (written down in shorter form than the first half of the proof):
If *d(P, M) = = a + r,*
and *R* is the point on the circle on the line segment *PM*, then also:
d(P, E) = d(P, R) = d(P, M) – d(R, M) = a + r – r = a,
Thus *P* on the iso-*a*-line of *E.*

7 ***Scale 1 : 2.***

a.



b. The dotted lines are the edges of the zones in this drwaing.

c. Zone I, III and V (the zones at the vertices): segments of the circle around the vertex.
Zone II, IV and IV (the zones of the line segments): pieces of straight line parallel to the sides.

d. That the straight pieces have the same length. That the circles run smoothly into the straight pieces.

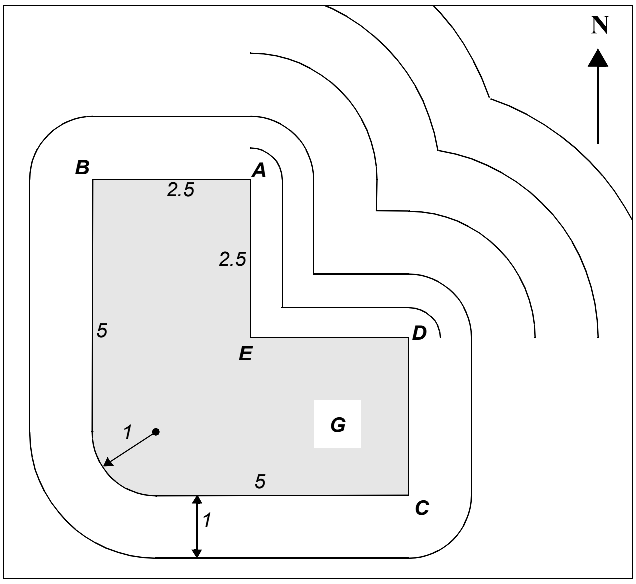
e. Only the arcs are extra compared to the triangle itself. These three arcs of each isodistance line can be put together as a circle. Thus the extra lengths are: 4π, 8π, 16π.

8 **a.** The points on the isodistance lines in the neighborhood of a cape all lie at the same distance from the cape. Thus the cape is the center of the circle on which that piece of isodistance line lies.

b. These lines are perpendicular to the pieces of edge, which lie next to the cape.

9

a.



b. On the angle bisector of angle *AED*

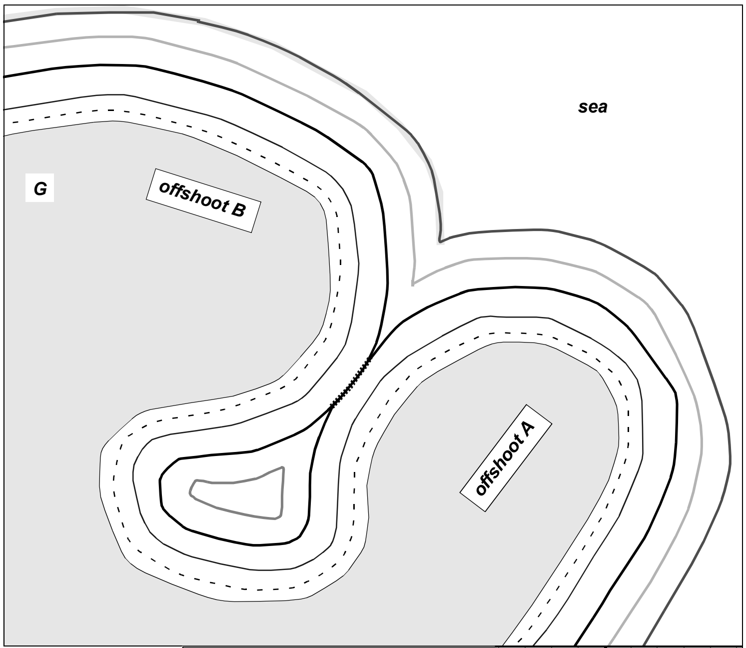
c. The iso-0.5-line and the iso-2-line show a kink of 90^{o}. The iso-3 and iso-4-line show kinks of approximately 108^{o} and 128^{o}.

d. Till 2.5. Because up to there the kink is formed by two straight line segments, which are perpendicular.

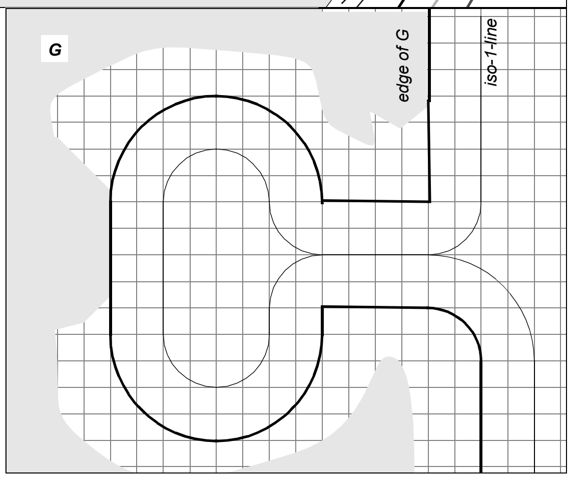
e. Since the kink is formed by the intersection of two circles and the isodistance line switches from one circle to the other in the kink, there always will be a kink.

10

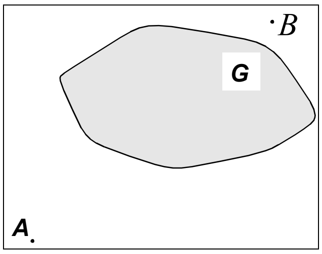
a. Something like the figure.
b. the iso-1.5-line falls apart into two pieces; the iso-0.2-line does not
c.
d.



11



12

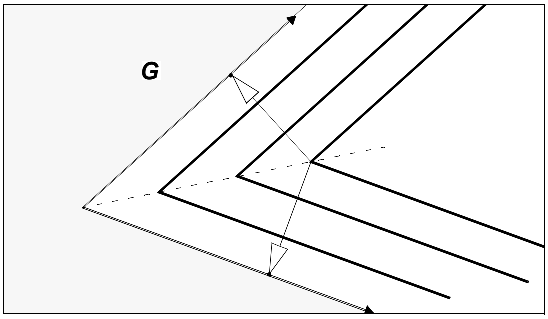


a. Yes.
b.

21: Angle bisectors

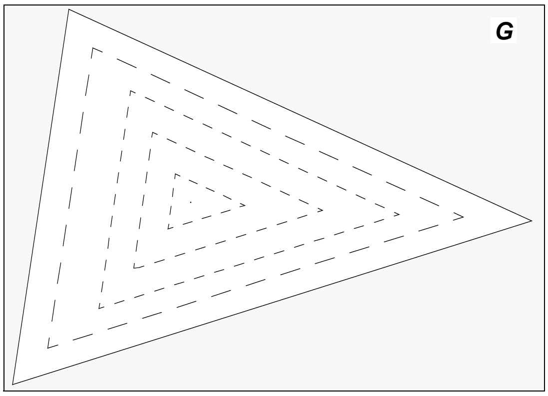
13 (scale 1:2)

a.
b. the angle bisector of the angle.
c. In this case 2.
d.



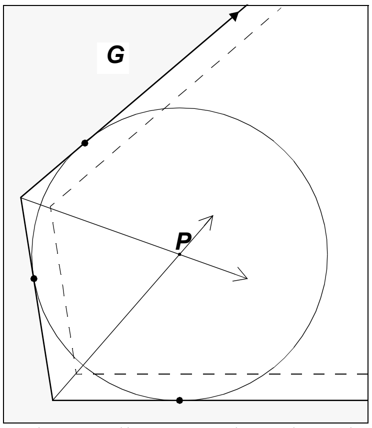
14

a.



b. They all are triangles with the same shape. The last triangle becomes one point.

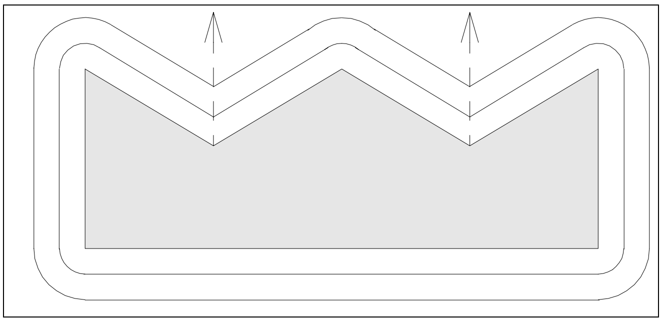
15



a. No. For larger distances there is only one kink.

b. Sketch the angle bisectors from both angles. If a point lies on one angle bisector, it has footpoints on two sides. If the point lies on both angle bisectors, it has footpoints on all three of the sides.

16

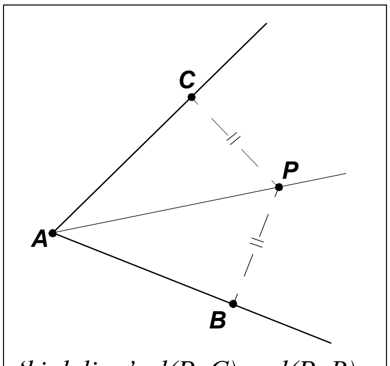


22: Theorems about angle bisectors

17 *Given:* P lies on the kink line of $\angle CAB$.

To prove: the line AP cuts $\angle CAB$ *in half.*

Proof: Since P lies on the kink line, like in the left figure, where C and B are footpoints of P, the triangles APC and APB are right. Since the hypotenuses are equal, and PC and PB also have the same length, the triangles are completely equal. Thus $\angle CAP = \angle BAP$.



'kink line': d(P, C) = d(P, B)

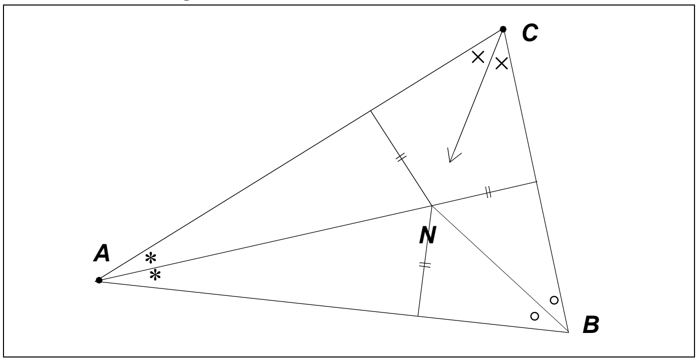
18 If two half lines a and b converge in an end point S and make an angle of less than 180°, the set of points for which holds:

$d(P, a) = d(P, b)$

is a half line c from S, which makes equal angles with a and b.
For points P inside the angle ASB hold: (Here A and B are points on a and b respectively, not equal to S)

if $d(P, a) < d(P, b)$, then $\angle ASP < \angle BSP$
if $d(P, a) = d(P, b)$, then $\angle ASP = \angle BSP$
if $d(P, a) > d(P, b)$, then $\angle ASP > \angle BSP$.

19 Let N be the intersection of the angle bisectors from A and B.



The proof that N also lies on the angle bisector from C, fits exactly into a scheme.

step 1

N on *angle bisector of* $\angle CAB$
Thus
$d(N, CA) = d(N, BA)$

step 1bis

N on *angle bisector of* $\angle CBA$
Thus
$d(N, CB) = d(N, AB)$

↓

connecting step

Thus

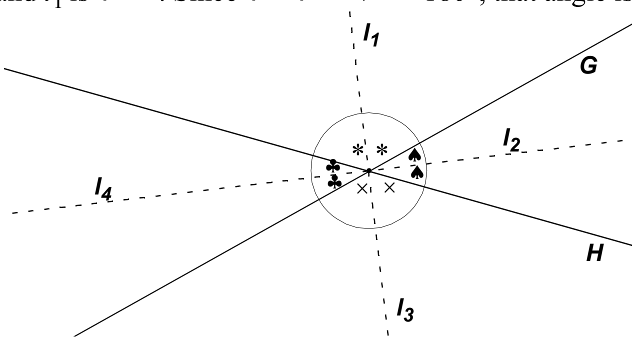
$d(N, CA) = d(N, CB)$

↓

conclusion step

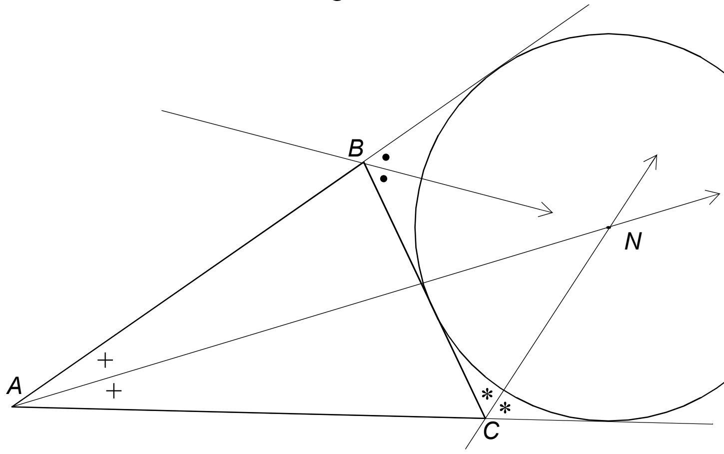
$d(N, CA) = d(N, CB)$
Thus
N on *angle bisector of* $\angle ACB$

20 the angle between l_4 and l_1 is ♣ + ∗. Since ♣ + ♣ + ∗+ ∗ = 180°, that angle is 90°. Likewise, the angle



between l_1 and l_2 is 90°. Thus in total is the angle between l_4 and l_2 180°. Thus the two halve l_4 and l_2 lines form one straight line.

21 Let *N* be the intersection of the external angle bisectors from *C* and the internal angle bisector from *A*.



Then again applies:

$$d(N, \text{line } AC) = d(N, \text{line } BC)$$

and also:

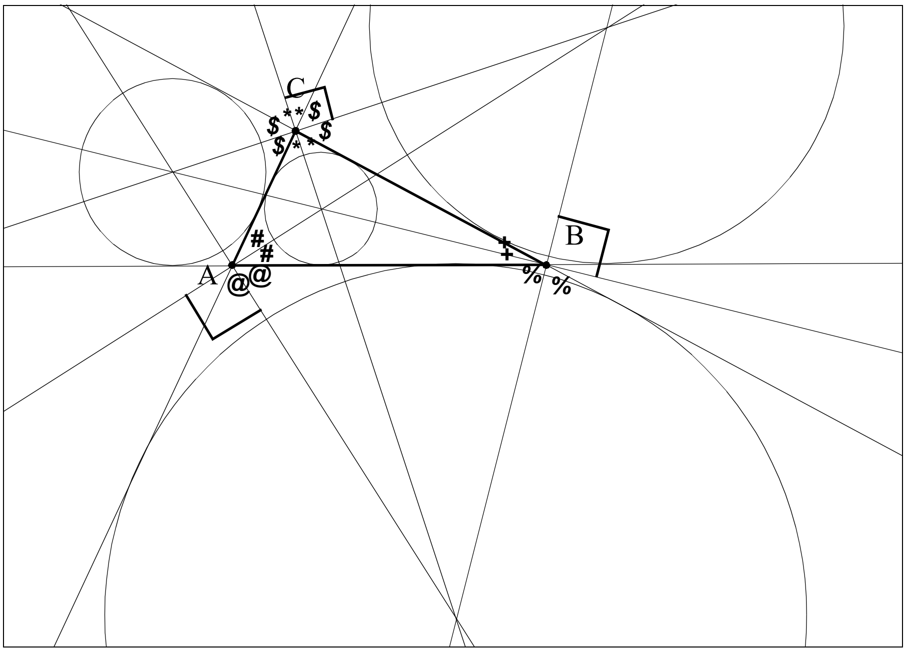
$$d(N, \text{line } AC) = d(N, \text{line } AB)$$

Thus:

$$d(N, \text{line } BC) = d(N, \text{line } AB).$$

Thus *N* also lies on the external angle bisector from *B*. (After all *N* lies outside the triangle).

22



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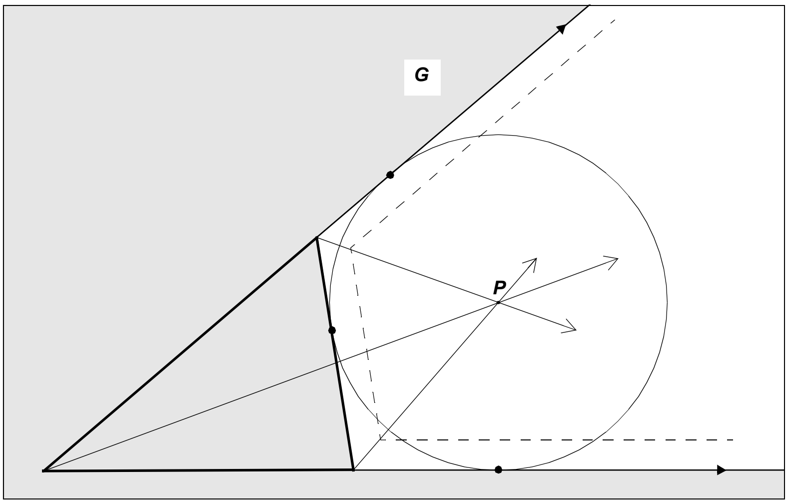
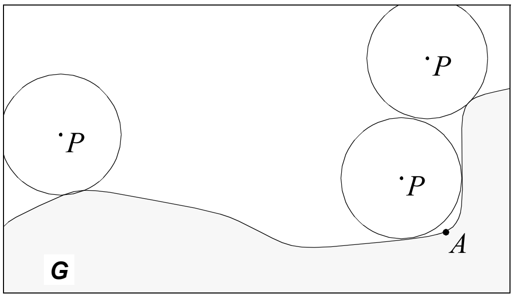


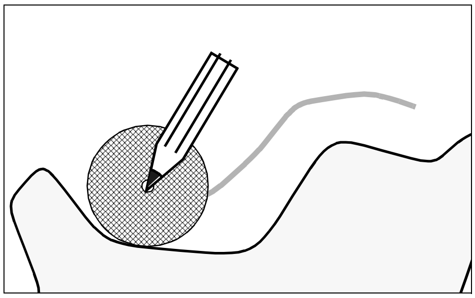
24 Mostly not.
For isosceles triangles the angle bisector between the equal sides does go through the tangent point on the third side. Thus for equilateral triangles it happens three times.

23: Bumping circles

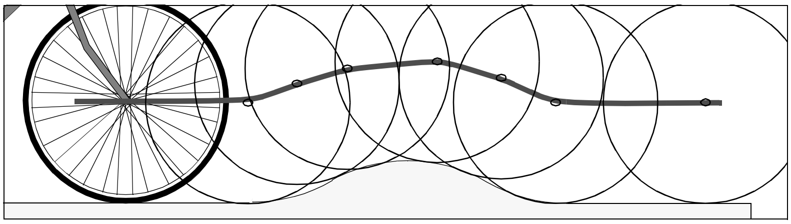
25 a.
b. yes.

26

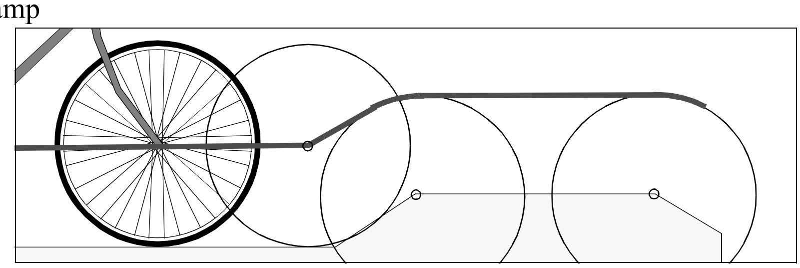




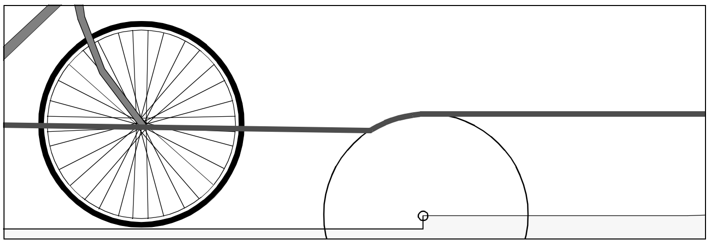
27
a.
1. bump



2. speed ramp



3. low pavement



b. Two. At the beginning and at the end.
c. It does not matter very much when you look at the kink angles. If the pavement was a little lower, the blow on the speed ramp would be bigger.

28 Yes, in exercise **25**. But not very formal.

29 It does not apply to the two bays with narrow entrances. Exercise **10** and **11**.

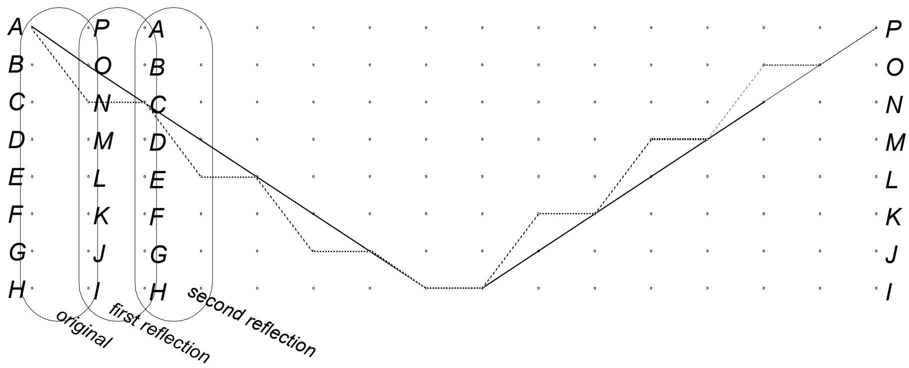
Chapter 6: shortest paths

24: From shoelaces to shortest paths

1

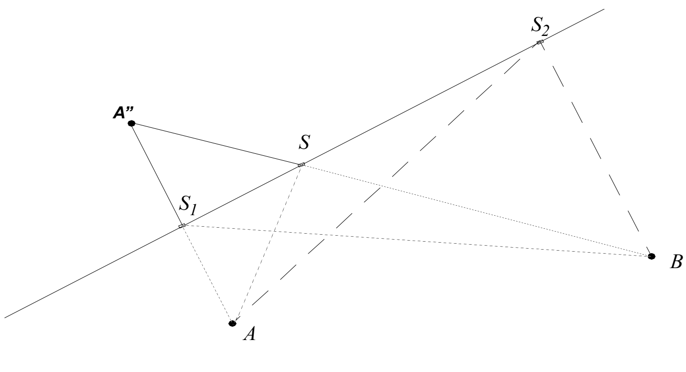
2

a.



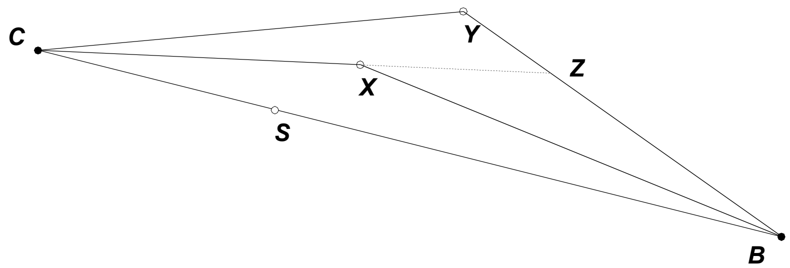
b. The lengths stay equal when reflecting.
c. The American. Since it uses the shortest path from A to the lowest point I and also back up from H to P.

3



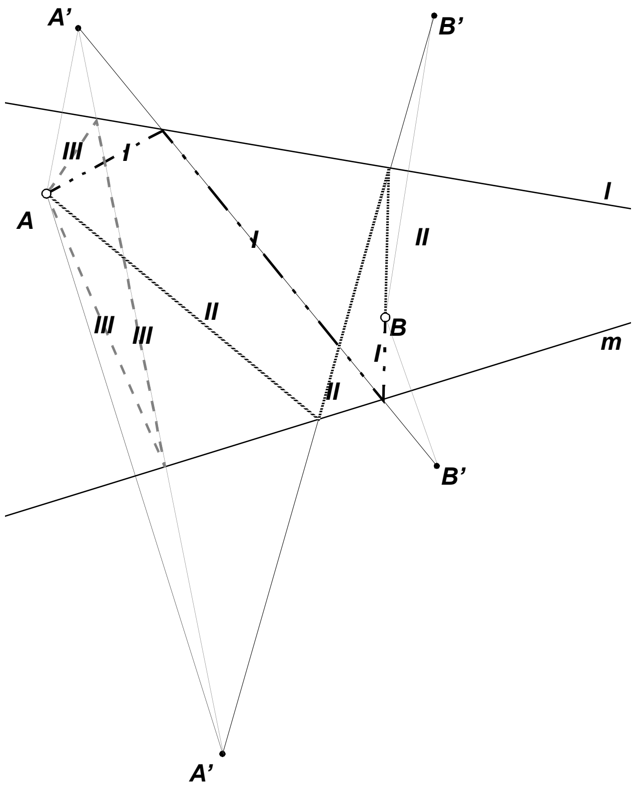
a.
b. *$\angle A'SS_1 = \angle S_2SB$ (Opposite angles are equal for intersecting lines).*
$\angle A'SS_1 = \angle ASS_1$ (Since the triangles are equal, due to the reflecting).
Thus $\angle ASS_1 = \angle S_2SB$.

4 All that is needed is mentioned in the figure and *A*' is renamed as *C*.



Given: *X* lies inside the triangle *CBY*.
To proof: *d(C, X) + d(X, B) < d(C, Y) + d(Y, B)*
Proof:
Compare the *X*- route with the route via *Z*:
d(C, X) + d(X, B) < d(C, X) + d(X, Z) + d(Z, B) = d(C, Z) + d(Z, B)
Compare the *Z*- route with the route via *Y*:
d(C, Z) + d(Z, B) < d(C, Y) + d(Y, Z) + d(Z, B) < d(C, Y) + d(Y, B)
Done!

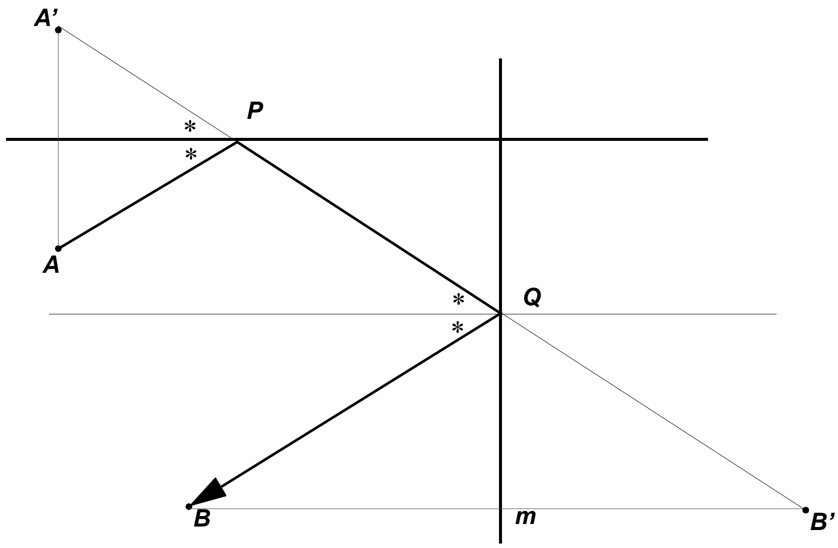
5
a. route I - I - I.
b. route II - II - II
c. route III - III - III.



6

a.

b. *AP* and *QB* are parallel. In the figure all angles with * are equal.



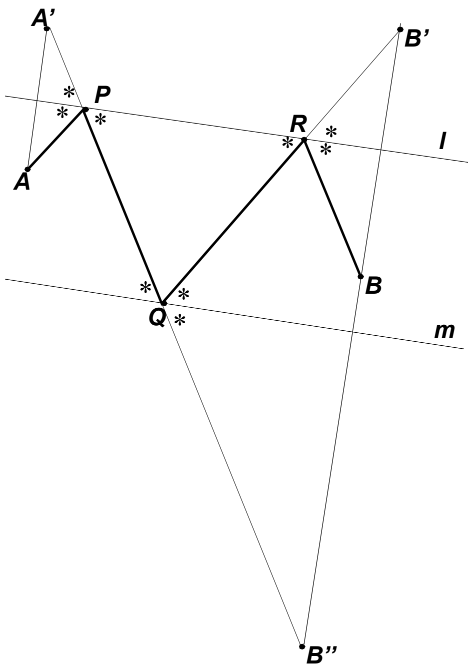
7

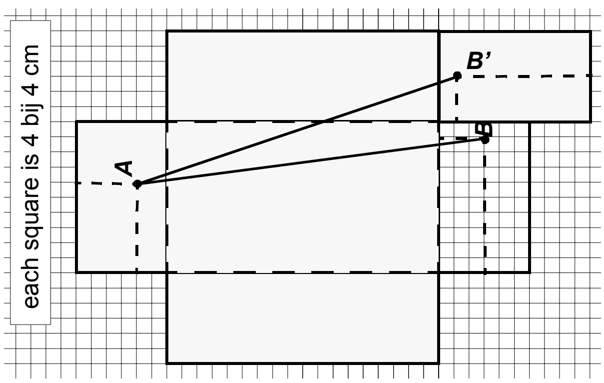
a. On the reflection of *B* in *l*. Call that *B'*.

b. On the reflection of *B'* in *m*. Call that *B"*.

c. *A'* is the reflection of *A* in *l*. *A'B"* is sketched. You find *P* and *Q*.
Then draw *QB'*. You find *R*.

d. Since all angles with a∗ are equal. There are many *Z*-angles!

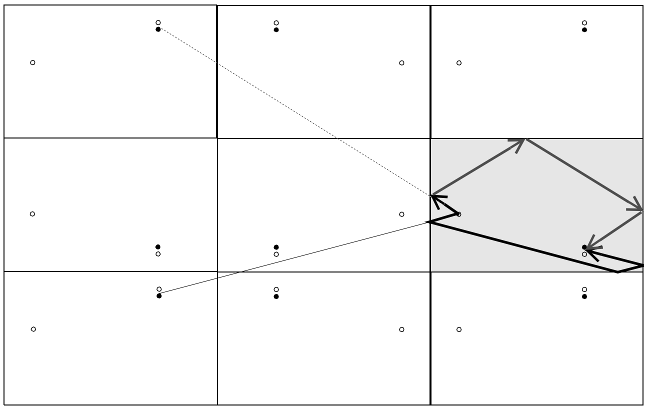




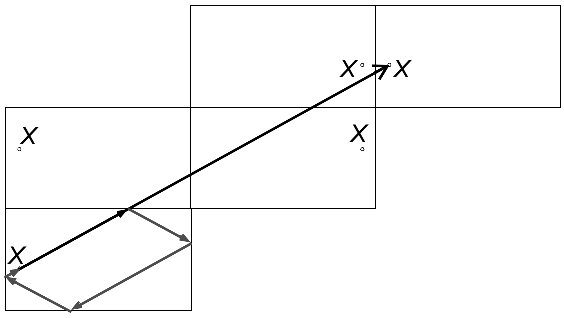
8 On the net shortest routes can be sketched as straight lines.
However, there are many ways to make a net.
In the figure connecting the square with ant B to the long side yields minor profit.

$$|AB] = \sqrt{46^2 + 6^2} = 46{,}389 \qquad |AB'| = \sqrt{42^2 + 14^2} = 44{,}271$$

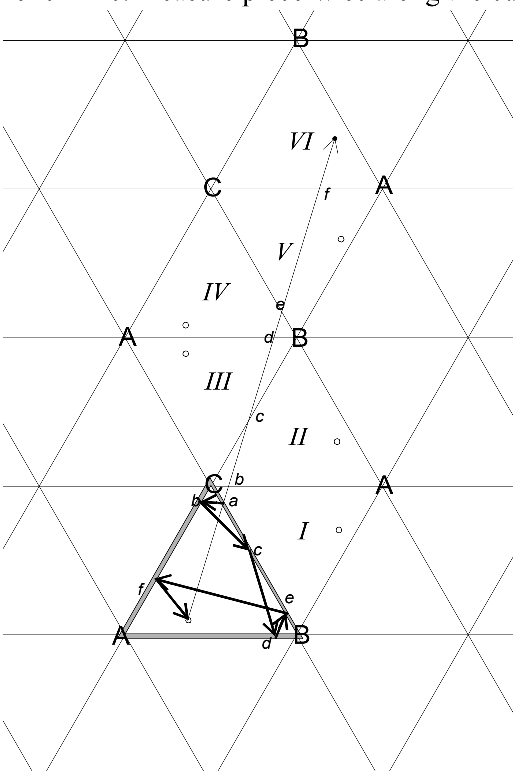
9 The figure shows the billiards reflected, reflected, reflected, reflected.
Each line from *A* to a reflection of the goal, which is reached via three cushions is correct. One example is shown.



10 The solution to this question is practically the same as the last one.
Of course there are other possibilities, which you can find by making a different mirror-mirror-mirror-mirror-image of *X*.

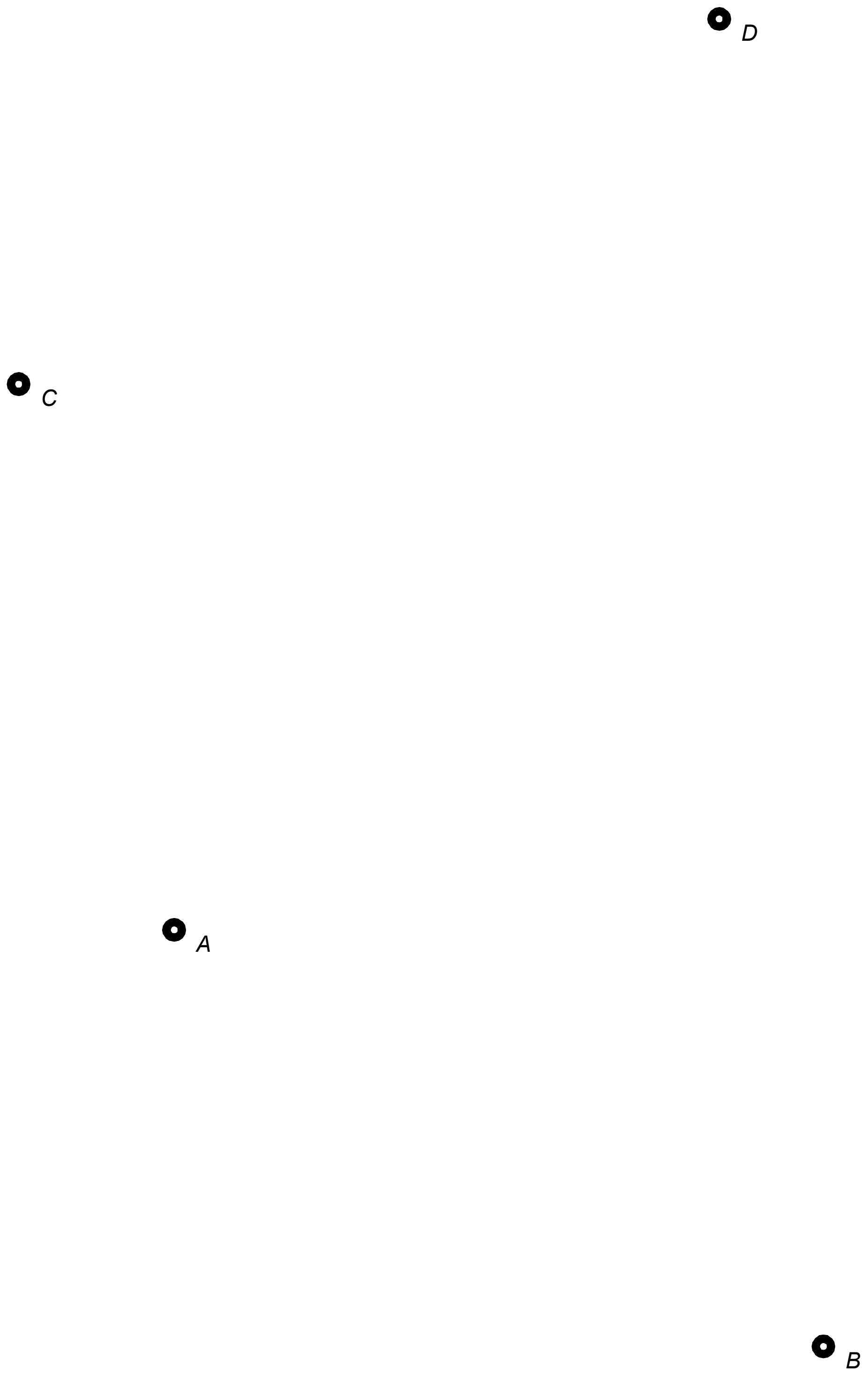


11 When using worksheet F it is not so hard. Putting letters on the angles of the reflections of the triangle will help you in the right direction. The reflections are numbered I, II, III etcetera.
From the long line to the broken line: measure piece-wise along the cushions of this billiards.

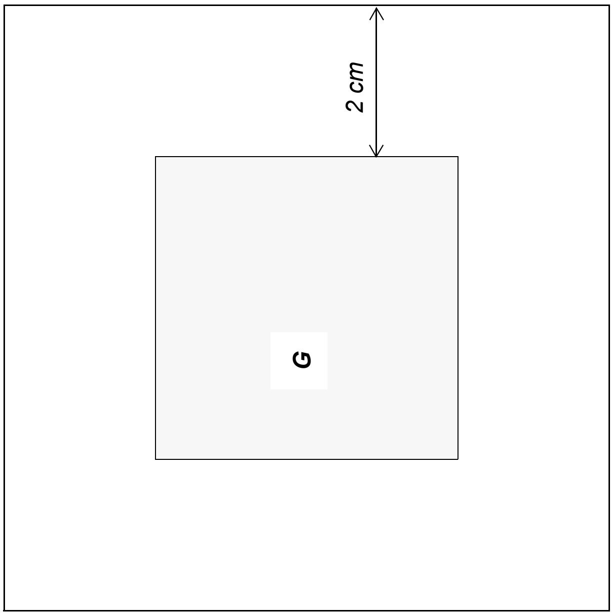


Worksheets

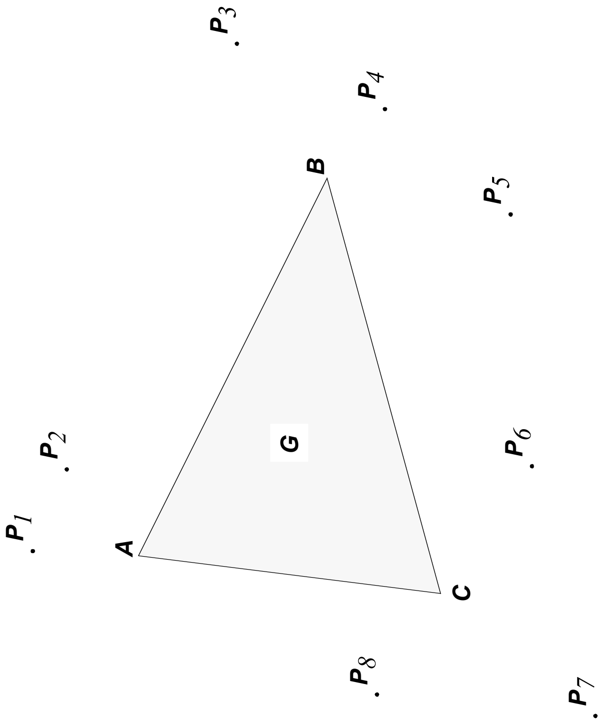
worksheet A: Folding to Voronoi (chapter 1, exercise 3)



worksheet B: Netherlands (chapter 3, exercise 12)

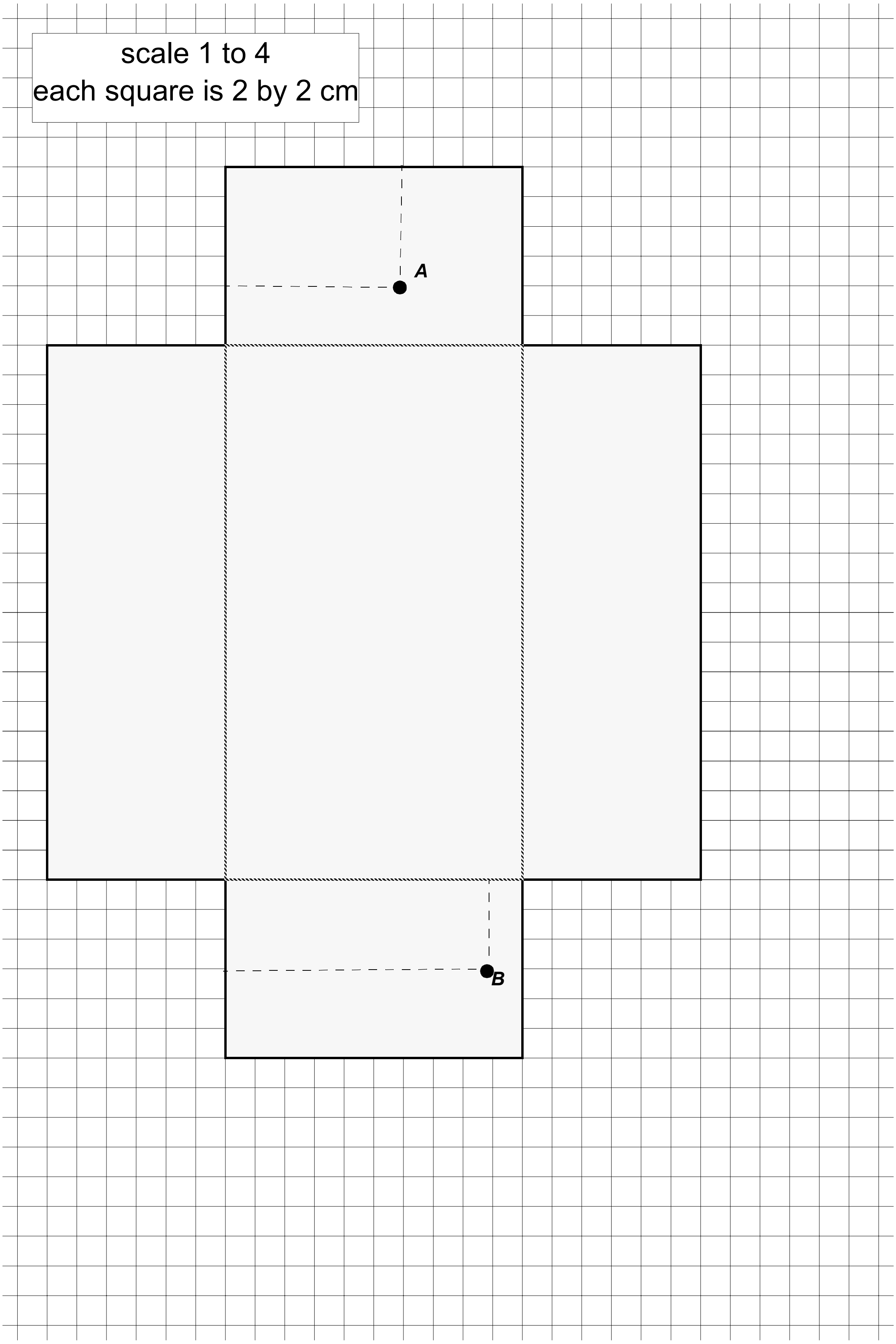

2 cm
G

worksheet D: triangle, feet, sectors (chapter 5, exercise 7)



worksheet E: ant on shoebox (chapter 6, exercise 8)

scale 1 to 4
each square is 2 by 2 cm



worksheet F: triangles (chapter 6, exercise 11)