THE CLASS OF THE BIELLIPTIC LOCUS IN GENUS 3

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Abstract. Let the bielliptic locus be the closure in the moduli space of stable curves of the locus of curves that admit a double covering map to a smooth genus 1 curve. In this paper we compute the class of the bielliptic locus in $\overline{\mathcal{M}}_3$ in terms of a standard basis of the rational Chow group of codimension 2 classes in the moduli space. Our method is to test the class on the hyperelliptic locus: this gives the desired result up to two free parameters, which are then determined by intersecting the locus with two surfaces in $\overline{\mathcal{M}}_3$.

1. The main result

A smooth bielliptic curve is a genus g curve that admits a $2:1$ map to a smooth genus 1 curve, with $2g-2$ ramification points by the Riemann-Hurwitz formula. It follows that the locus of bielliptic curves has codimension $g-1$ in $\overline{\mathcal{M}}_g$, the moduli stack of stable curves of genus g. In terms of enumerative geometry, we are interested in the following general problem: given a family over a base of dimension $g-1$, how many bielliptic curves occur in it? We solve this problem by expressing the class of the bielliptic locus in terms of standard classes in the case of genus 3 (and 2).

The main result of this paper is the following.

Theorem 1. *The rational stack class of the bielliptic locus $[\overline{\mathcal{B}}_3]_Q$ in $\overline{\mathcal{M}}_3$ equals*

$$[\overline{\mathcal{B}}_3]_Q = \frac{2673}{2}\lambda^2 - 267\lambda\delta_0 - 651\lambda\delta_1 + \frac{27}{2}\delta_0^2 + 69\delta_0\delta_1 + \frac{177}{2}\delta_1^2 - \frac{9}{2}\kappa_2.$$

Proof. In [Fa90] the first author has studied in particular the codimension 2 rational Chow group of $\overline{\mathcal{M}}_3$. In [Fa90, Thm. 2.9] he proves that

$$\lambda^2, \quad \lambda\delta_0, \quad \lambda\delta_1, \quad \delta_0^2, \quad \delta_0\delta_1, \quad \delta_1^2, \quad \kappa_2 \tag{1}$$

is a basis for $A^2_{\mathbb{Q}}(\overline{\mathcal{M}}_3)$.

We obtain the result by considering the pull-back via the map from the moduli stack of admissible hyperelliptic curves

$$\phi\colon \overline{\mathcal{H}}_3^{adm} \to \overline{\mathcal{M}}_3. \tag{2}$$

We prove in Proposition 3 that the pull-back $\phi^*([\overline{\mathcal{B}}_3]_Q)$ is a multiple of $[\overline{I}_8^{inv}]$: the class of the locus in $\overline{\mathcal{H}}_3^{adm}$ of the curves admitting an involution that acts without fixed points on the set of Weierstrass points. In Section 7 we prove by computing the class $[\overline{\mathcal{B}}_3]_Q$ on a suitable test surface Σ_5 that in fact $\phi^*([\overline{\mathcal{B}}_3]_Q) = [\overline{I}_8^{inv}]$.

The moduli stack $\overline{\mathcal{H}}_3^{adm}$ admits a moduli map to $\overline{\mathcal{M}}_{0,8}$; we identify the Chow groups of the former with those of the latter via the pull-back map. In Proposition 5 we compute the class $[\overline{I}_8^{inv}]$ in terms of boundary strata classes.

Observe that the inverse image loci via the map ϕ are invariant under permutation of the Weierstrass points. The linear map

$$\phi^*\colon A^2_{\mathbb{Q}}(\overline{\mathcal{M}}_3) \to A^2_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,8})^{S_8}$$

is surjective onto the S_8-invariant classes and thus it has 1-dimensional kernel since the image has dimension 6 (cf. the beginning of Section 5). To express the class $[\overline{\mathcal{B}}_3]_Q$ in the chosen basis (1) we write in Lemma 9 the matrix associated with the above linear map, where we have fixed the invariant boundary strata classes as a basis for the image. To conclude, we need to calculate the missing parameter coming from the kernel of ϕ^*. This is done in Section 7, by evaluating the class of $[\overline{\mathcal{B}}_3]_Q$ on a test surface Σ_1 containing bielliptic non-hyperelliptic curves. □

As explained in Section 7, we obtain as a corollary that the degree of the bielliptic locus in the $\mathbb{P}^{14}$ parametrizing plane quartic curves equals 225; a classical enumerative geometry result obtained via the moduli space.

Let us observe that an easy but nontrivial check of Theorem 1 can be made on a suitable test surface Σ_2 where the number of bielliptic curves is evident; this is done in Section 7.

Note that, with exactly the same method, we can compute the class of the bielliptic locus in $\overline{\mathcal{M}}_2$ with the simplifying difference that all genus 2 curves are hyperelliptic. Therefore, in the same way as before, but much simpler, we obtain the result:

Proposition 2. *The class of the bielliptic locus $[\overline{\mathcal{B}}_2]_Q$ in $\overline{\mathcal{M}}_2$ can be written as*

$$[\overline{\mathcal{B}}_2]_Q = 15\lambda + 3\delta_1 = \frac{3}{2}\delta_0 + 6\delta_1.$$

This agrees with the result for the usual fundamental class stated in [Fa96, p. 6].

Proof. With the obvious adjustments of notation from the proof of the theorem above, we see in Proposition 3 that $\phi^*([\overline{\mathcal{B}}_2]_Q) = [\overline{I}_6^{inv}]$. The map ϕ^* is an isomorphism of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_2)$ with $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,6})^{S_6}$. The class $[\overline{I}_6^{inv}]$ is computed in Corollary 4, and the isomorphism ϕ^* at the level of rational Picard groups is recalled in (5). □

Throughout this paper we work with Chow groups with rational coefficients. We express our results in the Chow groups in terms of the *stack* classes.

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2. Admissible double covers

We begin by recalling admissible double covers. Admissible covers were introduced by Harris and Mumford in their seminal paper [HMu82]. The definition we give here is the one of [ACV03, Section 4.1], adapted for the special case of degree-two covers.

Definition 1. Let C be a semistable curve of genus g. An *admissible double cover* with source C is the datum of a nodal genus $g' < g$ stable curve $(C', x_1, \dots, x_n)$, and of a finite, degree 2 map $\phi \colon C \to C'$ such that:

(1) the restriction $\phi^{sm} \colon C^{sm} \to C'^{sm}$ is branched over the marked points;
(2) the image via ϕ of each node is a node.

An *admissible hyperelliptic structure* on C is an admissible double cover where the genus g' equals 0, while an *admissible bielliptic structure* corresponds to the case when g' is 1.

By using the Riemann-Hurwitz formula and an induction on the number of nodes of C', one can see that the number n of marked points in the above definition must be $2g+2-4g'$.

On a smooth curve of genus g an admissible double cover is just the datum of a curve of genus g' and of the $2g+2-4g'$ ordered points of ramification. One can define families of admissible double covers, and isomorphisms of families of admissible double covers. In particular we have the two moduli stacks $\overline{\mathcal{H}}_g^{adm}$ and $\overline{\mathcal{B}}_g^{adm}$ parametrizing admissible hyperelliptic and bielliptic curves. They are both smooth proper Deligne–Mumford stacks, the first of dimension $2g-1$, the second of dimension $2g-2$.

We will use the following two maps. To each family of admissible hyperelliptic covers one can associate the target family of stable genus 0 curves together with the ordered branch divisor. This gives the map (a μ_2-gerbe): $\overline{\mathcal{H}}_g^{adm} \to \overline{\mathcal{M}}_{0,2g+2}$. Given a family of admissible bielliptic covers, one can forget all the extra structure besides the source family C of semistable curves, and then contract all rational bridges (the rational components that intersect the closure of the complement in precisely two points). This gives a map: $\overline{\mathcal{B}}_g^{adm} \to \overline{\mathcal{M}}_g$. It follows from the properness of $\overline{\mathcal{B}}_g^{adm}$ that on every family of stable curves, the locus corresponding to stable bielliptic curves forms a closed subscheme. We have thus a well-defined class

$$[\overline{\mathcal{B}}_g]_Q \in A^{g-1}_{\mathbb{Q}}(\overline{\mathcal{M}}_g).$$

3. Loci in moduli spaces of pointed, stable genus 0 curves

The following loci in $\overline{\mathcal{M}}_{0,2g+2}$ will play a central role.

Definition 2. We define I_{2g+2} as the closed subscheme of $\mathcal{M}_{0,2g+2}$ that parametrizes curves $(C, x_1, \dots, x_{2g+2})$ admitting an involution σ whose induced permutation on the marked points is $(12)(34)\dots(2g+1, 2g+2)$. Let $\overline{I}_{2g+2}$ be the closure of I_{2g+2} in $\overline{\mathcal{M}}_{0,2g+2}$.

The condition of admitting such an involution is a codimension $g-1$ condition. Let us now consider the invariant notion associated with the previous one.

Definition 3. We define I^{inv}_{2g+2} as the closed subscheme in $\mathcal{M}_{0,2g+2}$ that parametrizes curves $(C, x_1, \dots, x_{2g+2})$ admitting a fixed-point-free involution σ. Let $\overline{I}^{inv}_{2g+2}$ be the closure of I_{2g+2} in $\overline{\mathcal{M}}_{0,2g+2}$.

Let us take the chance to fix the notation for the boundary strata classes of $\overline{\mathcal{M}}_{0,n}$.

Notation 4. Given a partition $[n] = A_1 \sqcup A_2$ where $|A_i| \geq 2$, the general element of the divisor $(A_1|A_2) = (A_2|A_1)$ is made of genus 0 curves with two irreducible components, one of them containing the marked points in A_1 and the other those in A_2. Given a partition $[n] = A_1 \sqcup A_2 \sqcup A_3$ with $|A_2| \geq 1$ and $|A_1|, |A_3| \geq 2$, the general element of the codimension 2 boundary stratum $(A_1|A_2|A_3) = (A_3|A_2|A_1)$ is made of genus 0 curves with three irreducible components, the central one containing the marked points in A_2 and the extreme ones those in A_1 and A_3.

We also fix the notation for the invariant boundary strata classes on $\overline{\mathcal{M}}_{0,n}$ or, equivalently, the classes in $[\overline{\mathcal{M}}_{0,n}/S_n]$ pulled-back to $\overline{\mathcal{M}}_{0,n}$. Given a partition of $n = \lambda_1 + \lambda_2$ with $\lambda_i \geq 2$, the invariant divisor $d_{\lambda_1,\lambda_2} = d_{\lambda_2,\lambda_1}$ is the sum of all the distinct divisors $(A_1|A_2)$ such that $|A_i| = \lambda_i$. Given a partition of $n = \lambda_1 + \lambda_2 + \lambda_3$ that satisfies $\lambda_2 \geq 1$ and $\lambda_1, \lambda_3 \geq 2$, the invariant codimension 2 boundary stratum $d_{\lambda_1,\lambda_2,\lambda_3} = d_{\lambda_3,\lambda_2,\lambda_1}$ is the sum of all distinct codimension 2 boundary strata $(A_1|A_2|A_3)$ such that $|A_i| = \lambda_i$. A picture of $d_{5,1,2}$ in $\overline{\mathcal{M}}_{0,8}$ is in Figure 1.

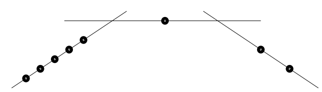

FIGURE 1. A picture of the boundary strata class $d_{5,1,2}$.

We now turn our attention to the genus 2 case. Vermeire in [Ve02] has computed

$$[\overline{I}_6] = (15|2346) + (25|1346) + (36|1245) + (46|1235) - (56|1234) + 2(125|346). \tag{3}$$

From this, it is immediate to compute the class of $\overline{I}_6^{inv}$ in terms of the boundary divisors. Let $d_{2,4}$ and $d_{3,3}$ be the two invariant divisor classes in $\overline{\mathcal{M}}_{0,6}$. The invariant divisor $\overline{I}_6^{inv}$ is the union of 15 irreducible divisors, each of them corresponding to an element in S_6 in the conjugacy class of $(12)(34)(56)$. Now since $d_{2,4}$ is the sum of 15 boundary divisor classes, and $d_{3,3}$ is the sum of the remaining 10, we obtain the equality in $A^1(\overline{\mathcal{M}}_{0,6})$

$$[\overline{I}_6^{inv}] = 3d_{2,4} + 3d_{3,3}. \tag{4}$$

4. THE BIELLIPTIC LOCUS AND THE INVARIANT LOCUS

We consider the moduli space $\overline{\mathcal{H}}_g^{adm}$, which parametrizes admissible hyperelliptic curves. We have a diagram:

$$\begin{tikzcd}
\overline{\mathcal{H}}_g^{adm} \arrow[r, "j"] \arrow[rr, bend left, "\phi"] \arrow[d, "\pi"] & \overline{\mathcal{H}}_g \arrow[r, "i"] & \overline{\mathcal{M}}_g \\
\overline{\mathcal{M}}_{0,2g+2} & &
\end{tikzcd}$$

where the map j forgets the structure of admissible double cover, and the representable map i is a closed embedding. The map j in particular forgets the ordering on the branch divisor and it stabilizes the strictly semistable components. We will implicitly assume the isomorphism π^* at the level of the Chow groups. Note also that the pull-back j^* is an isomorphism between the Chow groups of $\overline{\mathcal{H}}_g$ and the S_{2g+2}-invariants of the Chow groups of $\overline{\mathcal{M}}_{0,2g+2}$.

Proposition 3. *The inverse images via ϕ of $\overline{\mathcal{B}}_2$ and $\overline{\mathcal{B}}_3$ are respectively $\overline{I}_6^{inv}$ and $\overline{I}_8^{inv}$. In other words, we have that $\phi^*([\overline{\mathcal{B}}_2]_Q) = [\overline{I}_6^{inv}]$, and there exists $\epsilon \in \mathbb{Q}$ such that:*

$$\phi^*([\overline{\mathcal{B}}_3]_Q) = \epsilon \cdot [\overline{I}_8^{inv}].$$

Proof. We study the case of genus 3: the other case is similar and simpler.

We start by proving that $I_8^{inv} = \phi^{-1}(\mathcal{B}_3)$ and that $\overline{I}_8^{inv}$ is contained in $\phi^{-1}(\overline{\mathcal{B}}_3)$. Let C be a smooth genus 3 curve with a hyperelliptic quotient map $\psi\colon C \to C'$. If C also admits a bielliptic involution, this descends to an involution of C' because the hyperelliptic involution commutes with all automorphisms of C. The action of this involution on the branch locus of ψ swaps the 8 unordered points two-by-two. Vice versa, from a smooth curve C' of genus 0 with 8 distinct points on it, one can reconstruct the genus 3 hyperelliptic curve by taking the double cover branched at the 8 points. If C' admits an involution that exchanges the branch points two-by-two, this can be lifted to a bielliptic involution of C. Finally, if C is in $\overline{\mathcal{H}}_3^{adm}$, the same argument goes through after substituting "bielliptic involution" with "admissible bielliptic structure".

To conclude the proof, one has to check the following combinatorial statements.

(1) In each of the three irreducible boundary divisors of $\overline{\mathcal{H}}_3$, consider the open locus of curves that have the minimum number of singular points. On each of these open loci, the condition of having a bielliptic structure cuts out a locus of codimension *strictly greater than one*.
(2) None of the six codimension-two boundary strata classes of $\overline{\mathcal{H}}_3$ admits a bielliptic structure generically.

□

We will eventually be able to prove that $\epsilon = 1$ in Section 7, by enumerating bielliptic curves on the test surface Σ_5. We remark that when the genus is higher than 3, there are no smooth bielliptic-hyperelliptic curves. Therefore our method fails for computing the class of the bielliptic locus in higher genus.

By using the previous result, we can immediately compute the class of the bielliptic locus in genus 2. Let us take the boundary strata classes $d_{2,4}$ and $d_{3,3}$ as a basis for the S_6-invariant Picard group of $\overline{\mathcal{M}}_{0,6}$. We have the following equalities ([HMo98, 6.17, 6.18]):

$$\begin{cases} \phi^*\delta_0 = 2d_{2,4}; \\ \phi^*\delta_1 = \frac{1}{2}d_{3,3}. \end{cases} \tag{5}$$

In view of (4) and Proposition 3, these relations give the expression for the class of the bielliptic locus in terms of δ_0 and δ_1 stated in Proposition 2.

5. The class of the invariant locus in $\overline{\mathcal{M}}_{0,8}$

In this section, we express the class $[\overline{I}_8^{inv}]$ in terms of the generators of the invariant boundary strata classes in $\overline{\mathcal{M}}_{0,8}$ (see Notation 4):

$$d_{5,1,2}, \quad d_{4,2,2}, \quad d_{4,1,3}, \quad d_{3,3,2}, \quad d_{3,2,3}, \quad d_{2,4,2}. \tag{6}$$

We see from [Ge94, Theorem 5.9] that the S_8-invariant Chow ring of codimension 2 classes in $\overline{\mathcal{M}}_{0,8}$ has dimension 6, so that these 6 generators form a basis. We first compute the class of $\overline{I}_8$ in terms of boundary strata classes in $\overline{\mathcal{M}}_{0,8}$.

Let π_{12} and π_{78} be the two forgetful maps from $\overline{\mathcal{M}}_{0,8}$ to $\overline{\mathcal{M}}_{0,6}$. The map π_{12} forgets the marked points 1 and 2, and then renames $7,8$ to $1,2$. We have the equality

$$I_8 = \pi_{12}^{-1}(I_6) \cap \pi_{78}^{-1}(I_6). \tag{7}$$

Indeed, the right hand side contains curves admitting an involution σ that permutes the last six marked points as (34)(56)(78), and an involution τ that permutes the first six points as (12)(34)(56). The composition $\sigma \circ \tau$ must be the identity, as it fixes 4 points on a smooth genus 0 curve, and this means that both σ and τ do actually permute the eight marked points as (12)(34)(56)(78).

Equality (7) does not hold if one naïvely puts closures on both sides; anyway the argument above shows that $\overline{I}_8$ is an irreducible component of

$$\pi_{12}^{-1}(\overline{I}_6) \cap \pi_{78}^{-1}(\overline{I}_6). \tag{8}$$

We introduce the other components in $\overline{\mathcal{M}}_{0,8}$ that are contained in this intersection.

(1) Consider the locus (1278|3456) whose generic element is a curve with a node separating $\{1,2,7,8\}$ from $\{3,4,5,6\}$. The locus Div is the closure of the locus of curves in (1278|3456) with the property that the node is invariant under the involution that exchanges the points (34) and (56).
(2) Consider the 210 codimension 2 irreducible components of $d_{(2,4,2)}$. Two of them have $\{1,2,7,8\}$ as marked points on the separating component and occur in (8). We call the union of these strata Type I.
(3) Finally, consider the 280 irreducible components of $d_{(3,2,3)}$. Four of these boundary strata have the property that $\{3,4\}$ are the markings on the separating component, and $\{1,2\}$ are on two different components, and the same for $\{5,6\}$ and $\{7,8\}$. Four other ones come by exchanging the role of $\{3,4\}$ and that of $\{5,6\}$ in the previous sentence. In total, we call the union of these eight strata Type II.

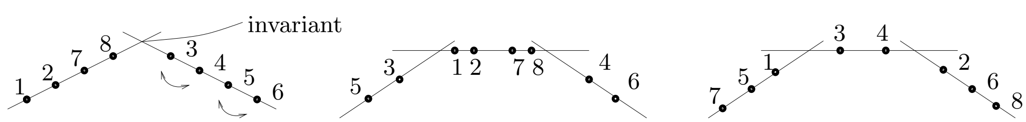

Figure 2. The locus Div, one component of Type I, and one component of Type II.

It is clear that these components are in the intersection (8).

Lemma 4. *The following equality holds in $A^2(\overline{\mathcal{M}}_{0,8})$:*

$$\pi_{12}^*([\overline{I}_6]) \cdot \pi_{78}^*([\overline{I}_6]) = \alpha[\overline{I}_8] + \beta[\mathrm{Div}] + \gamma[\text{Type I}] + \delta[\text{Type II}].$$

We will prove at the end of this section that the coefficients α, β, γ and δ equal 1. Assuming this for the moment, Lemma 4 gives a way to express the class of $\overline{I}_8$ as an explicit linear combination of boundary strata classes in $A^2(\overline{\mathcal{M}}_{0,8})$.

Corollary 5. *The class $[\overline{I}_8^{inv}]$ equals:*

$$[\overline{I}_8^{inv}] = \frac{5}{2}d_{5,1,2} + \frac{7}{4}d_{4,2,2} + \frac{3}{4}d_{4,1,3} + \frac{15}{4}d_{3,3,2} + 3d_{3,2,3} + \frac{3}{2}d_{2,4,2}.$$

Proof. In Lemma 4 we have expressed $[\overline{I}_8^{inv}]$ in terms of the other classes; we will prove that the coefficients $\alpha, \beta, \gamma, \delta$ are all equal to 1, see (6), (7), (8). So let us say how one can express all other classes in terms of boundary.

(1) By pulling back equality (3), it is not difficult to express $\pi_{12}^*([\overline{I}_6])$ and $\pi_{78}^*([\overline{I}_6])$ in terms of boundary strata classes in $\overline{\mathcal{M}}_{0,8}$. It is then lengthy but straightforward to express the product of the latter classes in terms of boundary.
(2) Let us study the class of Div. On $\overline{\mathcal{M}}_{0,5}$ with marked points $\{3, 4, 5, 6, \bullet\}$ there is a divisor corresponding to the condition of $\bullet$ being fixed by the involution (34)(56). The class of the latter divisor, by identifying $\overline{\mathcal{M}}_{0,5}$ with the blow-up of $\mathbb{P}^2$ in four general points, is the proper transform of the class of an hyperplane. Its class is therefore equal to $\psi_\bullet = (346) + (345) + (34)$. The class of Div is the push-forward of the class of this locus under the map that glues $\overline{\mathcal{M}}_{0,5}$ to the $\overline{\mathcal{M}}_{0,5}$ with marked points $\{1, 2, 7, 8, \star\}$:
$$[\mathrm{Div}] = (1278|5|346) + (1278|6|345) + (1278|56|34).$$
(3) The loci Type I and Type II are already boundary.

Once this is settled, the class of $\overline{I}_8^{inv}$ can be computed in terms of the invariant classes $d_{5,1,2}$, $d_{4,3,3}$, $d_{4,1,3}$, $d_{3,3,2}$, $d_{3,2,3}$, and $d_{2,4,2}$ by symmetrizing, similarly to what was done in (4). The inverse image of the locus I_8^{inv} in $\overline{\mathcal{M}}_{0,8}$ is the union of 105 irreducible components, each of them corresponding to an element in S_8 in the conjugacy class of (12)(34)(56)(78). The numbers of irreducible components in $\overline{\mathcal{M}}_{0,8}$ of the invariant loci are 168, 420, 280, 560, 280, and 210, respectively. □

Proof (of Lemma 4). We want to show that

$$\pi_{12}^{-1}(\overline{I}_6) \cap \pi_{78}^{-1}(\overline{I}_6) = \overline{I}_8 \cup \mathrm{Div} \cup \text{Type I} \cup \text{Type II}.$$

That the right hand side is included in the left hand side is a straightforward check. To prove the other inclusion, we consider the stratification of $\overline{\mathcal{M}}_{0,8}$ given by number of nodes. We have already observed that the restriction of the left hand side to the open part $\overline{\mathcal{M}}_{0,8}$ is precisely I_8. To conclude the proof, one has to check the following combinatorial statements involving boundary strata classes of $\overline{\mathcal{M}}_{0,8}$ of codimension 1 and 2.

(1) Among the boundary divisors of $\overline{\mathcal{M}}_{0,8}$, only one has the property that $\pi_{12}^{-1}(\overline{I}_6) \cap \pi_{78}^{-1}(\overline{I}_6)$ cuts a codimension 1 locus on the open part of the divisor that parametrizes curves with precisely one singular point. This boundary divisor is $(1234|5678)$, and $\pi_{12}^{-1}(\overline{I}_6) \cap \pi_{78}^{-1}(\overline{I}_6)$ cuts out in it precisely the locus Div.
(2) The only boundary strata classes of codimension 2 in $\overline{\mathcal{M}}_{0,8}$ that are included in $\pi_{12}^{-1}(\overline{I}_6) \cap \pi_{78}^{-1}(\overline{I}_6)$ are precisely those in Type I and Type II.

□

We now prove the equalities

$$\alpha = \beta = \gamma = \delta = 1,$$

for the coefficients that appear in Lemma 4. This is needed to complete the proof of Corollary 5. We have to perform computations of multiplicities of intersections that take place in $\overline{\mathcal{M}}_{0,8}$.

Lemma 6. *The coefficient α in (4) is* 1.

Proof. We want to show that the intersection of $\pi_{12}^*([\overline{I}_6])$ and $\pi_{78}^*([\overline{I}_6])$ has generically a reduced scheme structure.

We recall a description of $\overline{I}_6$ due to Vermeire in [Ve02], that uses Kapranov's description of $\overline{\mathcal{M}}_{0,6}$. From Kapranov's construction, there is a blow-down map $\overline{\mathcal{M}}_{0,6} \to \mathbb{P}^3$. Vermeire has proved that $\overline{I}_6$ is the proper transform of the divisor $x_0x_1 - x_2x_3$.

Similarly, there is a blow-down map $\overline{\mathcal{M}}_{0,8} \to \mathbb{P}^5$. After restricting to a Zariski open subset U of $\mathbb{P}^5$, $\pi_{12}^*([\overline{I}_6])$ is the proper transform of $x_0x_1 - x_2x_3$ and $\pi_{78}^*([\overline{I}_6])$ the proper transform of $x_0x_1 - x_4x_5$. It can then be checked that the two equations define a reduced subscheme of U. □

For the remaining coefficients, we construct test surfaces for $\overline{\mathcal{M}}_{0,8}$, and see that $\pi_{12}^*([\overline{I}_6])$ and $\pi_{78}^*([\overline{I}_6])$ intersect transversely on them.

Lemma 7. *The coefficients γ and δ in (4) are both* 1.

Proof. We construct a test surface over $\mathbb{P}^1 \times \mathbb{P}^1$. The general fiber is a genus 0 pointed stable curve with one node, which separates the odd markings from the even ones. The points 3 and 4 vary.

More precisely, we define a test surface for $\overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{M}}_{0,5}$ (marked points in $\{1,3,5,7,\star\}$ and $\{2,4,6,8,\bullet\}$ respectively), to obtain then a test surface for $\overline{\mathcal{M}}_{0,8}$ by gluing the last two marked points. So we take the product of two test $\mathbb{P}^1$'s on $\overline{\mathcal{M}}_{0,5}$ to obtain the test surface.

In $\mathcal{M}_{0,4}$ we fix the point p that corresponds to:

$$5 \to 0, \quad \star \to \infty, \quad 1 \to 1, \quad 7 \to 2.$$

The first family on $\mathbb{P}^1$ is obtained as the fiber over p of the last-point-forgetful map $\overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ (marked points $\{1,5,7,\star,3\}$), under the natural identification of the latter map with the universal curve. We call λ the free parameter on the first $\mathbb{P}^1$, corresponding to the variation of the marked point 3.

The second curve on $\mathbb{P}^1$ is constructed similarly, but starting from the point in $\mathcal{M}_{0,4}$ that corresponds to:

$$6 \to 0, \quad \bullet \to \infty, \quad 2 \to 1, \quad 8 \to 3.$$

We call μ the free parameter on the second $\mathbb{P}^1$, corresponding to the variation of the marked point 4.

Now the two divisors $\pi_{12}^*([\overline{I}_6])$ and $\pi_{78}^*([\overline{I}_6])$ define two divisors on the test surface. The divisor $\pi_{12}^*([\overline{I}_6])$ imposes the condition that the quadruples $5, \star, 7, 3$ and $6, \bullet, 8, 4$ define the same point on $\overline{\mathcal{M}}_{0,4}$. Thus it is given by the equation $3\lambda = 2\mu$ on $\mathbb{P}^1_\lambda \times \mathbb{P}^1_\mu$. The divisor $\pi_{78}^*([\overline{I}_6])$ instead imposes that the quadruples $5, \star, 1, 3$ and $6, \bullet, 2, 4$ identify the same point on $\overline{\mathcal{M}}_{0,4}$. Thus it corresponds to the equation $\lambda = \mu$ on $\mathbb{P}^1_\lambda \times \mathbb{P}^1_\mu$. The set-theoretic intersection is therefore in $\lambda = \mu = 0$ and $\lambda = \mu = \infty$. The solution $\lambda = \mu = 0$ corresponds to a curve with three nodes, each of them separating the curve in 2 connected components, and distribution of points $(35|17|28|46)$ (Type I). On the other hand, the second solution corresponds again to a curve with three nodes, each of them separating the curve in 2 connected components, and distribution of points $(157|3|4|268)$ (Type II). The fact that the two equations have degree 1 is enough to establish that both the intersection multiplicities are 1. $\square$

Lemma 8. *The coefficient β in (4) is* 1.

Proof. We construct the following test surface. Fix 4 distinct points $(3, 4, 5, 6)$ on a smooth genus 0 curve C, and let two points $1, \bullet$ vary on it. This defines a test surface for $\overline{\mathcal{M}}_{0,6}$ (markings $\{1, 3, 4, 5, 6, \bullet\}$). This also gives a test surface for $\overline{\mathcal{M}}_{0,8}$ once a choice of 4 distinct points $(2, 7, 8, \star)$ is fixed on a smooth genus 0 curve, by gluing $\bullet$ with $\star$.

We fix an isomorphism of C with $\mathbb{P}^1$ in such a way that

$$3 \to 0 \quad 4 \to \infty, \quad 5 \to 1, \quad 6 \to 4, \quad \bullet \to \lambda, \quad 1 \to \mu.$$

On this test surface, with this choice of coordinates, $\pi_{12}^*([\overline{I}_6])$ is given by the equation $\lambda^2 = 4$, and $\pi_{78}^*([\overline{I}_6])$ is given by $\lambda\mu = 4$, which clearly intersect transversely in

$$\mathrm{Div} = \{(2, 2), (-2, -2)\}.$$

$\square$

6. Pulling back from $\overline{\mathcal{M}}_3$ to the hyperelliptic locus

In this section we study the linear map

$$\phi^* \colon A^2(\overline{\mathcal{M}}_3) \to A^2(\overline{\mathcal{H}}_3^{adm})^{S_8}. \tag{9}$$

We have fixed (1) as the basis in the domain, and (6) as the basis in the image.

Remark 1. In the following lemma we will need an explicit expression of some tautological invariant classes in $\overline{\mathcal{M}}_{0,8}$ in terms of invariant boundary strata classes. These computations can be done using [FaMa].

Recall the Arbarello-Cornalba κ classes:

$$\kappa_i := \pi_* \left(c_1(\omega_\pi(D))^{i+1}\right)$$

where π is the universal curve over $\overline{\mathcal{M}}_{0,8}$, ω_π is the relative dualizing sheaf, and D is the divisor corresponding to the 8 disjoint sections in the universal curve. Another useful invariant class will be $\tilde{\psi}_j := \sum_{i=1}^{8} \psi_i^j$.

In codimension 1 we have (observe that $\kappa_1 = \tilde{\psi}_1 - d_{2,6} - d_{3,5} - d_{4,4}$):

$$\begin{cases} \kappa_1 = & \frac{5}{7} d_{2,6} + \frac{8}{7} d_{3,5} + \frac{9}{7} d_{4,4}, \\ \tilde{\psi}_1 = & \frac{12}{7} d_{2,6} + \frac{15}{7} d_{3,5} + \frac{16}{7} d_{4,4}. \end{cases} \tag{10}$$

In codimension 2, we obtain:

$$\begin{cases} \kappa_2 = & \frac{1}{7} d_{5,1,2} + \frac{1}{7} d_{4,2,2} + \frac{6}{35} d_{4,1,3} + \frac{1}{10} d_{3,3,2} + \frac{6}{35} d_{3,2,3} + \frac{1}{21} d_{2,4,2}, \\ \tilde{\psi}_2 = & \frac{11}{21} d_{5,1,2} + \frac{16}{35} d_{4,2,2} + \frac{3}{7} d_{4,1,3} + \frac{3}{10} d_{3,3,2} + \frac{3}{7} d_{3,2,3} + \frac{16}{105} d_{2,4,2}. \end{cases} \tag{11}$$

Finally, we can express the products of the invariant codimension 1 classes in terms of the invariant codimension 2 classes:

$$\begin{cases} d_{2,6}^2 & = -\frac{2}{3} d_{5,1,2} - \frac{2}{5} d_{4,2,2} - \frac{1}{5} d_{3,3,2} + \frac{28}{15} d_{2,4,2}, \\ d_{2,6} d_{3,5} & = d_{5,1,2} + d_{3,3,2}, \\ d_{2,6} d_{4,4} & = d_{4,2,2}, \\ d_{3,5}^2 & = -\frac{1}{3} d_{5,1,2} - \frac{3}{5} d_{4,1,3} - \frac{1}{10} d_{3,3,2} + \frac{7}{5} d_{3,2,3}, \\ d_{3,5} d_{4,4} & = d_{4,1,3}, \\ d_{4,4}^2 & = -\frac{1}{6} d_{4,2,2} - \frac{1}{2} d_{4,1,3}. \end{cases} \tag{12}$$

We are now in the position of computing the matrix associated with ϕ^*:

Lemma 9. *The following 7×6 matrix is associated to ϕ^* in the bases (1) and (6).*

$$\begin{pmatrix} \frac{1}{42} & \frac{19}{210} & \frac{1}{35} & \frac{1}{20} & \frac{3}{35} & \frac{1}{35} \\ 0 & \frac{11}{15} & 0 & \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{1}{12} & 0 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 \\ -\frac{8}{3} & \frac{86}{15} & -2 & -\frac{4}{5} & 0 & \frac{112}{15} \\ 1 & 0 & 1 & 1 & 0 & 0 \\ -\frac{1}{12} & 0 & -\frac{3}{20} & -\frac{1}{40} & \frac{7}{20} & 0 \\ \frac{13}{84} & \frac{6}{35} & \frac{33}{140} & \frac{1}{8} & \frac{33}{140} & \frac{2}{35} \end{pmatrix}$$

Proof. Let us first study the pull-back ϕ^* at the level of codimension 1 classes. We fix $d_{2,6}$, $d_{3,5}$ and $d_{4,4}$ as a basis of $A^1(\overline{\mathcal{M}}_{0,8})^{S_8}$. We have:

$$\begin{cases} \phi^* \lambda = & \frac{3}{14} d_{2,6} + \frac{1}{7} d_{3,5} + \frac{2}{7} d_{4,4}, \\ \phi^* \delta_0 = & 2 d_{2,6} + 2 d_{4,4}, \\ \phi^* \delta_1 = & \frac{1}{2} d_{3,5}. \end{cases} \tag{13}$$

The last two are well-known equalities (see [HMo98, 6.17, 6.18]), while the first is obtained from the equally well-known equation $12\lambda = \kappa_1 + \delta_0 + \delta_1$. Indeed from [FP00, p. 234], we can compute $\phi^*(\kappa_1) = 2\kappa_1 - \frac{1}{2}\tilde{\psi}_1$, and the latter terms are expressed in codimension 1 boundary strata classes in (10).

From this and from (12), it is immediate to compute the pull-back of the basis (1) in terms of the basis (6). We need only compute $\phi^*(\kappa_2)$, and this can be done again from the equality $\phi^*(\kappa_2) = 2\kappa_2 - \frac{1}{4}\tilde{\psi}_2$ ([FP00, p. 234]). The two terms on the right of the equality can be expressed in the basis (6) by using (11). □

By putting together Corollary 5, Proposition 3 and Lemma 9, we have an explicit expression of $[\overline{\mathcal{B}}_3]_Q$ in the basis (1) up to two parameters. For example, the coordinates for $[\overline{\mathcal{B}}_3]_Q$ in the basis (1) can be written in terms of the coefficient of δ_0^2 (that we call d) and of ϵ, the parameter introduced in Proposition 3:

$$\left(\frac{459+560d\epsilon}{6\epsilon}, \frac{18+58d\epsilon}{3\epsilon}, \frac{-117-136d\epsilon}{3\epsilon}, d, \frac{18+14d\epsilon}{3\epsilon}, \frac{99+32d\epsilon}{6\epsilon}, -\frac{9}{2\epsilon}\right). \tag{14}$$

7. Test surfaces

In this section we study three families of genus 3 stable curves over surfaces Σ_5, Σ_1 and Σ_2. These test surfaces for $\overline{\mathcal{M}}_3$ were first studied by the first author, we refer to [Fa90, Section 2] for their precise definition. We will be able to count the number of bielliptic curves on each of these families by means of elementary considerations. They will provide the following information (in order).

(1) The computation of the number of bielliptic curves on Σ_5 will prove that the coefficient ϵ in Proposition 3 equals 1.
(2) The computation of the number of bielliptic curves on Σ_1 will complete the proof of our main result: Theorem 1.
(3) The computation of the number of bielliptic curves on Σ_2 gives us a consistency check on Corollary 5.

Remark 2. On every family of stable curves the intersection between the loci of admissible (double) covers and the locus of singular curves is transversal. This is a general consequence of the fact that admissible covers are smoothable. In particular, we will use the transversality of the bielliptic locus and the locus parametrizing singular curves. See for example [ACG11, Lemma 6.15, p. 211], where the authors work out this transversality result for the hyperelliptic locus and the boundary of $\overline{\mathcal{M}}_g$.

<u>For the fifth test surface</u> ([Fa90, Section 2.5]), let us consider (E, p), (F, q), two 1-pointed curves of genus 1. On the surface $\Sigma_5 := E \times F$ there is a family of genus 3 curves, whose fiber over (e, f) is obtained by gluing E and F at p, q and at e, f. For a curve in the fiber to admit an admissible bielliptic involution, both e and f need to be points of 2−torsion of the elliptic curves (E, p) and (F, q). Thus we have $9 = 3 \times 3$ such fibers; this is also the value of $[\overline{\mathcal{B}}_3]_Q$ on Σ_5 as the test surface parametrizing singular curves is transversal to the bielliptic locus (cf. Remark 2).

We read in [Fa90, Proposition 2.5] that on this surface the following equalities hold:

$$\lambda^2 = \lambda\delta_0 = \lambda\delta_1 = \kappa_2 = 0, \quad \delta_0^2 = 8, \quad \delta_0\delta_1 = -4, \quad \delta_1^2 = 2.$$

After substituting these values in equation (14) we deduce that $\epsilon = 1$. So *a posteriori* we obtain that the bielliptic locus and the hyperelliptic locus are transversal on all of $\overline{\mathcal{M}}_3$ (cf. Proposition 3).

For the first test surface ([Fa90, Section 2.1]), we consider a curve C of genus 2. On the surface $\Sigma_1 = C \times C$ there is a family whose fiber over (p, q) is obtained by gluing on C the two points p and q. The fibers admit an admissible bielliptic involution when p and q are distinct Weierstrass points of C, thus there are $30 = 6 \times 5$ such fibers. For the same reason as above, we have that 30 is the value of the class $[\overline{\mathcal{B}}_3]_Q$ restricted to Σ_1. We read in [Fa90, Proposition 2.1] that on this surface the following equalities hold:

$$\lambda^2 = \lambda\delta_0 = \lambda\delta_1 = \delta_0\delta_1 = 0, \quad \delta_0^2 = 16, \quad \delta_1^2 = -2, \quad \kappa_2 = 2.$$

Since we now know that ϵ equals 1, Equation 14 gives the last parameter $d = \frac{27}{2}$.

The second test surface ([Fa90, Section 2.2]) is the product $C \times \mathbb{P}^1$, where C is a general genus 2 curve. Given a pencil of elliptic curves over $\mathbb{P}^1$, the fiber over a point (p, q) is obtained by gluing the curve C at p and the elliptic curve over q in the chosen pencil at the origin. No curve in the fiber admits an admissible bielliptic involution. We read in [Fa90, Proposition 2.2] that on this surface the following equalities hold:

$$\lambda^2 = \lambda\delta_0 = \delta_0^2 = \kappa_2 = 0, \quad \lambda\delta_1 = -2, \quad \delta_0\delta_1 = -24, \quad \delta_1^2 = 4, \quad \kappa_2 = 2.$$

These numbers, substituted in Equation 14, give a nontrivial consistency check of Corollary 5 and of Lemma 9.

Note that after the results of [Fa90, Section 2], it is equivalent to know the class of the bielliptic locus in the basis (1), and to know the restriction of the bielliptic class to the seven test surfaces. While we can compute $[\overline{\mathcal{B}}_3]_Q$ on $\Sigma_1, \ldots, \Sigma_5$, we do not know a direct way to compute it on Σ_6 and Σ_7.

After Theorem 1 and [Fa90, Section 2] however, a straightforward computation gives:

$$[\overline{\mathcal{B}}_3]_Q|_{\Sigma_3} = -24, \qquad [\overline{\mathcal{B}}_3]_Q|_{\Sigma_4} = 33, \qquad [\overline{\mathcal{B}}_3]_Q|_{\Sigma_6} = 225, \qquad [\overline{\mathcal{B}}_3]_Q|_{\Sigma_7} = 675.$$

Let us consider the sixth test surface Σ_6 ([Fa90, Section 2.6]). This surface is obtained by applying stable reduction to a linear $\mathbb{P}^2$ inside the $\mathbb{P}^{14}$ of plane quartics. Since the bielliptic locus has codimension 2, and the locus of singular curves has codimension 1, for a generic choice of the linear $\mathbb{P}^2$ the points corresponding to bielliptic curves are all smooth. Again by genericity we have that the number 225 enumerates the number of smooth bielliptic curves on the linear $\mathbb{P}^2$, and that the codimension 2 bielliptic locus in $\mathbb{P}^{14}$ has degree 225.

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