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Hodge integrals and Gromov-Witten theory

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0. Introduction

Let $\overline{M}_{g,n}$ be the nonsingular moduli stack of genus g, n-pointed, Deligne-Mumford stable curves. For each marking i, there is an associated cotangent line bundle $\mathbb{L}_i \rightarrow \overline{M}_{g,n}$ with fiber T^*_{C,p_i} over the moduli point $[C, p_1, \ldots, p_n]$. Let $\psi_i = c_1(\mathbb{L}_i) \in H^*(\overline{M}_{g,n}, \mathbb{Q})$. The integrals of products of the ψ classes over $\overline{M}_{g,n}$ are determined by Witten's conjecture (Kontsevich's theorem): their natural generating function satisfies the Virasoro constraints [W], [K]. Let ω_C denote the dualizing sheaf of a curve C. The Hodge bundle $\mathbb{E} \rightarrow \overline{M}_{g,n}$ is the rank g vector bundle with fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. Let $\lambda_j = c_j(\mathbb{E})$. A Hodge integral over $\overline{M}_{g,n}$ is defined to be an integral of products of the ψ and λ classes. It is the Hodge integrals that are studied here.

Hodge integrals arise naturally in Gromov-Witten theory. There are two specific occurrences which motivated this work. First, let $X = \mathbf{G}/\mathbf{P}$ be a compact algebraic homogeneous space. The virtual localization formula established in [GrP] reduces all Gromov-Witten invariants (and their descendents) of X to explicit graph sums involving only Hodge integrals over $\overline{M}_{g,n}$. For example, the classical Severi degrees – the numbers of degree d, genus g algebraic plane curves passing through $3d + g - 1$ points – are Gromov-Witten invariants of $\mathbf{P}^2$ and may be expressed in terms of Hodge integrals. Formulas for Hodge integrals therefore play a role in Gromov-Witten theory.

Second, let X be an arbitrary nonsingular projective variety of dimension r. Consider the stack $\overline{M}_{g,n}(X, 0)$ of stable *constant* maps from genus g,

n-pointed curves to X. There is a natural isomorphism:

$$\overline{M}_{g,n}(X,0) \stackrel{\sim}{=} \overline{M}_{g,n} \times X. \tag{1}$$

The virtual class $[\overline{M}_{g,n}(X,0)]^{vir}$ is equal to $c_{rg}(\mathbb{E}^* \boxtimes T_X) \cap [\overline{M}_{g,n}(X,0)]$ via the identification (1). Hence, the degree 0 Gromov-Witten invariants of X involve only the classical cohomology ring $H^*(X,\mathbb{Q})$ and Hodge integrals over $\overline{M}_{g,n}$. In [GeP], this observation is combined with the conjectural Virasoro constraints of Eguchi, Hori, and Xiong [EHX] to yield simple formulas for certain Hodge integrals. For example, the following relation is derived in [GeP] as a consequence of the Virasoro constraints applied to $\mathbf{P}^1$:

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \dots, k_n} b_g, \tag{2}$$

where $k_i \geq 0$ and

$$b_g = \begin{cases} 1, & g = 0, \\ \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g, & g > 0. \end{cases} \tag{3}$$

The methods of [GeP] also yield conjectural relations for Hodge integrals with a single λ_{g-1} factor. The simplest of these predictions is: for $g \geq 1$,

$$\begin{aligned} c_g &= \int_{\overline{M}_{g,1}} \psi_1^{2g-1} \lambda_{g-1} \\ &= \Big(\sum_{k=1}^{2g-1} \frac{1}{k}\Big) b_g - \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,\, g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2}. \end{aligned} \tag{4}$$

Remarkably, the integrals b_g seem to be unconstrained by the degree 0 Virasoro conjecture.

More generally, it is natural to consider Hodge integrals over stacks of stable maps $\overline{M}_{g,n}(X,\beta)$ for nonsingular projective varieties X:

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{a_i} \cup e_i^*(\gamma_i) \cup \prod_{j=1}^{g} \lambda_j^{b_j}. \tag{5}$$

The classes ψ_i here are the cotangent line classes on $\overline{M}_{g,n}(X,\beta)$, the maps e_i are the evaluation maps to X corresponding to the markings, and the classes γ_i satisfy $\gamma_i \in H^*(X,\mathbb{Q})$. The gravitational descendents are the integrals (5) for which all $b_j = 0$ (no λ classes appear). The first result proven in this paper is the following Reconstruction Theorem.

Theorem 1. *The set of Hodge integrals over moduli stacks of maps to X may be uniquely reconstructed from the set of descendent integrals.*

The method of proof is to utilize Mumford's Grothendieck-Riemann-Roch calculations in [Mu]. Mumford's results may be interpreted in Gromov-Witten theory to yield differential equations for suitably defined generating functions of Hodge integrals. A consequence of these equations is a direct geometric construction of the $g = 0$ relation $\tilde{L}_1$ which plays an important role in the proof of the $g = 0$ Virasoro constraints (see [EHX], [DZ], [Ge], [LiuT]). As the required generating function involves the Chern character of the Hodge bundle, it seems quite difficult to obtain closed formulas for the Hodge integrals (5) via Theorem 1. The reconstruction result was obtained in case X is a point in [F2].

In order to find closed solutions in certain cases, we introduce here a new method of obtaining relations among Hodge integrals. The idea is to use the localization formula of [GrP] in reverse: localization computations of known equivariant integrals against $[\overline{M}_{g,n}(\mathbf{G}/\mathbf{P}, \beta)]^{\text{vir}}$ yield relations among Hodge integrals over $\overline{M}_{g,n}$. A variant of this technique is to compute an equivariant integral against the virtual class via two different linearizations of the torus action. A relation among Hodge integrals is then obtained by the two results of the localization formula. A simpler case of these ideas provides motivation: application of the Bott residue formula to integrals over the Grassmannian yields nontrivial combinatorial identities when linearizations are altered.

Hodge integrals over $\overline{M}_{g,n}$ also arise naturally in the study of tautological degeneracy loci of the Hodge bundle. Formulas for these degeneracy loci are used here to find new relations among Hodge integrals. The geometry involved is closely related to classical curve theory: special linear series, Weierstrass points, and hyperelliptic curves.

The main result of this paper is the following formula proven by the localization method together with a degeneracy calculation. Define $F(t,k) \in \mathbb{Q}[k][[t]]$ by

$$F(t,k) = 1 + \sum_{g \geq 1} \sum_{i=0}^{g} t^{2g} k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}.$$

Theorem 2.

$$F(t,k) = \left(\frac{t/2}{\sin(t/2)} \right)^{k+1}.$$

In particular, the integrals b_g and c_g are determined by:

$$\sum_{g \geq 0} b_g t^{2g} = F(t,0) = \left(\frac{t/2}{\sin(t/2)} \right), \tag{6}$$

$$\sum_{g \geq 1} c_g t^{2g} = \frac{\partial F}{\partial k}(t,0) = \left(\frac{t/2}{\sin(t/2)} \right) \cdot \log \left(\frac{t/2}{\sin(t/2)} \right).$$

D. Zagier has provided us with a proof of the Virasoro prediction (4) from (6) and identities among Bernoulli numbers. M. Shapiro and A. Vainshtein informed us of another approach to Theorem 2 which will be pursued in [ELSV], see also [SSV].

Theorem 2 has a direct application in Gromov-Witten theory to a multiple cover formula for Calabi-Yau 3-folds. Under suitable conditions, the integral

$$C(g,d) = \int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{vir}} c_{\text{top}}(R^1\pi_*\mu^* N) \tag{7}$$

is the contribution to the genus g Gromov-Witten invariant of a Calabi-Yau 3-fold of multiple covers of a fixed rational curve (with normal bundle $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$). The genus 0 case is determined by the Aspinwall-Morrison formula

$$C(0,d) = 1/d^3,$$

[AM], [Ma], [V]. The genus 1 case was computed in physics [BCOV] and mathematics [GrP] to yield

$$C(1,d) = 1/12d.$$

Virtual localization and Theorem 2 determine this multiple cover contribution in the general case.

Theorem 3. *For $g \geq 2$,*

$$C(g,d) = \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!} = |\chi(M_g)| \cdot \frac{d^{2g-3}}{(2g-3)!},$$

where B_{2g} is the $2g^{th}$ Bernoulli number and $\chi(M_g) = B_{2g}/2g(2g-2)$ is the Harer-Zagier formula for the orbifold Euler characteristic of M_g.

Theorem 3 was conjectured in [GrP] from data obtained from the Hodge integral algorithm of [F2].

Another consequence of Theorem 2 is the determination of the following Hodge integral.

Theorem 4. *For $g \geq 2$,*

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}|}{2g} \frac{|B_{2g-2}|}{2g-2} \frac{1}{(2g-2)!}.$$

The genus $g \geq 2$, degree 0 Gromov-Witten invariant of a Calabi-Yau 3-fold X is simply

$$< 1 >_{g,0}^X = (-1)^g \frac{\chi}{2} \int_{\overline{M}_g} \lambda_{g-1}^3,$$

where χ is the topological Euler characteristic of X (see [GeP]). Theorem 4 was conjectured previously in [F1]. It implies Conjecture 1 in [F2].

Theorems 3 and 4 were very recently derived in string theory by physicists [MM], [GoV]. The method of [GoV] is to consider limits of type IIA string theory which may be conjecturally analyzed in M-theory. The degree 0 invariant of Theorem 4 is the leading order term in this limit. In M-theory, this leading term is evaluated via an explicit sum over states (the Bernoulli numbers arise via values of the ζ-function). The multiple cover formula is also derived in the M-theoretic framework.

We mention finally an interesting connection between Gromov-Witten theory and the intrinsic geometry of M_g via the Hodge integrals. The ring $\mathcal{R}^*(M_g)$ of tautological Chow classes in M_g has been conjectured in [F1] to be a Gorenstein ring with socle in degree $g-2$. The Hodge integrals

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g \lambda_{g-1} \tag{8}$$

determine the top intersection pairings in $\mathcal{R}^*(M_g)$. The study of $\mathcal{R}^*(M_g)$ in [F1] led to a simple combinatorial conjecture for the integrals (8):

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g \lambda_{g-1} = \frac{(2g+n-3)!(2g-1)!!}{(2g-1)! \prod_{i=1}^n (2k_i-1)!!} \int_{\overline{M}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}, \tag{9}$$

where $g \geq 2$ and $k_i > 0$. This prediction was shown in [GeP] to be implied by the degree 0 Virasoro conjecture applied to $\mathbf{P}^2$.

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1. Reconstruction equations

1.1. Mumford's calculation. We start by interpreting Mumford's beautiful Grothendieck-Riemann-Roch calculation [Mu] in Gromov-Witten theory. Let $\mathcal{M}$ be a nonsingular variety (or Deligne-Mumford stack). Let $\pi : \mathcal{C} \to \mathcal{M}$ be a flat family of genus g pre-stable curves (the fibers of π are complete, connected, and reduced, with only nodal singularities). Assume the variation of this family is maximal in the following sense: the Kodaira-Spencer map

$$T\mathcal{M}_m \to \mathrm{Ext}^1(\Omega_{\mathcal{C}_m}, \mathcal{O}_{\mathcal{C}_m}) \tag{10}$$

is surjective for every point $m \in \mathcal{M}$. In this case, the following facts are well-known from the deformation theory of nodal curves:

(i) $\mathcal{C}$ is a nonsingular variety (or Deligne-Mumford stack).
(ii) The singular locus of π (the locus of nodes of the fibers) is a nonsingular variety Z of pure codimension 2. The image $\pi(Z) = \partial\mathcal{M}$ is a divisor with normal crossings in $\mathcal{M}$.
(iii) There is a natural étale double cover $\epsilon : \tilde{Z} \to Z$ obtained from the 2-fold choice of branches incident at the nodes corresponding to points of Z.
(iv) There are natural line bundles $\mathbb{L}, \overline{\mathbb{L}}$ on $\tilde{Z}$ corresponding to the cotangent directions along the branches.
(v) There is a canonical isomorphism $\epsilon^*(\mathrm{Nor}_{Z/\mathcal{C}}) = \mathbb{L}^* \oplus \overline{\mathbb{L}}^*$.

Let $\iota : \tilde{Z} \to \mathcal{M}$ denote the natural composition. Let $\psi, \overline{\psi} \in A^1(\tilde{Z})$ denote the first Chern classes of $\mathbb{L}, \overline{\mathbb{L}}$ respectively (Chow groups will always be taken with $\mathbb{Q}$-coefficients). The morphism ι is generically $2-1$ onto the divisor $\partial\mathcal{M}$. Let $\kappa_l = \pi_*(c_1(\omega_\pi)^{l+1}) \in A^l(\mathcal{M})$.

The Hodge bundle is defined on $\mathcal{M}$ by $\mathbb{E} = \pi_*\omega_\pi$. Mumford calculates $\mathrm{ch}(\mathbb{E})$ in $A^*(\overline{M}_g)$ via the Grothendieck-Riemann-Roch formula. As he uses only properties (i-v) above for the family $\pi : \overline{M}_{g,1} \to \overline{M}_g$, his argument applies verbatim to the more general setting considered here.

Theorem (Mumford).

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \cdot \left(\kappa_{2l-1} + \frac{1}{2}\iota_* \sum_{i=0}^{2l-2} (-1)^i \psi^i \overline{\psi}^{2l-2-i}\right)$$

in $A^*(\mathcal{M})$.

The discrepancies between the above formula and [Mu] are due to a differing Bernoulli number convention and a typographic error in the κ term of [Mu]. In our formulas, the Bernoulli numbers are defined by:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

1.2. Gromov-Witten theory. Let X be a nonsingular projective variety over $\mathbb{C}$. Let $\overline{M} = \overline{M}_{g,n}(X, \beta)$ be the moduli stack of stable maps to X representing the class $\beta \in H_2(X, \mathbb{Z})$. Let $[\overline{M}]^{vir} \in A_*(\overline{M})$ denote the virtual class in the expected dimension [BF], [B], [LiT].

A direct analogue of Mumford's result will be proven for the universal family over $\overline{M}$ (with respect to the virtual fundamental class). The method essentially is to consider the morphism

$$\overline{M} \to \mathfrak{M}_g$$

to the Artin stack of pre-stable curves. However, this is done explicitly by finding an embedding

$$\overline{M} \subset \mathcal{M}$$

where $\mathcal{M}$ is a nonsingular base of a family of pre-stable curves satisfying (10). Such embeddings are not strictly necessary for those familiar with properties of the Artin stack $\mathfrak{M}_g$ (smoothness, representability of the universal curve), but are included here to make the presentation more accessible. Mumford's relations on $\mathcal{M}$ may then be pulled-back to $\overline{M}$ and intersected with the virtual class. The main technical tools involved are the splitting axioms of the virtual class. In case $2g - 2 + n > 0$, the space $\overline{M}$ also admits a morphism to $\overline{M}_{g,n}$, the Deligne-Mumford moduli space of stable curves. The latter morphism does not respect the cotangent lines (as there is contraction involved). Moreover, we do not restrict ourselves to the case $2g-2+n > 0$. For these reasons, the latter morphism is not pursued here.

Virtual divisors in $\overline{M}$ are of two types. First, stable splittings

$$\xi = (g_1 + g_2 = g, A_1 \cup A_2 = [n], \beta_1 + \beta_2 = \beta) \tag{11}$$

index virtual divisors in $\overline{M}$ corresponding to maps with reducible domain curves. Define

$$\Delta_\xi = \overline{M}_{g_1, A_1+*}(X, \beta_1) \times_X \overline{M}_{g_2, A_2+\bullet}(X, \beta_2) \rightarrow \overline{M} \tag{12}$$

to be the virtual divisor corresponding to the data ξ. The fibered product in (12) is taken with respect to the evaluation maps e_*, $e_\bullet$ corresponding to the markings $*$, $\bullet$. The virtual class of Δ_ξ is determined by:

$$[\Delta_\xi]^{vir} = [\overline{M}_{g_1, A_1+*}(X, \beta_1)]^{vir} \times [\overline{M}_{g_2, A_2+\bullet}(X, \beta_2)]^{vir} \cap (e_* \times e_\bullet)^{-1}(\delta)$$

where $\delta \subset X \times X$ is the diagonal (this is Axiom 4 of [BM]).

For $g \geq 1$, there is an additional virtual divisor Δ_0 corresponding to irreducible nodal domain curves:

$$\Delta_0 = \overline{M}_{g-1, [n]+\{*, \bullet\}}(X, \beta) \cap (e_* \times e_\bullet)^{-1}(\delta) \rightarrow \overline{M}$$

where $\delta \subset X \times X$ is the diagonal. By Axiom 4,

$$[\Delta_0]^{vir} = [\overline{M}_{g-1, [n]+\{*, \bullet\}}(X, \beta)]^{vir} \cap (e_* \times e_\bullet)^{-1}(\delta).$$

Let Δ be the set of all *ordered* splittings (11) indexing reducible divisors (with repetition) union $\{0\}$ for the irreducible divisor. There is a natural map

$$\iota : \bigcup_{\xi \in \Delta} \Delta_\xi \rightarrow \overline{M}$$

where the domain is the disjoint union.

Consider the morphism:

$$\overline{M} \rightarrow \mathfrak{M}_g$$

where the right side is the Artin stack of pre-stable genus g curves. For $0 \leq j \leq g$, let

$$B_j = \mathfrak{M}_{j,*} \times \mathfrak{M}_{g-j,\bullet}.$$

Let $B_{irr} = \mathfrak{M}_{g-1,\{*,\bullet\}}$. These Artin stacks admit natural maps $\nu_0, \ldots, \nu_g$, ν_{irr} to $\mathfrak{M}_g$. Let $\Delta^j \subset \Delta$ be the subset with (ordered) genus splitting $g_1 = j$, $g_2 = g - j$. Let $\Delta^{irr} = \{0\}$. Certainly,

$$\bigcup_{\xi \in \Delta^j} \Delta_\xi \stackrel{\sim}{=} B_j \times_{\mathfrak{M}_g} \overline{M}$$

for $j \in \{0, \ldots, g, irr\}$ (see [B]). The Isogeny Axiom of [BM] implies for each such j,

$$\nu_j^! [\overline{M}]^{vir} = \sum_{\xi \in \Delta^j} [\Delta_\xi]^{vir} \tag{13}$$

(used here in the form of Lemma 10 of [B]). This is one of the most important properties of the virtual class.

The analogue of Mumford's result required for Theorem 1 is the following Proposition.

Proposition 1.

$$\mathrm{ch}(\mathbb{E}) \cap [\overline{M}]^{vir} = g[\overline{M}]^{vir}$$
$$+ \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \cdot \left(\kappa_{2l-1} \cap [\overline{M}]^{vir} + \frac{1}{2} \iota_* \sum_{\xi \in \Delta} \sum_{i=0}^{2l-2} (-1)^i \psi_*^i \psi_\bullet^{2l-2-i} \cap [\Delta_\xi]^{vir} \right)$$

in $A_*(\overline{M})$.

Proof. We will find a nonsingular Deligne-Mumford stack $\mathcal{M}$ with a family of curves $\pi : \mathcal{C} \to \mathcal{M}$ satisfying assumption (10) and an embedding

$$\overline{M} \to \mathcal{M}$$

such that $\mathcal{C}$ restricts to the universal family over $\overline{M}$:

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{C} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ \overline{M} & \longrightarrow & \mathcal{M}. \end{array}$$

Following the notation of Sect. 1.1, we see

$$\tilde{Z} = \bigcup_{j \in \{irr, 0, \ldots, g\}} B_j \times_{\mathfrak{M}_g} \mathcal{M},$$

$$\tilde{Z} \times_{\mathcal{M}} \overline{M} = \bigcup_{j \in \{irr, 0, \ldots, g\}} B_j \times_{\mathfrak{M}_g} \overline{M}.$$

We may then apply Mumford's Theorem to the map $\pi : \mathcal{C} \to \mathcal{M}$. Intersecting Mumford's formula with $[\overline{M}]^{vir}$ yields:

$$\mathrm{ch}(\mathbb{E}) \cap [\overline{M}]^{vir} = g[\overline{M}]^{vir}$$

$$+\sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \cdot \Big(\kappa_{2l-1} \cap [\overline{M}]^{vir} + \frac{1}{2} \iota_* \sum_{j \in \{irr, 0, \ldots, g\}} \sum_{i=0}^{2l-2} (-1)^i \psi_*^i \psi_\bullet^{2l-2-i} \cap \nu_j^! [\overline{M}]^{vir} \Big)$$

in $A_*(\overline{M})$. The proposition then follows immediately from (13).

The construction of the required family $\pi : \mathcal{C} \to \mathcal{M}$ starts with a general observation. Let

$$S \subset \mathbf{P}^r \times B \to B \tag{14}$$

be a projective flat family of genus g, degree d pre-stable curves over a quasi-projective base scheme B. We show how to embed (14) in a family of curves over a nonsingular base satisfying assumption (10).

Let $\mathcal{L} = \mathcal{O}_{\mathbf{P}^r}(1)$. By standard boundedness arguments, there exists an integer α satisfying

$$H^1(S_b, \mathcal{L}_b^\alpha) = 0 \tag{15}$$

for all $b \in B$. Consider the Veronese embedding

$$\mathbf{P}^r \to \mathbf{P}^{\binom{r+\alpha}{\alpha}-1}$$

Then, there is a canonical map

$$\phi_1 : B \to \mathcal{H},$$

where $\mathcal{H}$ is the Hilbert scheme of genus g, degree αd curves in $\mathbf{P}^{\binom{r+\alpha}{\alpha}-1}$. The vanishing (15) easily implies $\mathcal{H}$ is nonsingular of expected dimension in a Zariski open set $\mathcal{H}^0 \subset \mathcal{H}$ containing $\mathrm{Im}(\phi_1)$. Assumption (10) for the universal family $\mathcal{U}^0 \to \mathcal{H}^0$ also is a direct consequence of (15). Let $\phi_2 : B \to X$ be a closed embedding in a nonsingular scheme X. Finally, the diagram

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{U}^0 \times X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi_1 \times \phi_2} & \mathcal{H}^0 \times X. \end{array}$$

is the required construction for the given family $S \to B$.

In [FP], $\overline{M}$ is constructed as a Deligne-Mumford quotient stack Hilb$/\mathbf{G}$ of a reductive group action on a Hilbert scheme of pointed graphs. The universal family $U \to \overline{M}$ is simply the stack quotient of the universal family $\mathcal{U} \to$ Hilb. The above construction applied $\mathbf{G}$-equivariantly to $\mathcal{U} \to$ Hilb directly yields the required construction for the Proposition (see also [GrP] where embeddings of $\overline{M}$ in nonsingular Deligne-Mumford stacks are constructed). □

1.3. Theorem 1. Let X be a nonsingular projective variety of dimension r. Let $\gamma_0, \ldots, \gamma_N$ be a graded $\mathbb{Q}$-basis of $H^*(X, \mathbb{Q})$. We take γ_0 to be the identity element. Let $g_{ef} = \int_X \gamma_e \cup \gamma_f$, and let g^{ef} be the inverse matrix. The descendent Gromov-Witten invariants of X

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_{a_i}) \right\rangle_{g,\beta}^{X} = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \cup e_i^*(\gamma_{a_i})$$

may be organized in a generating function

$$F^X = \sum_{g \geq 0} \hbar^{g-1} F_g^X,$$

where

$$F_g^X = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \sum_{n \geq 0} \frac{1}{n!} \sum_{\substack{a_1 \ldots a_n \\ k_1 \ldots k_n}} t_{k_n}^{a_n} \ldots t_{k_1}^{a_1} \left\langle \tau_{k_1}(\gamma_{a_1}) \ldots \tau_{k_n}(\gamma_{a_n}) \right\rangle_{g,\beta}^{X}.$$

We introduce here an analogous generating function $F_{\mathbb{E}}^X$ for the Hodge integrals over moduli stacks of maps to X. For each odd positive integer, let the variable s_i correspond to $\mathrm{ch}_i(\mathbb{E})$. By Mumford's relations [Mu], the even components of $\mathrm{ch}(\mathbb{E})$ vanish (for all genera). Let t, s denote the sets of variables $\{t_i^j\}$, $\{s_i\}$ respectively. The Hodge integrals

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_{\alpha_i}) \prod_{j=1}^{m} \mathrm{ch}_{b_j}(\mathbb{E}) \right\rangle_{g,\beta}^{X} =$$

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \cup e_i^*(\gamma_{\alpha_i}) \cup \prod_{j=1}^{m} \mathrm{ch}_{b_j}(\mathbb{E})$$

define formal functions

$$F_{g,\mathbb{E}}^X(t,s) =$$

$$\sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \sum_{n,m \geq 0} \frac{1}{n!m!} \sum_{\substack{k_1 \ldots k_n \\ a_1 \ldots a_n \\ b_1 \ldots b_m}} \prod_{i=1}^{n} t_{k_i}^{a_i} \prod_{j=1}^{m} s_{b_j} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\gamma_{a_i}) \prod_{j=1}^{m} \mathrm{ch}_{b_j}(\mathbb{E}) \right\rangle_{g,\beta}^{X}.$$

As before, we define $F_{\mathbb{E}}^X = \sum_{g \geq 0} \hbar^{g-1} F_{g,\mathbb{E}}^X$. This function is related to the descendent generating function by restriction: $F_{\mathbb{E}}^X|_{s=0} = F^X$. Finally, let $Z_{\mathbb{E}}^X = \exp(F_{\mathbb{E}}^X)$.

Formulas involving the cotangent line classes and the Chern character of the Hodge bundle yield the following consequence of Proposition 1. For $l \geq 1$, define a formal differential operator:

$$D_{2l-1} =$$

$$-\frac{\partial}{\partial s_{2l-1}} + \frac{B_{2l}}{(2l)!}\Big(\frac{\partial}{\partial t^0_{2l}} - \sum_{i=0}^{\infty}\sum_{j=0}^{N} t^j_i \frac{\partial}{\partial t^j_{i+2l-1}} + \frac{\hbar}{2}\sum_{i=0}^{2l-2}(-1)^i g^{ef}\frac{\partial}{\partial t^e_i}\frac{\partial}{\partial t^f_{2l-2-i}}\Big),$$

as usual the sum over the indices e, f is suppressed.

Proposition 2. *For all* $l \geq 1$, $D_{2l-1}Z^X_{\mathbb{E}} = 0$.

Proof. Let $\overline{M} = \overline{M}_{g,n}(X, \beta)$ as in Sect. 1.1. Three formulas are needed to deduce this vanishing from Proposition 1.

Let d be the virtual dimension of $\overline{M}$. The Chow class $\kappa_{2l-1} \cap [\overline{M}]^{vir}$ has dimension $d - 2l + 1$. The first formula is:

$$\prod_{i=1}^{n} \psi_i^{k_i} \cup e_i^*(\gamma_{a_i}) \ \cap (\kappa_{2l-1} \cap [\overline{M}]^{vir}) = \tag{16}$$

$$\Big\langle \tau_{2l}(\gamma_0)\prod_{i=1}^{n}\tau_{k_i}(\gamma_{a_i})\Big\rangle^X_{g,\beta} - \sum_{i=1}^{n}\Big\langle \tau_{k_i+2l-1}(\gamma_{a_i})\prod_{j\neq i}\tau_{k_j}(\gamma_{a_j})\Big\rangle^X_{g,\beta},$$

where the cohomology product on the left side has codimension $d-2l+1$. It follows from viewing the universal family over $\overline{M}$ as $\overline{M}_{g,n+1}(X, \beta)$ and applying the standard comparison results for cotangent lines (see [W]). The only virtual class property needed is the equality

$$[\overline{M}_{g,n+1}(X, \beta)]^{vir} = \pi^*[\overline{M}]^{vir}$$

which is an Axiom in [BM].

The second and third required formulas address the behavior of the Chern character of the Hodge bundle when restricted to the virtual boundary divisors. Let $\xi \in \Delta$ correspond to a virtual boundary divisor with genus splitting $g_1 + g_2 = g$. Let $\mathbb{E}_g$ denote the Hodge bundle on $\overline{M}$. Let $\mathbb{E}_{g_1}, \mathbb{E}_{g_2}$ denote the Hodge bundles obtained from the two factors in (12). The natural restriction sequence on Δ_ξ:

$$0 \to \mathbb{E}_{g_1} \to \iota^*\mathbb{E}_g \to \mathbb{E}_{g_2} \to 0$$

implies the formula

$$\mathrm{ch}(\mathbb{E}_{g_1}) + \mathrm{ch}(\mathbb{E}_{g_2}) = \iota^*\mathrm{ch}(\mathbb{E}_g) \in A^*(\Delta_\xi). \tag{17}$$

Similarly, for the irreducible virtual divisor Δ_0, the residue sequence

$$0 \to \mathbb{E}_{g-1} \to \iota^*\mathbb{E}_g \to \mathcal{O}_{\Delta_0} \to 0$$

implies the formula

$$\mathrm{ch}(\mathbb{E}_{g-1}) = \iota^*\mathrm{ch}(\mathbb{E}_g) \in A^*(\Delta_0). \tag{18}$$

Proposition 2 is a formal consequence of Proposition 1 and equations (16-18). □

The generating function $F^X_{\mathbb{E}}$ is determined by the initial $s = 0$ conditions (specified by F^X) and the differential equations from Proposition 2. Thus, Theorem 1 is proven.

We end this section with some remarks following from Proposition 2. All the Chern classes of the Hodge bundle vanish in genus 0. Hence, $\partial F^X_{0,\mathbb{E}}/\partial s_{2l-1} = 0$. The vanishing $D_{2l-1} Z^X_{\mathbb{E}} = 0$ analyzed at order $\hbar^{-1}$ then yields universal relations among genus 0 descendent invariants of X. The relation obtained for $l = 1$ coincides precisely with $\tilde{L}_1$ (defined in [EHX] and used in the proof of the genus 0 Virasoro constraints). Proposition 2 also yields geometric interpretations of several related equations in the latter proof (see [Ge]).

In fact, Proposition 2 yields many more new universal relations among pure descendent invariants. For example, the classes $\mathrm{ch}_{2l-1}(\mathbb{E})$ vanish in $A^*(\overline{M}_g)$ for $l > g$. Hence, generalizations of the above $g = 0$ equations to higher genus are obtained from

$$\frac{\partial F^X_{g,\mathbb{E}}}{\partial s_{2l-1}} = 0 \quad (l > g),$$

and the vanishing at order $\hbar^{g-1}$ in $D_{2l-1} Z^X_{\mathbb{E}} = 0$. The resulting relation is an efficient topological recursion relation (TRR) for τ_{2l} in genus $g < l$. Note the Bernoulli number drops out of these relations.

A more sophisticated method of obtaining pure descendent equations from Proposition 2 is to construct combinations of the operators D_{2l-1} that serve to introduce the Chern classes of $\mathbb{E}$. The Chern classes of $\mathbb{E}$ certainly vanish in degrees greater than g on $\overline{M}_g$. One obtains from Proposition 2 relations in degree greater than g (for each g). It would be interesting to understand these equations and their relation to TRR and the Virasoro constraints even in the point case.

Finally, while the Hodge integrals

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{a_i} \cup e_i^*(\gamma_i) \cup \prod_{j=1}^{g} \lambda_j^{b_j}$$

are determined by Proposition 2 and F^X, the relations satisfied by the natural generating functions of these integrals do not appear easy to write.

2. Relations via virtual localization

2.1. The localization formula. We review here the virtual localization formula in Gromov-Witten theory [GrP] in the special case of degree 1 maps to $\mathbf{P}^1$. While our strategy for obtaining relations among Hodge integrals may be pursued in much greater generality, only this special case is required for Theorem 2.

Let $\mathbf{P}^1 = \mathbf{P}(V)$ where $V = \mathbb{C} \oplus \mathbb{C}$. Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{19}$$

Let p_1, p_2 be the fixed points of the corresponding action on $\mathbf{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbf{P}(V)$ is uniquely determined by the weights $[l_1, l_2]$ of the fiber representations at the fixed points

$$L_1 = L|_{p_1}, \quad L_2 = L|_{p_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle, Tan, has weights $[1, -1]$. There is a scaling lifting of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbf{P}(V)}$ for each integer α with weights $[\alpha, \alpha]$. For each integer β, there is a $\mathbb{C}^*$-lifting to $\mathcal{O}_{\mathbf{P}(V)}(-1)$ with weights $[\beta, \beta+1]$.

Let $g \geq 1$. Let $\text{Map}_g = \overline{M}_{g,0}(\mathbf{P}(V), 1)$ be the moduli stack of stable, genus g, unpointed maps to $\mathbf{P}(V)$ of degree 1. Let

$$\pi : U \to \text{Map}_g, \quad \mu : U \to \mathbf{P}(V) \tag{20}$$

be the universal curve and universal map over the moduli stack. The representation (19) canonically induces $\mathbb{C}^*$-actions on U and Map_g compatible with the maps π and μ.

The virtual dimension of Map_g is $2g$. There are two natural rank g bundles on Map_g: $R^1\pi_*(\mu^*\mathcal{O}_{\mathbf{P}(V)})$ and $R^1\pi_*(\mu^*\mathcal{O}_{\mathbf{P}(V)}(-1))$. Let x, y denote the respective top Chern classes of these bundles in $A^g(\text{Map}_g)$. The following two integrals against the virtual class $[\text{Map}_g]^{vir} \in A_{2g}(\text{Map}_g)$ will be considered:

$$\int_{[\text{Map}_g]^{vir}} x \cup y, \quad \int_{[\text{Map}_g]^{vir}} y \cup y. \tag{21}$$

The virtual localization formula will be used to compute these integrals with respect to various linearizations on $\mathcal{O}_{\mathbf{P}(V)}$ and $\mathcal{O}_{\mathbf{P}(V)}(-1)$.

The fixed locus X of the $\mathbb{C}^*$-action on Map_g is a disjoint union of irreducible components

$$X = \bigcup_{\substack{g_1+g_2=g \\ g_1,\, g_2 \geq 0}} X_{g_1,g_2}.$$

The component X_{g_1,g_2} corresponds to the loci of maps where subcurves of genus g_1 and g_2 are contracted to the fixed points p_1 and p_2 respectively. The fixed locus is naturally isomorphic to $\overline{M}_{g_1,1} \times \overline{M}_{g_2,1}$ (where $\overline{M}_{0,1}$ is defined to be a point). Moreover, the induced fixed stack structure on X_{g_1,g_2} is simply the reduced nonsingular structure [GrP]. The cotangent line and λ classes of the two factors yield cohomology classes on X_{g_1,g_2} via pull-back.

Let ψ_1, ψ_2 denote the cotangent line classes from the factors $\overline{M}_{g_1,1}$ and $\overline{M}_{g_2,1}$ respectively. For $k \in \mathbb{Z}$, let

$$\Lambda_1(k) = \sum_{i=0}^{g_1} k^i \lambda_{g_1-i} \in A^*(\overline{M}_{g_1,1}),$$

$$\Lambda_2(k) = \sum_{i=0}^{g_2} k^i \lambda_{g_2-i} \in A^*(\overline{M}_{g_2,1}).$$

We note Mumford's formula $c(\mathbb{E}) \cdot c(\mathbb{E}^*) = 1$ implies

$$\Lambda_i(-1)\Lambda_i(1) = (-1)^{g_i}, \qquad (22)$$

$$\Lambda_i(0)\Lambda_i(0) = \delta_{g_i 0}.$$

These sums $\Lambda_i(k)$ will be convenient for the formulas below.

Let $\iota : X \to \mathrm{Map}_g$ be the inclusion. The virtual localization formula is:

$$\iota_* \sum_{g_1+g_2=g} \frac{[X_{g_1,g_2}]}{c_{\mathrm{top}}(\mathrm{Nor}^{vir}_{g_1,g_2})} = [\mathrm{Map}_g]^{vir} \ \in H^*_*(\mathrm{Map}_g)[1/t]. \qquad (23)$$

The virtual normal bundle $\mathrm{Nor}^{vir}_{g_1,g_2}$ is isomorphic in equivariant K-theory on X_{g_1,g_2} to the sum:

$$[\psi_1 \otimes \mathrm{Tan}_1] + [\psi_2 \otimes \mathrm{Tan}_2] + [\pi_* \mu^* \mathrm{Tan}] - [R^1 \pi_* \mu^* \mathrm{Tan}] - [\mathrm{Aut}]$$

(see [GrP]). Let $\gamma \in H^{4g}_*(\mathrm{Map}_g)$. After an expansion of the virtual normal contribution, equation (23) yields an explicit integration formula for γ:

$$\int_{[\mathrm{Map}_g]^{vir}} \gamma = \sum_{g_1+g_2=g} \int_{X_{g_1,g_2}} (-1)^g \ \iota^*(\gamma) \ \frac{\Lambda_1(-1)}{1-\psi_1} \ \frac{\Lambda_2(1)}{1+\psi_2}. \qquad (24)$$

2.2. Relations. Application of formula (24) to the integrals (21) yields the following linearization dependent equations. We find

$$\int_{[\mathrm{Map}_g]^{vir}} x \cup y = (-1)^g I_g(\alpha, \beta)$$

with respect to the linearizations $[\alpha, \alpha]$ on $\mathcal{O}_{\mathbf{P}(V)}$ and $[\beta, \beta+1]$ on $\mathcal{O}_{\mathbf{P}(V)}(-1)$ where

(25)
$$I_g(\alpha, \beta) = \sum \int_{X_{g_1,g_2}} \frac{\Lambda_1(-1)\Lambda_1(-\alpha)\Lambda_1(-\beta)}{1-\psi_1} \, \frac{\Lambda_2(-1)\Lambda_2(\alpha)\Lambda_2(\beta+1)}{1-\psi_2}.$$

Similarly,

$$\int_{[\mathrm{Map}_g]^{vir}} y \cup y = (-1)^g J_g(\alpha, \beta)$$

with respect to the linearizations $[\alpha, \alpha+1]$, $[\beta, \beta+1]$ on the two copies of $\mathcal{O}_{\mathbf{P}(V)}(-1)$ where

(26) $$J_g(\alpha,\beta) = \sum \int_{X_{g_1,g_2}} \frac{\Lambda_1(-1)\Lambda_1(-\alpha)\Lambda_1(-\beta)}{1-\psi_1} \frac{\Lambda_2(-1)\Lambda_2(\alpha+1)\Lambda_2(\beta+1)}{1-\psi_2}.$$

Hence, we have obtained the relations

(27) $$I_g(\alpha,\beta) = I_g(\alpha',\beta'), \quad J_g(\alpha,\beta) = J_g(\alpha',\beta')$$

for all integers $\alpha, \alpha', \beta, \beta'$.

For $\xi \in \mathbb{Z}$, define the series $f_\xi(t) \in \mathbb{Q}[[t]]$ by:

$$f_\xi(t) = 1 + \sum_{g\geq 1} t^{2g} \int_{\overline{M}_{g,1}} \frac{\Lambda(\xi)}{1-\psi_1} = 1 + \sum_{g\geq 1}\sum_{i=0}^{g} t^{2g}\xi^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i}\lambda_{g-i}.$$

Proposition 3. *For $\xi \in \mathbb{Z}$, $f_\xi(t) = f_0(t)^{\xi+1}$.*

Proof. By the integration formulas (25-26) together with Mumford's relations (22), we find:

$$1 + \sum_{g\geq 1} t^{2g} I_g(0,0) = f_0(it),$$

$$1 + \sum_{g\geq 1} t^{2g} J_g(0,-1) = f_0^2(it).$$

We will consider the relations:

$$1 + \sum_{g\geq 1} t^{2g} I_g(\xi,0) = f_0(it),$$

$$1 + \sum_{g\geq 1} t^{2g} J_g(0,\xi) = f_0^2(it).$$

Define a new series for $\xi \in \mathbb{Z}$:

$$g_\xi(t) = 1 + \sum_{g\geq 1} t^{2g} \int_{\overline{M}_{g,1}} \frac{\Lambda(-1)\Lambda(0)\Lambda(-\xi)}{1-\psi_1}.$$

The integration formulas imply:

$$1 + \sum_{g\geq 1} t^{2g} I_g(\xi,0) = g_\xi(t)\, f_\xi(it),$$

$$1 + \sum_{g\geq 1} t^{2g} J_g(0,\xi) = g_\xi(t)\, f_{\xi+1}(it).$$

We then deduce the equations:

$$g_\xi(t)\, f_\xi(it) = f_0(it), \quad g_\xi(t)\, f_{\xi+1}(it) = f_0^2(it).$$

Hence, $f_{\xi+1}(it) = f_\xi(it)\, f_0(it)$ for all $\xi \in \mathbb{Z}$. The proposition now follows easily by induction (as it is true for $\xi = 0$). □

In order to determine the functions $f_\xi(t)$, it suffices to compute only $f_{-2}(t) = f_0(t)^{-1}$. This calculation too may be accomplished via localization relations, but a shorter and more elegant derivation by classical curve theory will be given in Proposition 4.

To show the flavor of Hodge relations obtained from localization, we mention two further examples. The formula:

$$1 + \sum_{g\geq 1} t^g \int_{\overline{M}_{g,1}} \psi^{3g-2} = \exp(t/24) \tag{28}$$

is a well known consequence of Witten's conjecture (Kontsevich's theorem). It is a nice exercise to prove this formula via Hodge relations obtained from localization on the stack of maps to $\mathbf{P}^1$. A geometric proof of (28) will be given in the next section.

Let $\gamma \in H^2(\mathbf{P}^1)$ be the point class. The integral

$$\int_{[\overline{M}_{g,1}(\mathbf{P}^1,d)]^{vir}} x \cup y \cup e_1^*(\gamma^d) \tag{29}$$

clearly vanishes for $d \geq 2$ (as before x and y are the top Chern classes of the vector bundles obtained from the higher direct images of $\mu^*(\mathcal{O}_{\mathbf{P}(V)})$ and $\mu^*(\mathcal{O}_{\mathbf{P}(V)}(-1))$ respectively, e_1 is the evaluation map corresponding to the marking). When (29) is computed by localization with an appropriate choice of linearization, the following Hodge relation is found:

$$\sum_{m\in \mathrm{Part}(d)} \frac{(-1)^{d+l(m)} \prod_i m_i^{m_i}}{\mathrm{Aut}(m) \prod_i m_i \prod_i m_i!} \int_{\overline{M}_{g,l(m)+1}} \frac{\lambda_g}{\prod_i (1 - m_i \psi_i)} = 0$$

where $m = \{m_1, \ldots, m_{l(m)}\}$ is a partition of d. We have checked algebraically that the Virasoro prediction (2) of [GeP] satisfies these relations. As yet, we are unable to prove (2) via Hodge relations of this type.

3. Relations via classical curve theory

3.1. Relations via the canonical system. In this section, we derive several relations among Hodge integrals from classical curve theory. The starting point is [Mu]. The base-point-freeness of the canonical system on a smooth curve can be formulated as the surjectivity of the natural map $\pi^*\mathbb{E} \to \mathbb{L}_1$ on $C_g = M_{g,1}$. This gives rise to an exact sequence

$$0 \to F \to \pi^*\mathbb{E} \to \mathbb{L}_1 \to 0$$

with F locally free of rank $g-1$. Hence one finds on $M_{g,1}$ the relations

$$\left(\frac{c(\mathbb{E})}{1+\psi_1}\right)_j = 0 \qquad (j \geq g).$$

If we want to extend these relations to $\overline{M}_{g,1}$, we must take into account the stable pointed curves for which $\mathbb{L}_1$ is not generated by global sections. As Mumford observes, the global sections generate the subsheaf of $\mathbb{L}_1$ that is zero at all disconnecting nodes and on all smooth rational curves all of whose nodes are disconnecting. Let us denote for $2 \leq i \leq g$ by X_i the locus of stable one-pointed curves of genus g consisting of a stable $(i+1)$-pointed rational curve with i tails (stable one-pointed curves of positive genus; the i genera sum to g) attached to the last i marked points. It follows that the relations above hold on $\overline{M}_{g,1}$ modulo a class supported on the loci $X_2, \ldots, X_g$. (Note that X_2 is the locus of disconnecting nodes in the universal curve.)

Since the moduli stack of $(i+1)$-pointed rational curves has dimension $i-2$ we have that ψ_1^{i-1} is 0 on X_i. Hence ψ_1^{g-1} is 0 on all these loci; we find the relations

$$\left(\frac{c(\mathbb{E})}{1+\psi_1}\right)_j = 0 \qquad (j \geq 2g-1)$$

on $\overline{M}_{g,1}$. For $j = 3g-2$, we find

(30) $$\int_{\overline{M}_{g,1}} \frac{\Lambda(1)}{1+\psi_1} = 0$$

(in the notation of Sect. 2). This identity implies $f_{-1}(t) = 1$ which is also a consequence of Proposition 3.

If instead we intersect the relation for $j = g$ with ψ_1^{g-2}, we find

(31) $$\left(\frac{c(\mathbb{E})}{1+\psi_1}\right)_{2g-2} = *\psi_1^{g-2}[X_g]_Q\,.$$

Here $[\ \ ]_Q$ denotes the Q-class or fundamental class in the sense of stacks as in [Mu]. The coefficient $*$ can be determined by intersecting with the locus Y parametrizing one-pointed irreducible curves with g nodes (hence with rational normalization) and their degenerations. Let $Z = X_g \cap Y$; this is the locus of one-pointed curves consisting of a stable $(g+1)$-pointed rational curve with g singular elliptic tails attached. The intersection is transverse in the universal deformation space, so that $[X_g]_Q \cdot [Y]_Q = [Z]_Q$; it is easy to see that ψ_1^{g-2} times this class equals $\dfrac{1}{2^g g!}$.

As the restriction of $\mathbb{E}$ to Y is trivial, the intersection of the left side of (31) with $[Y]_Q$ is $\psi_1^{2g-2}[Y]_Q$. This product evaluates to $\dfrac{1}{2^g g!}$ as well, since the natural map $\overline{M}_{0,2g+1} \to Y$ has degree $2^g g!$. We conclude that the coefficient $*$ in (31) is equal to 1:

(32)
$$\left(\frac{c(\mathbb{E})}{1+\psi_1}\right)_{2g-2} = \psi_1^{g-2}(\psi_1^g - \psi_1^{g-1}\lambda_1 + \cdots + (-1)^g\lambda_g) = \psi_1^{g-2}[X_g]_Q\,.$$

Intersecting this relation with $\psi_1^g+\psi_1^{g-1}\lambda_1+\cdots+\lambda_g$ gives just ψ_1^{3g-2} on the left side, since $c(\mathbb{E})\cdot c(\mathbb{E}^*)=1$. On the right side we obtain $\lambda_g\psi_1^{g-2}[X_g]_Q$ which easily evaluates to $1/(24^g g!)$. We find another proof of the identity (28),

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}.$$

3.2. Relations via Weierstrass loci. Above, our starting point was the base-point-freeness of the canonical system on a smooth curve. We then extended some of the relations so obtained to the moduli stack of stable curves. Below, we study hyperelliptic Weierstrass points; this may be viewed as a first step in analyzing the very-ampleness of the canonical system. We obtain the following result.

Proposition 4.

$$f_{-2}(t) = 1+\sum_{g\geq 1} t^{2g}\int_{\overline{M}_{g,1}} \psi_1^{2g-2}(\lambda_g - 2\psi_1\lambda_{g-1}+\cdots+(-2\psi_1)^g) = \frac{\sin(t/2)}{t/2}\,.$$

Proof. In [Mu] Mumford computed the class in C_g of the locus WH_g of hyperelliptic Weierstrass points:

$$\begin{aligned}[WH_g]_Q &= \left(c(\mathbb{E}^*)\frac{1}{1-\psi_1}\frac{1}{1-2\psi_1}\right)_{g-1}\\ &= (2^g-1)\psi_1^{g-1} - (2^{g-1}-1)\psi_1^{g-2}\lambda_1 + \ldots\\ &\quad +(-1)^{g-1}(2^1-1)\lambda_{g-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\psi_1[WH_g]_Q = &\big((2\psi_1)^g - \lambda_1(2\psi_1)^{g-1} + \cdots + (-1)^g\lambda_g\big)\\ &- \big((\psi_1)^g - \lambda_1(\psi_1)^{g-1} + \cdots + (-1)^g\lambda_g\big)\,.\end{aligned}$$

Let us suppose this identity continues to hold on $\overline{C}_g = \overline{M}_{g,1}$ modulo classes on which ψ_1^{2g-2} is zero. Then, by the vanishing (30), the formula for $f_{-2}(t)$ is equivalent to

(33)
$$\psi_1^{2g-1}[\overline{WH}_g] = \frac{1}{2^{2g-1}(2g+1)!}\,.$$

Note the usual fundamental class appears on the left in (33), as this is more convenient in the sequel. We will first prove identity (33) and then verify the required assumption.

The space $\overline{M}_{0,2g+2}$ may be viewed as the moduli space of stable hyperelliptic curves of genus g with an ordering of the $2g+2$ Weierstrass points. (The hyperelliptic automorphism is lost in this identification, however.) The universal (ordered) hyperelliptic curve is a double cover of $\overline{M}_{0,2g+3}$ (the universal curve over $\overline{M}_{0,2g+2}$). The ramification locus is $\overline{WH}_g$ (ordered); the branch locus B is

$$\sum_{j=1}^{2g+2} D_{j,2g+3}\,,$$

where $D_{j,2g+3}$ is the boundary divisor corresponding to the partition $\{j, 2g+3\} \cup \{j, 2g+3\}^c$ (note that the $2g+2$ divisors are disjoint). The reason we can compute $\psi_1^{2g-1}[\overline{WH}_g]$ is that ψ_1 on the double cover is a pullback from $\overline{M}_{0,2g+3}$. Denote the double cover map by f, then $\psi_1 = f^*(\psi_{2g+3} - B/2)$. This follows from the Riemann-Hurwitz formula; note that ψ_{2g+3} has degree $-2 + (2g+2) = 2g$ on the fibers of the map to $\overline{M}_{0,2g+2}$. Hence

$$\begin{aligned}\psi_1^{2g-1}[\overline{WH}_g] &= f_*(\psi_1^{2g-1}[\overline{WH}_g]) \\ &= \left(\psi_{2g+3} - \frac{1}{2}B\right)^{2g-1} B = \left(-\frac{1}{2}\right)^{2g-1} B^{2g}.\end{aligned}$$

The last equality holds because ψ_{2g+3} is zero on every component of B. Now B consists of $2g+2$ disjoint components, each isomorphic to $\overline{M}_{0,2g+2}$; the restriction of B to itself is then $-\psi_*$ if $*$ is the marked point corresponding to the node. Hence

$$\psi_1^{2g-1}[\overline{WH}_g] = (2g+2)\left(\frac{1}{2}\right)^{2g-1} \psi_*^{2g-1} = \frac{2g+2}{2^{2g-1}}\,.$$

This is the answer in the ordered case; the formula for the unordered case follows immediately.

It remains to verify the assumption made: that Mumford's formula for $[WH_g]_Q$ valid on C_g holds on $\overline{C}_g$ after multiplying by ψ_1^{2g-1}. One may prove Mumford's formula by observing that the locus of hyperelliptic Weierstrass points is the degeneracy locus $\{\mathrm{rk}\,\phi_2 \le 1\}$, where $\phi_2 : \mathbb{E} \to \mathbb{F}_2$ is the natural evaluation map from the Hodge bundle to the jet bundle $\mathbb{F}_2$ whose fiber at $[C, p]$ is the vector space $H^0(C, K/K(-2p))$ of dimension 2. The class of the locus is then given by Porteous's formula.

In order to verify the assumption, we must analyze the irreducible loci $\{L_i\}$ of singular stable pointed curves included in the degeneracy locus $\{\mathrm{rk}\,\phi_2 \le 1\}$ and show $\psi_1^{2g-1}[L_i] = 0$. If $[C, p]$ lies in the degeneracy locus, it is easy to see one of the following two possibilities must be satisfied:

(a) p lies on a nonsingular rational component X;
(b) p is a hyperelliptic Weierstrass point: the component X containing p is a possibly nodal hyperelliptic curve (of arithmetic genus $h \geq 1$), and the point p is a Weierstrass point on X.

Let L_i be an irreducible boundary component of the degeneracy locus. If (a) holds generically on L_i, naive estimates show the moduli of the component X (with marked nodes and point p) is bounded by $(3g-4)/2$ parameters. Hence, $\psi_1^{2g-1}[L_i]$ is certainly 0 in this case. Suppose (b) holds generically on L_i. We may assume the generic total curve C is not hyperelliptic, otherwise L_i lies in the closure of WH_g and is of dimension less than $2g-1$. In particular, C must be reducible. We will show the marked component X has fewer than $2g-1$ moduli. We may assume X is nonsingular and meets the rest of the curve in m points. We have to show that $2h-1+m < 2g-1$. Since $h < g$ this is clear when $m = 1$. When $m = 2$, $h = g-1$ doesn't result in a stable curve of genus g, so we are done. For $m \geq 3$, the maximal h is obtained when rational curves are attached. But attaching a k-pointed rational curve lowers $2h-1+m$ by $k-2$, so $2h-1+m$ is always smaller than $2g-1$. This finishes the proof of Proposition 4. □

3.3. Proof of Theorem 2. Define the series $F(t,k) \in \mathbb{Q}[k][[t]]$ by

$$F(t,k) = 1 + \sum_{g\geq 1}\sum_{i=0}^{g} t^{2g}k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i}\lambda_{g-i}.$$

By Propositions 3 and 4

$$F(t,\xi) = f_\xi(t) = \left(\frac{t/2}{\sin(t/2)}\right)^{\xi+1}$$

for all $\xi \in \mathbb{Z}$. The equality of formal series

$$F(t,k) = \left(\frac{t/2}{\sin(t/2)}\right)^{k+1}$$

then follows immediately. Theorem 2 is proven. □

4. Bernoulli identities and Theorems 3-4

4.1. Proof of Theorem 3. Let $\overline{M}_{g,0}(\mathbf{P}^1, d)$ be the moduli stack of genus g, degree d maps to $\mathbf{P}^1$. Consider the $\mathbb{C}^*$ action on $\mathbf{P}(V) = \mathbf{P}^1$ as defined in Sect. 2. As before, there are canonical maps

$$\pi : U \to \overline{M}_{g,0}(\mathbf{P}^1, d), \quad \mu : U \to \mathbf{P}(V)$$

where U is the universal curve over the moduli stack. Let N denote the bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Let

$$C(g,d) = \int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{vir}} c_{\text{top}}(R^1\pi_*\mu^* N).$$

For each pair of linearizations $[\alpha, \alpha+1]$, $[\beta, \beta+1]$, the virtual localization formula yields an explicit computation of $C(g,d)$.

For general choices of linearization, $C(g,d)$ is expressed as a complicated sum over connected graphs Γ (see [GrP]) indexing the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,0}(\mathbf{P}^1, d)$. The vertices of these graphs lie over the fixed points $p_1, p_2 \in \mathbf{P}^1$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbf{P}^1$ and are labelled with degrees (which sum over the graph to d). However, for the natural linearization $[0,1], [0,1]$, a vanishing result holds: if a graph Γ contains a vertex lying over p_1 of genus greater than 0 or valence greater than 1, then the contribution of Γ to $C(g,d)$ vanishes. As a result, the sum over graphs reduces to a more manageable sum over partitions of d. This linearization was found by Manin and used to compute $C(0,d) = 1/d^3$ in [Ma]. In [GrP], the same choice was used to compute $C(1,d) = 1/12d$.

A dramatic improvement occurs if the linearization $[0,1], [-1,0]$ is chosen. In this case, a stronger vanishing holds: if a graph Γ contains *any* vertex of valence greater than 1, then the contribution of Γ to $C(g,d)$ vanishes. Hence, contributing graphs have exactly 1 edge. The graph sum then reduces simply to a sum over partitions $g_1 + g_2 = g$ of the genus. The localization formula yields the following result for $g \geq 0$:

$$C(g,d) = d^{2g-3} \sum_{\substack{g_1+g_2=g \\ g_1,\, g_2 \geq 0}} b_{g_1} b_{g_2} \tag{34}$$

where b_g is defined by (3). In particular, the computations of $C(0,d)$ and $C(1,d)$ now require no series manipulations of the type pursued in [Ma], [GrP]. Note equation (34) implies

$$\sum_{g \geq 0} C(g,1) t^{2g} = f_0(t)^2 = f_1(t). \tag{35}$$

In Sect. 4.2, the formula (for $g \geq 1$)

$$\sum_{\substack{g_1+g_2=g \\ g_1,\, g_2 \geq 0}} b_{g_1} b_{g_2} = \frac{|B_{2g}|}{2g} \frac{1}{(2g-2)!}$$

will be proven from Theorem 2 and Bernoulli identities to complete the proof of Theorem 3.

4.2. Identities. Recall, the Bernoulli numbers B_m are defined by the series expansion

$$\beta(t) = \frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{36}$$

We start by computing b_g explicitly in terms of Bernoulli numbers.

Lemma 1.

$$\frac{t/2}{\sin(t/2)} = 1 + \sum_{g \geq 1} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g}.$$

Proof. This is well-known. We include a proof only for the reader's convenience.

$$\begin{aligned}
\frac{t/2}{\sin(t/2)} &= \frac{it}{e^{it} - 1} e^{it/2} = \frac{it}{e^{it/2} - 1} - \frac{it}{e^{it} - 1} = 2\beta(it/2) - \beta(it) \\
&= 2 - \frac{1}{2} it - \sum_{g \geq 1} \frac{1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g} - \left(1 - \frac{1}{2} it - \sum_{g \geq 1} \frac{|B_{2g}|}{(2g)!} t^{2g} \right) \\
&= 1 + \sum_{g \geq 1} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g}.
\end{aligned}$$

□

By Theorem 2, we see (for $g \geq 1$)

$$b_g = \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$

The series $f_1(t) = f_0^2(t)$ is determined by the following lemma.

Lemma 2.

$$\sum_{h=0}^{g} b_h b_{g-h} = \frac{|B_{2g}|}{2g} \frac{1}{(2g-2)!} .$$

Proof. Set $\beta_g = (2 - 2^{2g}) \dfrac{B_{2g}}{(2g)!}$. The identity to be proved (for $g \geq 1$) is then

$$2\beta_g + \sum_{h=1}^{g-1} \beta_h \beta_{g-h} = -\frac{2^{2g}}{2g} \frac{B_{2g}}{(2g-2)!} . \tag{37}$$

Since $\sum_{g=0}^{\infty} \beta_g x^{2g-1} = \dfrac{1}{\sinh(x)}$ and $\sum_{g=0}^{\infty} \dfrac{2^{2g} B_{2g}}{(2g)!} x^{2g-1} = \coth(x)$, equation (37) is an immediate consequence of $(\coth x)' = -\sinh^{-2} x$. □

Lemma 2 yields the equality

$$f_1(t) = 1 + \sum_{g\geq 1} \frac{|B_{2g}|}{2g} \frac{t^{2g}}{(2g-2)!}.$$

This result together with equations (34-35) completes the proof of Theorem 3. □

4.3. Proof of Theorem 4. The equality (for $g \geq 2$)

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}|}{2g} \frac{|B_{2g-2}|}{2g-2} \frac{1}{(2g-2)!}$$

now may be established by manipulating Mumford's Grothendieck-Riemann-Roch formulas and using Lemma 2.

Proof. The formula $\sum_{k\geq 1}(-1)^{k-1}(k-1)!\,\mathrm{ch}_k(V)t^k = \log(\sum_{k\geq 0} c_k(V)t^k)$ gives for $V = \mathbb{E}$

$$\sum_{k\geq 1}(-1)^{k-1}k!\,\mathrm{ch}_k(\mathbb{E})t^{k-1} = \left(\sum_{k=1}^{g} k\lambda_k t^{k-1}\right)\left(\sum_{k=0}^{g} \lambda_k(-t)^k\right)$$

since $c(\mathbb{E})^{-1} = c(\mathbb{E}^*)$. (Note that both sides are even polynomials in t.) In particular $(2g-3)!\,\mathrm{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1}(3\lambda_g\lambda_{g-3} - \lambda_{g-1}\lambda_{g-2})$ so that

$$\lambda_g\lambda_{g-1}\lambda_{g-2} = (-1)^g(2g-3)!\,\lambda_g \mathrm{ch}_{2g-3}(\mathbb{E}).$$

Mumford's formula [Mu] for $\mathrm{ch}(\mathbb{E})$ gives

$$(2g-3)!\,\mathrm{ch}_{2g-3}(\mathbb{E}) = \frac{B_{2g-2}}{2g-2}\left[\kappa_{2g-3} + \frac{1}{2}\sum_{h=0}^{g-1} i_{h,*}\left(\sum_{i=0}^{2g-4} \psi_1^i(-\psi_2)^{2g-4-i}\right)\right].$$

Since $\lambda_g = 0$ on Δ_0 while $i_h^*\lambda_g = \mathrm{pr}_1^*\lambda_h \mathrm{pr}_2^*\lambda_{g-h}$ for $h > 0$, this implies

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \int_{\overline{M}_g} 2\lambda_g\lambda_{g-1}\lambda_{g-2} = \frac{|B_{2g-2}|}{2g-2}\left[2b_g + \sum_{h=1}^{g-1} b_h b_{g-h}\right]$$

(where the first equality follows from $c(\mathbb{E})c(\mathbb{E}^*) = 1$). Hence, it remains to prove

$$\sum_{h=0}^{g} b_h b_{g-h} = \frac{|B_{2g}|}{2g} \frac{1}{(2g-2)!}.$$

But, this is Lemma 2. □

4.4. The Virasoro prediction for c_g. We include here D. Zagier's proof of the prediction (for $g \geq 1$):

$$\Big(\sum_{k=1}^{2g-1} \frac{1}{k}\Big) b_g = c_g + \frac{1}{2} \sum_{g=g_1+g_2,\, g_i>0} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2}.$$

From Theorem 2, we obtain

$$\sum_{g\geq 1} c_g t^{2g} = \Big(\frac{t/2}{\sin(t/2)}\Big) \cdot \log\Big(\frac{t/2}{\sin(t/2)}\Big).$$

Lemma 3 below (together with Lemma 1) expresses c_g in terms of Bernoulli numbers. Then, the Virasoro prediction for c_g is equivalent to an identity among Bernoulli numbers proven in Lemma 4.

Lemma 3.

$$\log\left(\frac{t/2}{\sin(t/2)}\right) = \sum_{k\geq 1} \frac{|B_{2k}|}{(2k)(2k)!} t^{2k}.$$

Proof. Let $f(t) = \dfrac{t/2}{\sin(t/2)}$. It suffices to prove

$$t\frac{f'(t)}{f(t)} = \sum_{k\geq 1} \frac{|B_{2k}|}{(2k)!} t^{2k}. \tag{38}$$

By definition (36), the right side of (38) equals $1 - \dfrac{1}{2} it - \dfrac{it}{e^{it}-1}$. The left side equals $1 - \dfrac{t}{2}\cot(t/2) = 1 - i\dfrac{t}{2}\dfrac{e^{it}+1}{e^{it}-1}$. □

Lemma 4.

$$\begin{aligned}\Big(\sum_{l=1}^{2g-1} \frac{1}{l}\Big) \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} &= \sum_{k=0}^{g-1} \frac{|2^{2k-1}-1|}{2^{2k-1}} \frac{|B_{2k}|}{(2k)!} \frac{|B_{2g-2k}|}{(2g-2k)(2g-2k)!} \\ &+ \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,\, g_2>0}} \frac{1}{(2g-1)!} \frac{2^{2g_1-1}-1}{2^{2g_1-1}} \frac{2^{2g_2-1}-1}{2^{2g_2-1}} \frac{|B_{2g_1}|}{2g_1} \frac{|B_{2g_2}|}{2g_2}.\end{aligned}$$

Proof (Zagier). Set $\beta_g = (2-2^{2g})\,\dfrac{B_{2g}}{(2g)!}$. The identity to be proved is $a(g)+b(g)=c(g)$, where

$$\begin{aligned}
a(g) &:= \left(1+\frac{1}{2}+\cdots+\frac{1}{2g-1}\right)\beta_g\,,\\
b(g) &:= \sum_{n=1}^{g}\frac{2^{2n}B_{2n}}{2n\,(2n)!}\,\beta_{g-n}\,,\\
c(g) &:= \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,\,g_2>0}}\frac{(2g_1-1)!\,(2g_2-1)!}{(2g-1)!}\,\beta_{g_1}\beta_{g_2}\,.
\end{aligned}$$

Using the generating function identity $\sum\limits_{g=0}^{\infty}\beta_g\,x^{2g-1}=\dfrac{1}{\sinh x}$, we find

$$\begin{aligned}
A(x) &:= \sum_{g=1}^{\infty}a(g)\,x^{2g-1} = \sum_{g=1}^{\infty}\beta_g\int_0^x\frac{x^{2g-1}-t^{2g-1}}{x-t}\,dt\\
&= \int_0^x\left[\frac{1}{x-t}\left(\frac{1}{\sinh x}-\frac{1}{\sinh t}\right)+\frac{1}{xt}\right]dt\,,
\end{aligned}$$

$$B(x) := \sum_{g=1}^{\infty}b(g)\,x^{2g-1} = \frac{1}{\sinh x}\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{2n\,(2n)!}\,x^{2n} = \frac{1}{\sinh x}\,\log\left(\frac{\sinh x}{x}\right)\,,$$

$$\begin{aligned}
C(x) &:= \sum_{g=1}^{\infty}c(g)\,x^{2g-1} = \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,\,g_2>0}}\beta_{g_1}\beta_{g_2}\int_0^x t^{2g_1-1}\,(x-t)^{2g_2-1}\,dt\\
&= \frac{1}{2}\int_0^x\left(\frac{1}{t}-\frac{1}{\sinh t}\right)\left(\frac{1}{x-t}-\frac{1}{\sinh(x-t)}\right)dt
\end{aligned}$$

and hence

$$\begin{aligned}
2\,C(x)-2\,A(x) &= \int_0^x\Bigg\{\left(\frac{1}{t}-\frac{1}{\sinh t}\right)\left(\frac{1}{x-t}-\frac{1}{\sinh(x-t)}\right)\\
&\quad-\left(\frac{1}{x-t}+\frac{1}{t}\right)\left(\frac{1}{\sinh x}+\frac{1}{x}\right)+\frac{1}{x-t}\,\frac{1}{\sinh t}+\frac{1}{t}\,\frac{1}{\sinh(x-t)}\Bigg\}\,dt\\
&= \int_0^x\left(\frac{1}{\sinh(t)\,\sinh(x-t)}-\frac{x}{\sinh x}\,\frac{1}{t\,(x-t)}\right)dt\\
&= \frac{1}{\sinh x}\,\log\left(\frac{\sinh t}{t}\cdot\frac{x-t}{\sinh(x-t)}\right)\Bigg|_0^x \;=\; 2\,B(x)\,.
\end{aligned}$$

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Note added in proof:

We have recently proved the Virasoro prediction (2) using generalizations of the Hodge relations given at the end of Sect. 2 (“Hodge integrals, partition matrices, and the λ_g conjecture”, in preparation).