Tautological and non-tautological cohomology of the moduli space of curves

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Abstract. After a short exposition of the basic properties of the tautological ring of $\overline{M}_{g,n}$, we explain three methods of detecting non-tautological classes in cohomology. The first is via curve counting over finite fields. The second is by obtaining length bounds on the action of the symmetric group Σ_n on tautological classes. The third is via classical boundary geometry. Several new non-tautological classes are found.

Contents

0. Introduction

0.1. Overview

Let $\overline{M}_{g,n}$ be the moduli space of Deligne-Mumford stable curves of genus g with n marked points. The cohomology of $\overline{M}_{g,n}$ has a distinguished subring of tautological classes

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n}, \mathbb{Q})$$

studied extensively since Mumford's seminal article [51]. While effective methods for exploring the tautological ring have been developed over the years, the structure of the non-tautological classes remains mysterious. Our goal here, after reviewing the basic definitions and properties of $RH^*(\overline{M}_{g,n})$ in Section 1, is to present three approaches for detecting and studying non-tautological classes.

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We view $\overline{M}_{g,n}$ as a Deligne-Mumford stack. Not much is lost by considering the coarse moduli space (if the automorphism groups are not forgotten). We encourage the reader to take the coarse moduli point of view if stacks are unfamiliar.

0.2. Point counting and modular forms

Since the moduli space of Deligne-Mumford stable curves is defined over $\mathbb{Z}$, reduction to finite fields $\mathbb{F}_q$ is well-defined. Let $\overline{M}_{g,n}(\mathbb{F}_q)$ denote the set of $\mathbb{F}_q$-points. For various ranges of g, n, and q, counting the number of points of $\overline{M}_{g,n}(\mathbb{F}_q)$ is feasible. A wealth of information about $H^*(\overline{M}_{g,n}, \mathbb{Q})$ can be then obtained from the Lefschetz fixed point formula applied to Frobenius.

The first examples where point counting reveals non-tautological cohomology occur in genus 1. The relationship between point counting and elliptic modular forms is discussed in Section 2. By interpreting the counting results in genus 2, a conjectural description of the *entire* cohomology of $\overline{M}_{2,n}$ has been found by Faber and van der Geer [19] in terms of Siegel modular forms. The formula is consistent with point counting data for $n \leqslant 25$. In fact, large parts of the formula have been proven. The genus 2 results are presented in Section 3.

In genus 3, a more complicated investigation involving Teichmüller modular forms is required. The situation is briefly summarized in Section 3.7. As the genus increases, the connection between point counting and modular forms becomes more difficult to understand.

0.3. Representation theory

The symmetric group Σ_n acts on $\overline{M}_{g,n}$ by permuting the markings. As a result, an Σ_n-representation on $H^*(\overline{M}_{g,n}, \mathbb{Q})$ is obtained.

Studying the Σ_n-action on $H^*(\overline{M}_{g,n}, \mathbb{Q})$ provides a second approach to the non-tautological cohomology. In Section 4, we establish an upper bound for the length[1] of the irreducible Σ_n-representations occurring in the tautological ring $R^*(\overline{M}_{g,n})$. In many cases, the bound is sharp. Assuming the conjectural formulas for $H^*(\overline{M}_{2,n}, \mathbb{Q})$ obtained by point counting, we find several classes of Hodge type which are presumably algebraic (by the Hodge conjecture) but cannot possibly be tautological, because the length of the corresponding Σ_n-representations is too large. The first occurs in $\overline{M}_{2,21}$.

The proofs of the length bounds for the Σ_n-action on $R^*(\overline{M}_{g,n})$ are obtained by studying representations induced from the boundary strata. The strong vanishing of Proposition 2 of [23] with tautological boundary terms plays a crucial role in the argument.

[1] The number of parts in the corresponding partition.

0.4. Boundary geometry

The existence of non-tautological cohomology classes of Hodge type was earlier established by Graber and Pandharipande [36]. In particular, an explicit such class in the boundary of $\overline{M}_{2,22}$ was found. In Section 5, we revisit the non-tautological boundary constructions. The old examples are obtained in simpler ways and new examples in $\overline{M}_{2,21}$ are found. The method is by straightforward intersection theory on the moduli space of curves. Finally, we connect the boundary constructions in $\overline{M}_{2,21}$ to the representation investigation of Section 4.

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1. Tautological classes

1.1. Definitions

Let $\overline{M}_{g,n}$ be the moduli space of stable curves of genus g with n marked points defined over $\mathbb{C}$. Let $A^*(\overline{M}_{g,n}, \mathbb{Q})$ denote the Chow ring. The system of tautological rings is defined[2] to be the set of smallest $\mathbb{Q}$-subalgebras of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}, \mathbb{Q}),$$

satisfying the following two properties:

(i) The system is closed under push-forward via all maps forgetting markings:

$$\pi_* : R^*(\overline{M}_{g,n}) \to R^*(\overline{M}_{g,n-1}).$$

(ii) The system is closed under push-forward via all gluing maps:

$$\iota_* : R^*(\overline{M}_{g_1,n_1\cup\{\star\}}) \otimes_{\mathbb{Q}} R^*(\overline{M}_{g_2,n_2\cup\{\bullet\}}) \to R^*(\overline{M}_{g_1+g_2,n_1+n_2}),$$

$$\iota_* : R^*(\overline{M}_{g,n\cup\{\star,\bullet\}}) \to R^*(\overline{M}_{g+1,n}),$$

with attachments along the markings $\star$ and $\bullet$.

While the definition appears restrictive, natural algebraic constructions typically yield Chow classes lying in the tautological ring.

[2] We follow here the definition of tautological classes in Chow given in [23]. Tautological classes in cohomology are discussed in Section 1.7.

1.2. Basic examples

Consider first the cotangent line classes. For each marking i, let

$$\mathbb{L}_i \to \overline{M}_{g,n}$$

denote the associated cotangent line bundle. By definition

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{M}_{g,n}).$$

Let π denote the map forgetting the last marking,

$$\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}, \tag{1.1}$$

and let ι denote the gluing map,

$$\iota : \overline{M}_{g,\{1,2,\ldots,i-1,\star,i+1,\ldots,n\}} \times \overline{M}_{0,\{\bullet,i,n+1\}} \longrightarrow \overline{M}_{g,n+1}.$$

The $\mathbb{Q}$-multiples of the fundamental classes $[\overline{M}_{g,n}]$ are contained in the tautological rings (as $\mathbb{Q}$-multiples of the units in the subalgebras). A direct calculation shows:

$$-\pi_*\Big(\big(\iota_*([\overline{M}_{g,n}] \times [\overline{M}_{0,3}])\big)^2\Big) = \psi_i.$$

Hence, the cotangent line classes lie in the tautological rings,

$$\psi_i \in R^1(\overline{M}_{g,n})\ .$$

Consider next the κ classes defined via push-forward by the forgetful map (1.1),

$$\kappa_j = \pi_*(\psi_{n+1}^{j+1}) \in A^j(\overline{M}_{g,n})\ .$$

Since ψ_{n+1} has already been shown to be tautological, the κ classes are tautological by property (i),

$$\kappa_j \in R^j(\overline{M}_{g,n})\ .$$

The ψ and κ classes are very closely related.

For a nodal curve C, let ω_C denote the dualizing sheaf. The Hodge bundle $\mathbb{E}$ over $\overline{M}_{g,n}$ is the rank g vector bundle with fiber $H^0(C,\omega_C)$ over the moduli point

$$[C, p_1, \ldots, p_n] \in \overline{M}_{g,n}\ .$$

The λ classes are defined by

$$\lambda_k = c_k(\mathbb{E}) \in A^k(\overline{M}_{g,n})\ .$$

The Chern characters of $\mathbb{E}$ lie in the tautological ring by Mumford's Grothendieck-Riemann-Roch computation [51]. Since the λ classes are polynomials in the Chern characters,

$$\lambda_k \in R^k(\overline{M}_{g,n})\ .$$

The ψ and λ classes are basic elements of the tautological ring $R^*(\overline{M}_{g,n})$ arising very often in geometric constructions and calculations. The ψ integrals,

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}, \tag{1.2}$$

can be evaluated by KdV constraints due to Witten and Kontsevich [45, 65]. Other geometric approaches to the ψ integrals can be found in [49, 52]. In genus 0, the simple formula

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \ldots, a_n}$$

holds. Non-integer values occur for the first time in genus 1,

$$\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24} \, .$$

Hodge integrals, when the λ classes are included,

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_1^{b_1} \ldots \lambda_g^{b_g},$$

can be reduced to ψ integrals (1.2) by Mumford's Grothendieck-Riemann-Roch calculation. An example of a Hodge integral evaluation[3] is

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}|}{2g} \frac{|B_{2g-2}|}{2g-2} \frac{1}{(2g-2)!} \tag{1.3}$$

The proof of (1.3) and several other exact formulas for Hodge integrals can be found in [20].

1.3. Strata

The boundary strata of the moduli spaces of curves correspond to *stable graphs*

$$A = (V, H, L, g : V \to \mathbb{Z}_{\geqslant 0}, a : H \to V, i : H \to H)$$

satisfying the following properties:

(1) V is a vertex set with a genus function g,
(2) H is a half-edge set equipped with a vertex assignment a and fixed point free involution i,
(3) E, the edge set, is defined by the orbits of i in H,
(4) (V, E) define a *connected* graph,
(5) L is a set of numbered legs attached to the vertices,
(6) For each vertex v, the stability condition holds:

$$2g(v) - 2 + n(v) > 0,$$

where $n(v)$ is the valence of A at v including both half-edges and legs.

The data of the topological type of a generic curve in a boundary stratum of the moduli space is recorded in the stable graph.

Let A be a stable graph. The genus of A is defined by

$$g = \sum_{v \in V} g(v) + h^1(A).$$

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[3] Here, B_{2g} denotes the Bernoulli number.

Define the moduli space $\overline{M}_A$ by the product

$$\overline{M}_A = \prod_{v \in V(A)} \overline{M}_{g(v),n(v)}.$$

There is a canonical morphism[4]

$$\xi_A : \overline{M}_A \to \overline{M}_{g,n}$$

with image equal to the boundary stratum associated to the graph A. By repeated use of property (ii),

$$\xi_{A*}[\overline{M}_A] \in R^*(\overline{M}_{g,n}) .$$

We can now describe a set of additive generators for $R^*(\overline{M}_{g,n})$. Let A be a stable graph of genus g with n legs. For each vertex v of A, let

$$\theta_v \in R^*(\overline{M}_{g(v),n(v)})$$

be an arbitrary monomial in the ψ and κ classes of the vertex moduli space. The following result is proven in [36].

Theorem 1.4. $R^*(\overline{M}_{g,n})$ *is generated additively by classes of the form*

$$\xi_{A*}\Big(\prod_{v \in V(A)} \theta_v\Big).$$

By the dimension grading, the list of generators provided by Theorem 1.4 is finite. Hence, we obtain the following result.

Corollary 1.5. *We have* $\dim_{\mathbb{Q}} R^*(\overline{M}_{g,n}) < \infty$.

1.4. Further properties

The following two formal properties of the full system of tautological rings

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}, \mathbb{Q}),$$

are a consequence of properties (i) and (ii):

(iii) The system is closed under pull-back via the forgetting and gluing maps.
(iv) $R^*(\overline{M}_{g,n})$ is an $\mathbb{S}_n$-module via the permutation action on the markings.

Property (iii) follows from the well-known boundary geometry of the moduli space of curves. A careful treatment using the additive basis of Theorem 1.4 can be found in [36]. The meaning of property (iii) for the reducible gluing is that the Künneth components of the pull-backs of tautological classes are tautological. Since the defining properties (i) and (ii) are symmetric with respect to the marked points, property (iv) holds.

[4] To construct ξ_A, a family of stable pointed curves over $\overline{M}_A$ is required. Such a family is easily defined by attaching the pull-backs of the universal families over each of the $\overline{M}_{g(v),n(v)}$ along the sections corresponding to half-edges.

1.5. Pairing

Intersection theory on the moduli space $\overline{M}_{g,n}$ yields a canonical pairing

$$\mu : R^k(\overline{M}_{g,n}) \times R^{3g-3+n-k}(\overline{M}_{g,n}) \to \mathbb{Q}$$

defined by

$$\mu(\alpha, \beta) = \int_{\overline{M}_{g,n}} \alpha \cup \beta \ .$$

While the pairing μ has been speculated to be perfect in [21, 38, 53], very few results are known.

The pairing μ can be effectively computed on the generators of Theorem 1.4 by the following method. The pull-back property (iii) may be used repeatedly to reduce the calculation of μ on the generators to integrals of the form

$$\int_{\overline{M}_{h,m}} \psi_1^{a_1} \cdots \psi_m^{a_m} \cdot \kappa_1^{b_1} \cdots \kappa_r^{b_r} \ .$$

The κ classes can be removed to yield a sum of purely ψ integrals

$$\int_{\overline{M}_{h,m+r}} \psi_1^{a_1} \cdots \psi_{m+r}^{a_{m+r}},$$

see [3]. As discussed in Section 1.2, the ψ integrals can be evaluated by KdV constraints.

1.6. Further examples

We present here two geometric constructions which also yield tautological classes. The first is via stable maps and the second via moduli spaces of Hurwitz covers.

Let X be a nonsingular projective variety, and let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps[5] representing $\beta \in H_2(X, \mathbb{Z})$. Let ρ denote the map to the moduli of curves,

$$\rho : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}.$$

The moduli space $\overline{M}_{g,n}(X, \beta)$ carries a virtual class

$$[\overline{M}_{g,n}(X, \beta)]^{vir} \in A_*(\overline{M}_{g,n}(X, \beta))$$

obtained from the canonical obstruction theory of maps.

Theorem 1.6. *Let X be a nonsingular projective toric variety. Then,*

$$\rho_*[\overline{M}_{g,n}(X, \beta)]^{vir} \in R^*(\overline{M}_{g,n}).$$

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[5] We refer the reader to [27] for an introduction to the subject. A discussion of obstruction theories and virtual classes can be found in [4, 5, 47].

The proof follows directly from the virtual localization formula of [35]. If $[C] \in M_g$ is a general moduli point, then

$$\rho_*[\overline{M}_g(C,1)]^{vir} = [C] \in A^{3g-3}(\overline{M}_g).$$

Since not all of $A^{3g-3}(\overline{M}_g)$ is expected to be tautological, Theorem 1.6 is unlikely to hold for C. However, the result perhaps holds for a much more general class of varieties X. In fact, we do not know[6] any nonsingular projective variety defined over $\bar{\mathbb{Q}}$ for which Theorem 1.6 is expected to be false.

The moduli spaces of Hurwitz covers of $\mathbb{P}^1$ also define natural classes on the moduli space of curves. Let $\mu^1, \ldots, \mu^m$ be m partitions of equal size d satisfying

$$2g - 2 + 2d = \sum_{i=1}^{m} \Big(d - \ell(\mu^i)\Big),$$

where $\ell(\mu^i)$ denotes the length of the partition μ^i. The moduli space of Hurwitz covers,

$$H_g(\mu^1, \ldots, \mu^m)$$

parameterizes morphisms,

$$f : C \to \mathbb{P}^1,$$

where C is a complete, connected, nonsingular curve with marked profiles $\mu^1, \ldots, \mu^m$ over m ordered points of the target (and no ramifications elsewhere). The moduli space of Hurwitz covers is a dense open set of the compact moduli space of admissible covers [39],

$$H_g(\mu^1, \ldots, \mu^m) \subset \overline{H}_g(\mu^1, \ldots, \mu^m).$$

Let ρ denote the map to the moduli of curves,

$$\rho : \overline{H}_g(\mu^1, \ldots, \mu^m) \to \overline{M}_{g, \sum_{i=1}^m \ell(\mu^i)}.$$

The following is a central result of [23].

Theorem 1.7. *The push-forwards of the fundamental classes lie in the tautological ring,*

$$\rho_*[\overline{H}_g(\mu^1, \ldots, \mu^m)] \in R^*(\overline{M}_{g, \sum_{i=1}^m \ell(\mu^i)}) \ .$$

The admissible covers in Theorem 1.7 are of $\mathbb{P}^1$. The moduli spaces of admissible covers of higher genus targets can lead to non-tautological classes [36].

[6] We do not know any examples at all of nonsingular projective X where

$$\rho_*[\overline{M}_{g,n}(X,\beta)]^{vir} \notin RH^*(\overline{M}_{g,n}).$$

See [46] for a discussion.

1.7. Tautological rings

The tautological subrings

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n}, \mathbb{Q}) \tag{1.8}$$

are defined to be the images of $R^*(\overline{M}_{g,n})$ under the cycle class map

$$A^*(\overline{M}_{g,n}, \mathbb{Q}) \to H^*(\overline{M}_{g,n}, \mathbb{Q}) .$$

The tautological rings $RH^*(\overline{M}_{g,n})$ could alternatively be defined as the smallest system of subalgebras (1.8) closed under push-forward by all forgetting and gluing maps (properties (i) and (ii)). Properties (iii) and (iv) also hold for $RH^*(\overline{M}_{g,n})$.

Tautological rings in Chow and cohomology may be defined for all stages of the standard moduli filtration:

$$\overline{M}_{g,n} \supset M^c_{g,n} \supset M^{rt}_{g,n}. \tag{1.9}$$

Here, $M^c_{g,n}$ is the moduli space of stable curves of compact type (curves with tree dual graphs, or equivalently, with compact Jacobians), and $M^{rt}_{g,n}$ is the moduli space of stable curves with rational tails (the preimage of the moduli space of nonsingular curves $M_g \subset \overline{M}_g$). The tautological rings

$$R^*(M^c_{g,n}) \subset A^*(M^c_{g,n}, \mathbb{Q}), \quad RH^*(M^c_{g,n}) \subset H^*(M^c_{g,n}, \mathbb{Q}),$$

$$R^*(M^{rt}_{g,n}) \subset A^*(M^{rt}_{g,n}, \mathbb{Q}), \quad RH^*(M^{rt}_{g,n}) \subset H^*(M^{rt}_{g,n}, \mathbb{Q})$$

are all defined as the images of $R^*(\overline{M}_{g,n})$ under the natural restriction and cycle class maps.

Of all the tautological rings, the most studied case by far is $R^*(M_g)$. A complete (conjectural) structure of $R^*(M_g)$ is proposed in [18] with important advances made in [42, 48, 50]. Conjectures for the compact type cases $R^*(M^c_{g,n})$ can be found in [21]. See [22, 54] for positive results for compact type. Tavakol [59, 60] has recently proved that $R^*(M^c_{1,n})$ and $R^*(M^{rt}_{2,n})$ are Gorenstein, with socles in degrees $n-1$ and n, respectively.

1.8. Hyperelliptic curves

As an example, we calculate the class of the hyperelliptic locus

$$H_g \subset M_g$$

following [51].

Over the moduli point $[C, p] \in M_{g,1}$ there is a canonical map

$$\phi : H^0(C, \omega_C) \to H^0(C, \omega_C/\omega_C(-2p))$$

defined by evaluating sections of ω_C at the marking $p \in C$. If we define $\mathbb{J}$ to be the rank 2 bundle over $M_{g,1}$ with fiber $H^0(C, \omega_C/\omega_C(-2p))$ over $[C, p]$, then we obtain a morphism

$$\phi : \mathbb{E} \to \mathbb{J}$$

over $M_{g,1}$. By classical curve theory, the map ϕ fails to be surjective precisely when C is hyperelliptic and $p \in C$ is a Weierstrass point. Let

$$\Delta \subset M_{g,1}$$

be the degeneracy locus of ϕ of pure codimension $g-1$. By the Thom-Porteous formula [26],

$$[\Delta] = \left(\frac{c(\mathbb{E}^*)}{c(\mathbb{J}^*)}\right)_{g-1} .$$

By using the jet bundle sequence

$$0 \to \mathbb{L}_1^2 \to \mathbb{J} \to \mathbb{L}_1 \to 0,$$

we conclude

$$[\Delta] = \left(\frac{1-\lambda_1+\lambda_2-\lambda_3+\ldots+(-1)^g\lambda_g}{(1-\psi_1)(1-2\psi_1)}\right)_{g-1} \in R^{g-1}(M_{g,1}) .$$

To find a formula for the hyperelliptic locus, we view

$$\pi : M_{g,1} \to M_g$$

as the universal curve over moduli space. Since each hyperelliptic curve has $2g+2$ Weierstrass points,

$$[H_g] = \frac{1}{2g+2}\pi_*([\Delta]) \in R^{g-2}(M_g) .$$

By Theorem 1.7, the class of the closure of the hyperelliptic locus is also tautological,

$$[\overline{H}_g] \in R^{g-2}(\overline{M}_g),$$

but no simple formula is known to us.

2. Point counting and elliptic modular forms

2.1. Elliptic modular forms

We present here an introduction to the close relationship between modular forms and the cohomology of moduli spaces. The connection is most direct between elliptic modular forms and moduli spaces of elliptic curves.

Classically, a modular form is a holomorphic function f on the upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$$

with an amazing amount of symmetry. Precisely, a modular form of *weight* $k \in \mathbb{Z}$ satisfies the functional equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. By the symmetry for $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,

$$f(z) = (-1)^k f(z),$$

so k must be even. Using $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we see

$$f(z+1)=f(z),$$

so f can be written as a function of $q=\exp(2\pi iz)$. We further require f to be holomorphic at $q=0$. The Fourier expansion is

$$f(q)=\sum_{n=0}^{\infty}a_nq^n.$$

If a_0 vanishes, f is called a *cusp* form (of weight k for $SL(2,\mathbb{Z})$).

Well-known examples of modular forms are the Eisenstein series[7]

$$E_k(q)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

of weight $k\geqslant 4$ and the discriminant cusp form

$$\Delta(q)=\sum_{n=1}^{\infty}\tau(n)q^n=q\prod_{n=1}^{\infty}(1-q^n)^{24} \tag{2.1}$$

of weight 12 with the Jacobi product expansion (defining the Ramanujan τ-function). In both of the above formulas, the right sides are the Fourier expansions.

Let M_k be the vector space of holomorphic modular forms of weight k for $SL(2,\mathbb{Z})$. The ring of modular forms is freely generated [44, Exc. III.2.5] by E_4 and E_6,

$$\bigoplus_{k=0}^{\infty}M_k=\mathbb{C}[E_4,E_6]\ ,$$

and the ideal $\bigoplus_k S_k$ of cusp forms is generated by Δ. Let $m_k=\dim M_k$ and $s_k=m_k-1$.

For every positive integer n, there exists a naturally defined *Hecke operator*

$$T_n : M_k\to M_k\ .$$

The T_n commute with each other and preserve the subspace S_k. Moreover, S_k has a basis of simultaneous eigenforms, which are orthogonal with respect to the *Petersson inner product* and can be normalized to have $a_1=1$. The T_n-eigenvalue of such an eigenform equals its n-th Fourier coefficient [44, Prop. III.40].

2.2. Point counting in genus 1

While the moduli space $M_{1,1}$ of elliptic curves can be realized over $\mathbb{C}$ as the analytic space $\mathbb{H}/SL(2,\mathbb{Z})$, we will view $M_{1,1}$ here as an algebraic variety over a field k or a stack over $\mathbb{Z}$.

Over $\overline{k}$, the j-invariant classifies elliptic curves, so

$$M_{1,1}\ \cong_{\overline{k}}\ \mathbb{A}^1$$

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[7] Recall the Bernoulli numbers B_k are given by $\frac{x}{e^x-1}=\sum B_k\frac{x^k}{k!}$ and $\sigma_m(n)$ is the sum of the m-th powers of the positive divisors of n.

as a coarse moduli space. But what happens over a finite field $\mathbb{F}_p$, where p is a prime number? Can we calculate $\#M_{1,1}(\mathbb{F}_p)$? If we count elliptic curves E over $\mathbb{F}_p$ (up to $\mathbb{F}_p$-isomorphism) with weight factor

$$\frac{1}{\#\mathrm{Aut}_{\mathbb{F}_p}(E)}\,, \tag{2.2}$$

we obtain the expected result.

Proposition 2.3. *We have* $\#M_{1,1}(\mathbb{F}_p) = p$.

Proof. The counting is very simple for $p \neq 2$. Given an elliptic curve with a point $z \in E$ defined over $\mathbb{F}_p$, we obtain a degree 2 morphism

$$\phi : E \to \mathbb{P}^1$$

from the linear series associated to $\mathcal{O}_E(2z)$. If we view the image of z as $\infty \in \mathbb{P}^1$, the branched covering ϕ expresses E in Weierstrass form,

$$y^2 = x^3 + ax^2 + bx + c \quad \text{with} \;\; a, b, c \in \mathbb{F}_p,$$

where the cubic on the right has distinct roots in $\overline{\mathbb{F}}_p$. The number of monic cubics in x is p^3, the number with a single pair of double roots is $p^2 - p$, and the number with a triple root is p. Hence, the number of Weierstrass forms is $p^3 - p^2$.

Since only $\infty \in \mathbb{P}^1$ is distinguished, we must further divide by the group of affine transformations of

$$\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$$

over $\mathbb{F}_p$. Since the order of the affine transformation group is $p^2 - p$, we conclude

$$\#M_{1,1}(\mathbb{F}_p) = \frac{p^3 - p^2}{p^2 - p} = p\,.$$

We leave to the reader to check the above counting weights curves by the factor (2.2). We also leave the $p = 2$ case to the reader. □

Motivated by Proposition 2.3, we will study the number of points of $M_{1,n}(\mathbb{F}_p)$ weighted by the order of the $\mathbb{F}_p$-automorphism group of the marked curve. For example,

$$\#M_{1,6}(\mathbb{F}_2) = 0,$$

since an elliptic curve E over $\mathbb{F}_2$ contains at most 5 distinct points by the Weil bound [40, Exc. V.1.10],

$$|\#E(\mathbb{F}_2) - (2+1)| \leqslant 2\sqrt{2}\,.$$

To investigate the behavior of $M_{1,n}(\mathbb{F}_p)$, we can effectively enumerate n-pointed elliptic curves over $\mathbb{F}_p$ by computer. Interpreting the data, we find for $n \leqslant 10$,

$$\#M_{1,n}(\mathbb{F}_p) = f_n(p), \tag{2.4}$$

where f_n is a monic polynomial of degree n with integral coefficients. The moduli spaces $M_{1,n}$ are rational varieties [6] for $n \leqslant 10$. The polynomiality (2.4) has been proved directly by Bergström [8, §15] using geometric constructions (as in the proof of Proposition 2.3). However for $n = 11$, the computer counts exclude the possibility of a degree 11 polynomial for $\#M_{1,11}(\mathbb{F}_p)$. The data suggest

$$\#M_{1,11}(\mathbb{F}_p) = f_{11}(p) - \tau(p). \tag{2.5}$$

For higher n, the p-th Fourier coefficients of the other Hecke cusp eigenforms eventually appear in the counting as well (multiplied with polynomials in p).

2.3. Cohomology of local systems

To understand what is going on, we recall the number of points of $M_{1,n}$ over $\mathbb{F}_p$ equals the trace of Frobenius on the Euler characteristic of the compactly supported ℓ-adic cohomology of $M_{1,n}$. The Euler characteristic has good additivity properties and the multiplicities in applying the Lefschetz fixed point formula to Frob_p are all equal to 1. The cohomology of projective space yields traces which are polynomial in p,

$$\#\mathbb{P}^n(\mathbb{F}_p) = p^n + p^{n-1} + \cdots + p + 1,$$

since the trace of Frobenius on the Galois representation $\mathbb{Q}_\ell(-i)$ equals p^i.

As is well-known, the fundamental class of an algebraic subvariety is of Tate type (fixed by the Galois group). The Tate conjecture essentially asserts the converse. If equation (2.5) holds for all p, then $\#M_{1,11}(\mathbb{F}_p)$ will fail to fit a degree 11 polynomial (even for all but finitely many p). We can then conclude $M_{1,11}$ possesses cohomology not represented by algebraic classes defined over $\mathbb{Q}$. Such cohomology, in particular, can not be tautological.

In fact, the moduli space $M_{1,n}(\mathbb{C})$ has cohomology related to cusp forms. The simplest connection is the construction of a holomorphic differential form on $M_{1,11}$ from the discriminant (2.1). We view $M_{1,11}$ as an open subset[8]

$$M_{1,11} \subset \frac{\mathbb{H} \times \mathbb{C}^{10}}{\mathsf{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^{10}} .$$

The multiplication in the semidirect product $\mathsf{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^{10}$ is defined by the following formula

$$\begin{aligned}\Big(\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right), (x_1', y_1'), \ldots, (x_{10}', y_{10}')\Big) \cdot \Big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (x_1, y_1), \ldots, (x_{10}, y_{10})\Big) = \\ \Big(\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (x_1', y_1') + \rho\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \cdot (x_1, y_1), \ldots, \\ (x_{10}', y_{10}') + \rho\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)(x_{10}, y_{10})\Big)\end{aligned}$$

where ρ is conjugate to the standard representation

$$\rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) . \tag{2.6}$$

[8] In fact, $M_{1,11}$ is the open set obtained by removing lattice translates of zero sections and diagonal loci from the right side.

As usual, $\mathsf{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ via linear fractional transformations. At the point $(z,\zeta_1,\ldots,\zeta_{10}) \in \mathbb{H}\times\mathbb{C}^{10}$, the action is

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, (x_1,y_1),\ldots,(x_{10},y_{10})\right)\cdot(z,\zeta_1,\ldots,\zeta_{10}) =$$
$$\left(\left(\frac{az+b}{cz+d}\right), \frac{\zeta_1}{cz+d}+x_1+y_1\left(\frac{az+b}{cz+d}\right),\ldots,\frac{\zeta_{10}}{cz+d}+x_{10}+y_{10}\left(\frac{az+b}{cz+d}\right)\right).$$

A direct verification using the weight 12 functional equation for the discriminant shows the holomorphic 11-form

$$\Delta(e^{2\pi i z})\, dz\wedge d\zeta_1\wedge\ldots\wedge d\zeta_{10} \tag{2.7}$$

on $\mathbb{H}\times\mathbb{C}^{10}$ is invariant under the action and descends to $M_{1,11}$. Since the discriminant is a cusp form, the form (2.7) extends to a nontrivial element of $H^{11,0}(\overline{M}_{1,11},\mathbb{C})$. In fact,

$$H^{11,0}(\overline{M}_{1,11},\mathbb{C})\cong\mathbb{C}$$

by calculations of [32] (the Σ_{11}-representation on $H^{11,0}(\overline{M}_{1,11},\mathbb{C})$ is alternating). As a consequence, $\overline{M}_{1,11}$ is irrational.

Returning to the open locus $M_{1,n}$, we can consider the fibration

$$\pi : M_{1,n}\to M_{1,1}$$

with fibers given by open subsets of E^{n-1} (up to automorphisms). The only interesting cohomology of E is $H^1(E,\mathbb{Q})$. The motivic Euler characteristic of the cohomology of the fibers of π can be expressed in terms of the symmetric powers of $H^1(E,\mathbb{Q})$. Letting E vary, the a-th symmetric power gives rise to a *local system* $\mathbb{V}_a$ on $M_{1,1}$. The cohomology of these local systems was studied in detail by Shimura. A basic result [58] (see also [63, Thm. 4.2.6]) is the *Shimura isomorphism*,

$$H^1_!(M_{1,1},\mathbb{V}_a)\otimes\mathbb{C} = S_{a+2}\oplus\overline{S}_{a+2}\,.$$

Here, $H^i_!$ is the *inner cohomology*, the image of the cohomology H^i_c with compact support in the usual cohomology H^i. The Shimura isomorphism gives a connection between $M_{1,11}$ and the space of cusp forms S_{12}. The inner cohomology group $H^1_!(M_{1,1},\mathbb{V}_a)$ has a pure Hodge structure of weight $a+1$ and S_{a+2} has Hodge type $(a+1,0)$. We have found non-algebraic cohomology, but we still need to understand the contribution to the trace of Frobenius.

As shown by Deligne [16, Prop. 3.19], Hecke operators can be defined on the inner cohomology group $H^1_!(M_{1,1},\mathbb{V}_a)$ compatibly with the Shimura isomorphism and the earlier operators on the spaces of cusp forms. The Eichler-Shimura congruence relation [16, Thm. 4.9], [17, 58] establishes a connection between the Hecke operator T_p on $H^1_!(M_{1,1},\mathbb{V}_a)\otimes\mathbb{Q}_\ell$ and the Frobenius at p. More precisely, T_p equals the sum of Frobenius and Verschiebung at p (the adjoint of Frobenius with respect to the natural scalar product). As explained in [16, (5.7)], the results finally show the trace of Frobenius at p on $H^1_!(M_{1,1}, \mathbb{V}_a) \otimes \mathbb{Q}_\ell$ to be equal to the trace of the Hecke operator T_p on S_{a+2}, which is the sum of the p-th Fourier coefficients of the normalized Hecke cusp eigenforms.

We are now in a position to derive the mysterious counting formula

$$\#M_{1,11}(\mathbb{F}_p) = f_{11}(p) - \tau(p).$$

The kernel of the map

$$H^i_c(M_{1,1}, \mathbb{V}_a) \to H^i(M_{1,1}, \mathbb{V}_a)$$

is the *Eisenstein cohomology*. Like the inner cohomology, the Eisenstein cohomology for $a > 0$ is concentrated in degree 1. It consists of Hodge-Tate classes of weight 0 which over $\mathbb{C}$ are represented by suitably normalized Eisenstein series. Writing $S[a+2]$ for $H^1_!(M_{1,1}, \mathbb{V}_a)$, we obtain

$$e_c(M_{1,1}, \mathbb{V}_a) = -S[a+2] - 1 \tag{2.8}$$

for $a > 0$ even, while the Euler characteristic vanishes for a odd. A motivic construction of $S[a+2]$ can be found in [15, 57], see also [56]. By the discussion in the previous paragraph, the trace of Frobenius picks up $-\tau(p)$ from (2.8). As another consequence of (2.8), we find the following formula for the integer valued Euler characteristic,

$$E_c(M_{1,1}, \mathbb{V}_a) = -2s_{a+2} - 1.$$

In fact, formula (2.8) has been used by Getzler [32] to calculate the full Hodge decomposition of the cohomology of $\overline{M}_{1,n}$.

3. Point counting and Siegel modular forms

3.1. Summary

Siegel modular forms are related to moduli spaces of abelian varieties. As a consequence, a strong relationship between such modular forms and the cohomology of moduli spaces of curves of genus 2 is obtained, leading to a complete (conjectural) description of the cohomology of $\overline{M}_{2,n}$. Perhaps surprisingly, the resulting description in genus 3 is not complete. Elliptic and Siegel modular forms do not suffice to describe the cohomology of $M_{3,n}$ for n large enough — the Teichmüller modular forms introduced by Ichikawa [41] are needed. In genus 2 and 3, data from point counting over finite fields is used to formulate the conjectures and to explore the cohomology.

3.2. Siegel modular forms

In higher genus $g > 1$, the analogue of elliptic modular forms are *Siegel modular forms*. The latter are defined on $\mathbb{H}_g$, the space of $g \times g$ complex symmetric matrices with positive-definite imaginary part and satisfy the functional equation

$$f((az+b)(cz+d)^{-1}) = \rho(cz+d)f(z)$$

for all $z \in \mathbb{H}_g$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Sp}(2g, \mathbb{Z})$. Here, ρ is a representation of $\mathsf{GL}(g, \mathbb{C})$. Classically,

$$\rho(M) = \mathrm{Det}^k(M),$$

but other irreducible representations of $\mathsf{GL}(g, \mathbb{C})$ are just as relevant. The form f is vector valued. Since z has positive definite imaginary part, $cz + d$ is indeed invertible.

Hecke operators can be defined on Siegel modular forms [1, 2]. The Fourier coefficients of an eigenform here contain much more information than just the Hecke eigenvalues:

(i) the coefficients are indexed by positive semi-definite $g \times g$ symmetric matrices with integers on the diagonal and half-integers elsewhere,

(ii) the coefficients themselves are vectors.

Cusp forms are Siegel modular forms in the kernel of the Siegel Φ-operator:

$$(\Phi f)(z') = \lim_{t \to \infty} f \begin{pmatrix} z' & 0 \\ 0 & it \end{pmatrix},$$

for $z' \in \mathbb{H}_{g-1}$. As before, the Hecke operators preserve the space of cusp forms.

3.3. Point counting in genus 2

Since genus 2 curves are double covers of $\mathbb{P}^1$, we can effectively count them over $\mathbb{F}_p$. As before, each isomorphism class over $\mathbb{F}_p$ is counted with the reciprocal of the number of $\mathbb{F}_p$-automorphisms. For small n, the enumeration of $M_{2,n}(\mathbb{F}_p)$ can be done by hand as in Proposition 2.3. For large n, computer counting is needed.

Bergström [9] has proven the polynomiality in p of $\#M_{2,n}(\mathbb{F}_p)$ for $n \leqslant 7$ by geometric stratification of the moduli space. Experimentally, we find $\#M_{2,n}(\mathbb{F}_p)$ is a polynomial of degree $n+3$ in p for $n \leqslant 9$. In the range $10 \leqslant n \leqslant 13$, the function $\tau(p)$ multiplied by a polynomial appears in $\#M_{2,n}(\mathbb{F}_p)$. More precisely, computer counting predicts

$$\#M_{2,10}(\mathbb{F}_p) = f_{13}(p) + (p-9)\tau(p)\ .$$

If the above formula holds, then the possibility of a degree 13 polynomial for $\#M_{2,10}(\mathbb{F}_p)$ is excluded, so we conclude $M_{2,10}$ has cohomology which is not algebraic over $\mathbb{Q}$ (and therefore non-tautological).

Higher weight modular forms appear in the counting for $n = 14$. Computer counts predict

$$\#M_{2,14}(\mathbb{F}_p) = f_{17}(p) + f_5(p)\tau(p) - 13c_{16}(p) - 429c_{18}(p),$$

where $c_{16}(p)$ and $c_{18}(p)$ are the Fourier coefficients of the normalized Hecke cusp eigenforms of weights 16 and 18. The appearance of $c_{16}(p)$ is perhaps not so surprising since

$$\tau = c_{12}$$

occurs in the function $\#M_{2,10}(\mathbb{F}_p)$. By counting Σ_{14}-equivariantly, we find the coefficient 13 arises as the dimension of the irreducible representation corresponding to the partition $[2\,1^{12}]$ (the tensor product of the standard representation and the alternating representation).

More surprising is the appearance of $c_{18}(p)$. Reasoning as before in Section 2.2, we would expect the motive $S[18]$ to occur in the cohomology of $M_{2,14}$. Since the Hodge types are (17,0) and (0,17), the motive $S[18]$ cannot possibly come from the boundary of the 17 dimensional moduli space, but must come from the interior. We would then expect a Siegel cusp form to be responsible for the occurrence, with Hecke eigenvalues closely related to those of the elliptic cusp form of weight 18. There indeed exists such a Siegel modular form: the classical Siegel cusp form χ_{10} of weight 10, the product of the squares of the 10 even theta characteristics, is a Saito-Kurokawa lift of $E_6\Delta$. In particular, the Hecke eigenvalues are related via

$$\lambda_{\chi_{10}}(p) = \lambda_{E_6\Delta}(p) + p^8 + p^9 = c_{18}(p) + p^8 + p^9.$$

The coefficient 429 is the dimension of the irreducible representation corresponding to the partition $[2^7]$, as a Σ_{14}-equivariant count shows.

The function $\#M_{2,15}(\mathbb{F}_p)$ is expressible in terms of τ, c_{16}, and c_{18} as expected. For $n = 16$, the Hecke eigenvalues of a vector valued Siegel cusp form appear for the first time: the unique form of type

$$\rho = \mathrm{Sym}^6 \otimes \mathrm{Det}^8$$

constructed by Ibukiyama (see [19]) arises. For $n \leqslant 25$, the computer counts of $\#M_{2,n}(\mathbb{F}_p)$ have to a large extent been successfully fit by Hecke eigenvalues of Siegel cusp forms [37].

3.4. Local systems in genus 2

To explain the experimental results for $\#M_{2,n}(\mathbb{F}_p)$ discussed above, we study the cohomology of local systems on the moduli space M_2. Via the Jacobian, there is an inclusion

$$M_2 \subset A_2$$

as an open subset in the moduli space of principally polarized abelian surfaces. In fact, the geometry of local systems on A_2 is a more natural object of study.

The irreducible representations $V_{a,b}$ of $\mathrm{Sp}(4,\mathbb{Z})$ are indexed by integers

$$a \geqslant b \geqslant 0.$$

More precisely, $V_{a,b}$ is the irreducible representation of highest weight occurring in $\mathrm{Sym}^{a-b}(R) \otimes \mathrm{Sym}^b(\wedge^2 R)$, where R is the dual of the standard representation. Since the moduli space of abelian surfaces arises as a quotient,

$$A_2 = \mathbb{H}_2/\mathrm{Sp}(4,\mathbb{Z}),$$

we obtain local systems $\mathbb{V}_{a,b}$ for $\mathrm{Sp}(4,\mathbb{Z})$ on A_2 associated to the representations $V_{a,b}$. The local system $\mathbb{V}_{a,b}$ is *regular* if $a > b > 0$.

Faltings and Chai [25, Thm. VI.5.5] relate the cohomology of $\mathbb{V}_{a,b}$ on A_2 to the space $S_{a-b,b+3}$ of Siegel cusp forms of type $\mathrm{Sym}^{a-b} \otimes \mathrm{Det}^{b+3}$. To start, $H^i_c(A_2, \mathbb{V}_{a,b})$ has a natural mixed Hodge structure with weights at most $a+b+i$, and $H^i(A_2, \mathbb{V}_{a,b})$ has a mixed Hodge structure with weights at least $a+b+i$. The inner cohomology $H^i_!(A_2, \mathbb{V}_{a,b})$ therefore has a pure Hodge structure of weight $a+b+i$. Faltings [24, Cor. of Thm. 9] had earlier shown that $H^i_!(A_2, \mathbb{V}_{a,b})$ is concentrated in degree 3 when the local system is regular. For the Hodge structures above, the degrees of the Hodge filtration are contained in

$$\{0, b+1, a+2, a+b+3\}.$$

Finally, there are natural isomorphisms

$$F^{a+b+3}H^3(A_2, \mathbb{V}_{a,b}) \cong M_{a-b,b+3},$$
$$F^{a+b+3}H^3_c(A_2, \mathbb{V}_{a,b}) = F^{a+b+3}H^3_!(A_2, \mathbb{V}_{a,b}) \cong S_{a-b,b+3}$$

with the spaces of Siegel modular (respectively cusp) forms of the type mentioned above. The quotients in the Hodge filtration are isomorphic to certain explicit coherent cohomology groups, see [31, Thm. 17].

One is inclined to expect the occurrence of motives $S[a-b, b+3]$ corresponding to Siegel cusp forms in $H^3_!(A_2, \mathbb{V}_{a,b})$, at least in the case of a regular weight. Each Hecke cusp eigenform should contribute a 4-dimensional piece with four 1-dimensional pieces in the Hodge decomposition of types

$$(a+b+3{,}0),\ (a+2, b+1),\ (b+1, a+2),\ (0, a+b+3).$$

By the theory of automorphic representations, the inner cohomology may contain other terms as well, the so-called endoscopic contributions. In the present case, the endoscopic contributions come apparently only[9] with Hodge types $(a+2, b+1)$ and $(b+1, a+2)$.

Based on the results from equivariant point counts, Faber and van der Geer [19] obtain the following explicit conjectural formula for the Euler characteristic of the compactly supported cohomology:

Conjecture. *For* $a > b > 0$ *and* $a+b$ *even,*

$$\begin{aligned} e_c(A_2, \mathbb{V}_{a,b}) = {} & -S[a-b, b+3] - s_{a+b+4}S[a-b+2]L^{b+1} \\ & + s_{a-b+2} - s_{a+b+4}L^{b+1} - S[a+3] + S[b+2] + \tfrac{1}{2}(1+(-1)^a). \end{aligned}$$

The terms in the second line constitute the Euler characteristic of the Eisenstein cohomology. The second term in the first line is the endoscopic contribution. Just as in the genus 1 case, the hyperelliptic involution causes the vanishing of all cohomology when $a+b$ is odd. However, the above conjecture is not quite a precise

[9] However, our use of the term endoscopic is potentially non-standard.

statement. While L is the Lefschetz motive and S[k] has been discussed in Section 2.3, the motives $S[a-b, b+3]$ have not been constructed yet. Nevertheless, several precise predictions of the conjecture will be discussed.

First, we can specialize the conjecture to yield a prediction for the integer valued Euler characteristic $E_c(A_2, \mathbb{V}_{a,b})$. The formula from the conjecture is

$$E_c(A_2, \mathbb{V}_{a,b}) = -4s_{a-b,b+3} - 2s_{a+b+4}s_{a-b+2} \\ + s_{a-b+2} - s_{a+b+4} - 2s_{a+3} + 2s_{b+2} + \tfrac{1}{2}(1+(-1)^a).$$

for $a > b > 0$ and $a+b$ even. The lower case s denotes the dimension of the corresponding space of cusp forms. The dimension formula was proved by Grundh [37] for $b > 1$ using earlier work of Getzler [33] and a formula of Tsushima [61] for $s_{j,k}$, proved for $k > 4$ (and presumably true for $k = 4$). In fact, combining work of Weissauer [64] on the inner cohomology and of van der Geer [28] on the Eisenstein cohomology, one may deduce the implication of the conjecture obtained by taking the realizations as ℓ-adic Galois representations of all terms.

A second specialization of the conjecture yields a prediction for $\#M_{2,n}(\mathbb{F}_p)$ via the eigenvalues of Hecke cusp forms. The prediction has been checked for $n \leqslant 17$ and $p \leqslant 23$. The prediction for the trace of Frob_p on $e_c(M_2, \mathbb{V}_{a,b})$ has been checked in many more cases. Consider the local systems $\mathbb{V}_{a,b}$ with $a+b \leqslant 24$ and $b \geqslant 5$. There are 13 such local systems with $\dim S_{a-b,b+3} = 1$; the prediction has been checked for 11 of them, for $p \leqslant 23$. There are also 6 such local systems with $\dim S_{a-b,b+3} = 2$, the maximal dimension; the prediction has been checked for all 6 of them, for $p \leqslant 17$.

We believe the conjecture to be correct also when $a = b$ or $b = 0$ after a suitable re-interpretation of several terms. To start, we set $s_2 = -1$ and define

$$S[2] = -L - 1,$$

which of course is *not* equal to $H^1_!(M_{1,1}, \mathbb{V}_0)$ but does yield the correct answer for

$$e_c(M_{1,1}, \mathbb{V}_0) = L.$$

Let $S[k] = 0$ for odd k. Similarly, define

$$S[0,3] = -L^3 - L^2 - L - 1 .$$

By analogy, let $s_{0,3} = -1$, which equals the value produced by Tsushima's formula (the only negative value of $s_{j,k}$ for $k \geqslant 3$). We then obtain the correct answer for

$$e_c(A_2, \mathbb{V}_{0,0}) = L^3 + L^2.$$

Finally, $S[0,10]$ is defined as $L^8 + S[18] + L^9$, and more generally, $S[0, m+1]$ includes (for m odd) a contribution $SK[0, m+1]$ defined as $S[2m] + s_{2m}(L^{m-1} + L^m)$ — corresponding to the Saito-Kurokawa lifts, but not a part of $H^3_!(A_2, \mathbb{V}_{m-2,m-2})$.

3.5. Holomorphic differential forms

We can use the Siegel modular form χ_{10} to construct holomorphic differential forms on $M_{2,14}$. To start, we consider the seventh fiber product U^7 of the universal abelian surface

$$U \to A_2 \ .$$

We can construct U^7 as a quotient,

$$U^7 = \frac{\mathbb{H}_2 \times (\mathbb{C}^2)^7}{\mathsf{Sp}(4,\mathbb{Z}) \ltimes (\mathbb{Z}^4)^7} \ .$$

The semidirect product is defined by the conjugate of the standard representation following (2.6). We will take $z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{smallmatrix}\right) \in \mathbb{H}_2$ and $\zeta_i = (\zeta_{i1}, \zeta_{i2}) \in \mathbb{C}^2$ to be coordinates. At the point

$$(z, \zeta_1, \ldots, \zeta_7) \in \mathbb{H}_2 \times (\mathbb{C}^2)^7,$$

the action is

$$\begin{aligned} &\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (x_1, y_1), \ldots, (x_7, y_7)\right) \cdot (z, \zeta_1, \ldots, \zeta_7) = \\ &\qquad \Big((az+b)(cz+d)^{-1}, \zeta_1(cz+d)^{-1} + x_1 + y_1(az+b)(cz+d)^{-1}, \\ &\qquad\qquad \ldots, \zeta_7(cz+d)^{-1} + x_7 + y_7(az+b)(cz+d)^{-1}\Big) \ , \end{aligned}$$

where $x_i, y_i \in \mathbb{Z}^2$. A direct verification using the weight 10 functional equation for χ_{10} shows the holomorphic 17-form

$$\chi_{10}(z)\, dz_{11} \wedge dz_{12} \wedge dz_{22} \wedge d\zeta_{11} \wedge d\zeta_{12} \wedge \ldots \wedge d\zeta_{71} \wedge d\zeta_{72} \tag{3.1}$$

on $\mathbb{H}_2 \times (\mathbb{C}^2)^7$ is invariant under the action and descends to U^7.

To obtain a 17-form on $M_{2,14}$, we consider the Abel-Jacobi map $M_{2,14} \to U^7$ defined by

$$(C, p_1, \ldots, p_{14}) \mapsto (\mathrm{Jac}_0(C), \omega_C^*(p_1+p_2), \ldots, \omega_C^*(p_{13}+p_{14})) \ .$$

The pull-back of (3.1) yields a holomorphic 17-form on $M_{2,14}$. Since χ_{10} is a cusp form, the pull-back extends to a nontrivial element of $H^{17,0}(\overline{M}_{2,14}, \mathbb{C})$. Assuming the conjecture for local systems in genus 2,

$$H^{17,0}(\overline{M}_{2,14}, \mathbb{C}) \cong \mathbb{C}^{429}$$

with Σ_{14}-representation irreducible of type $[2^7]$. We leave as an exercise for the reader to show our construction of 17-forms naturally yields the representation $[2^7]$.

The existence of 17-forms implies the irrationality of $\overline{M}_{2,14}$. Both $M_{2,n\leqslant 12}$ and the quotient of $M_{2,13}$ obtained by unordering the last two points are rational by classical constructions [14]. Is $M_{2,13}$ rational?

3.6. The compactification $\overline{M}_{2,n}$

We now assume the conjectural formula for $e_c(A_2, \mathbb{V}_{a,b})$ holds for $a \geqslant b \geqslant 0$. As a consequence, we can compute the Hodge numbers of the compactifications $\overline{M}_{2,n}$ for all n.

We view, as before, $M_2 \subset A_2$ via the Torelli morphism. The complement is isomorphic to $\mathrm{Sym}^2 A_1$. The local systems $\mathbb{V}_{a,b}$ can be pulled back to M_2 and restricted to $\mathrm{Sym}^2 A_1$. The cohomology of the restricted local systems can be understood by combining the branching formula from Sp_4 to $\mathrm{SL}_2 \times \mathrm{SL}_2$ with an analysis of the effect of quotienting by the involution of $A_1 \times A_1$ which switches the factors, [13, 37, 55]. Hence, the Euler characteristic $e_c(M_2, \mathbb{V}_{a,b})$ may be determined. The motives $\wedge^2 S[k]$ and $\mathrm{Sym}^2 S[k]$ (which sum to $S[k]^2$) will occur. Note

$$\wedge^2 S[k] \cong L^{k-1}$$

when $s_k = 1$.

As observed by Getzler [31], the Euler characteristics $e_c(M_2, \mathbb{V}_{a,b})$ for $a+b \leqslant N$ determine and are determined by the Σ_n-equivariant Euler characteristics $e_c^{\Sigma_n}(M_{2,n})$ for $n \leqslant N$. Hence, the latter are (conjecturally) determined for all n. Next, by the work of Getzler and Kapranov [34], the Euler characteristics $e_c^{\Sigma_n}(\overline{M}_{2,n})$ are determined as well, since we know the Euler characteristics in genus at most 1. Conversely, the answers for $\overline{M}_{2,n}$ determine those for $M_{2,n}$. After implementing the Getzler-Kapranov formalism (optimized for genus 2), a calculation of $e_c^{\Sigma_n}(\overline{M}_{2,n})$ for all $n \leqslant 22$ has been obtained.[10] All answers satisfy Poincaré duality, which is a very non-trivial check.

The cohomology of $\overline{M}_{2,21}$ is of particular interest to us. The motives $\wedge^2 S[12]$ and $\mathrm{Sym}^2 S[12]$ appear for the first time for $n = 21$ (with a Tate twist). The coefficient of $L \wedge^2 S[12]$ in $e_c^{\Sigma_{21}}(\overline{M}_{2,21})$ equals

$$[3\,1^{18}] + [3\,2^2\,1^{14}] + [3\,2^4\,1^{10}] + [3\,2^6\,1^6] + [3\,2^8\,1^2] + [2\,1^{19}] + [2^2\,1^{17}]$$
$$+ [2^3\,1^{15}] + [2^4\,1^{13}] + [2^5\,1^{11}] + [2^6\,1^9] + [2^7\,1^7] + [2^8\,1^5] + [2^9\,1^3] + [2^{10}\,1^1]$$

as a Σ_{21}-representation. As mentioned,

$$\wedge^2 S[12] \cong L^{11}. \tag{3.2}$$

We find thus 1939938 independent Hodge classes L^{12}, which should be algebraic by the Hodge conjecture. However, 1058148 of these classes come with an irreducible representation of length at least 13. By Theorem 4.2 obtained in Section 4, the latter classes cannot possibly be tautological.

The 1058148 classes actually span a small fraction of the full cohomology $H^{12,12}(\overline{M}_{2,21})$. There are also

$$1243344485012727233336 91$$

[10] Thanks to Stembridge's symmetric functions package SF.

classes L^{12} not arising via the isomorphism (3.2) — all the irreducible representations of length at most 12 arise in the coefficients.

3.7. Genus 3

In recent work [11], Bergström, Faber, and van der Geer have extended the point counts of moduli spaces of curves over finite fields to the genus 3 case. For all (n_1, n_2, n_3) and all prime powers $q \leqslant 17$, the frequencies

$$\sum_C \frac{1}{\# \operatorname{Aut}_{\mathbb{F}_q}(C)}$$

have been computed, where we sum over isomorphism classes of curves C over $\mathbb{F}_q$ with exactly n_i points over $\mathbb{F}_{q^i}$. The Σ_n-equivariant counts $\#M_{3,n}(\mathbb{F}_q)$ are then determined.

The isomorphism classes of Jacobians of nonsingular curves of genus 3 form an open subset of the moduli space A_3 of principally polarized abelian varieties of dimension 3.[11] The complement can be easily understood. Defining the $\mathrm{Sp}(6, \mathbb{Z})$ local systems $\mathbb{V}_{a,b,c}$ in expected manner, the traces of Frob_q on $e_c(M_3, \mathbb{V}_{a,b,c})$ and $e_c(A_3, \mathbb{V}_{a,b,c})$ can be computed for $q \leqslant 17$ and arbitrary $a \geqslant b \geqslant c \geqslant 0$.

Interpretation of the results has been quite successful for A_3 : an explicit conjectural formula for $e_c(A_3, \mathbb{V}_{a,b,c})$ has been found in [11], compatible with all known results (the results of Faltings [24] and Faltings-Chai [25] mentioned above, the dimension formula for the spaces of classical Siegel modular forms [62], and the numerical Euler characteristics [13, §8]). Two features of the formula are quite striking. First, it has a simple structure in which the endoscopic and Eisenstein contributions for genus 2 play an essential role. Second, it predicts the existence of many vector valued Siegel modular forms that are lifts, connected to local systems of *regular* weight[12], see [11].

Interpreting the results for M_3 is much more difficult. In fact, the stack M_3 cannot be correctly viewed as an open part of A_3. Rather, M_3 is a (stacky) double cover of the locus J_3 of Jacobians of nonsingular curves, branched over the locus of hyperelliptic Jacobians. The double covering occurs because

$$\operatorname{Aut}(\operatorname{Jac}_0(C)) = \operatorname{Aut}(C) \times \{\pm 1\}$$

for C a non-hyperelliptic curve, while equality of the automorphism groups holds for C hyperelliptic. As a result, the local systems $\mathbb{V}_{a,b,c}$ of odd weight $a + b + c$ will in general have non-vanishing cohomology on M_3, while their cohomology on A_3 vanishes. We therefore can not expect that the main part of the cohomology of $\mathbb{V}_{a,b,c}$ on M_3 can be explained in terms of Siegel cusp forms when $a + b + c$ is odd. On the other hand, the local systems of even weight provide no difficulties.

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[11] For $g \geqslant 4$, Jacobians have positive codimension in A_g.

[12] An analogous phenomenon occurs for Siegel modular forms of genus 2 and level 2, see [10], §6.

Bergström [8] has proved that the Euler characteristics $e_c(M_3, \mathbb{V}_{a,b,c})$ are certain explicit polynomials in L, for $a+b+c \leqslant 7$.

In fact, calculations ([12], in preparation) show that motives not associated to Siegel cusp forms must show up in $e_c(M_3, \mathbb{V}_{11,3,3})$ and $e_c(M_3, \mathbb{V}_{7,7,3})$, and therefore in $e_c^{\Sigma_{17}}(\overline{M}_{3,17})$, with the irreducible representations of type $[3^3\,1^8]$ and $[3^3\,2^4]$. Teichmüller modular forms [41] should play an important role in accounting for the cohomology in genus 3 not explained by Siegel cusp forms.

4. Representation theory

4.1. Length bounds

Consider the standard moduli filtration

$$\overline{M}_{g,n} \supset M^c_{g,n} \supset M^{rt}_{g,n}$$

discussed in Section 1.7. We consider only pairs g and n which satisfy the stability condition

$$2g-2+n > 0\ .$$

If $g=0$, all three spaces are equal by definition

$$\overline{M}_{0,n} = M^c_{0,n} = M^{rt}_{0,n}\ .$$

If $g=1$, the latter two are equal

$$M^c_{1,n} = M^{rt}_{1,n}\ .$$

For $g \geqslant 2$, all three are different.

The symmetric group Σ_n acts on $\overline{M}_{g,n}$ by permuting the markings. Hence, Σ_n-actions are induced on the tautological rings

$$R^*(\overline{M}_{g,n}),\ \ R^*(M^c_{g,n}),\ \ R^*(M^{rt}_{g,n})\ . \tag{4.1}$$

By Corollary 1.5, all the rings (4.1) are finite dimensional representations of Σ_n.

We define the *length* of an irreducible representation of Σ_n to be the number of parts in the corresponding partition of n. The trivial representation has length 1, and the alternating representation has length n. We define the length $\ell(V)$ of a finite dimensional representation V of Σ_n to be the maximum of the lengths of the irreducible constituents.

Our main result bounds the lengths of the tautological rings in all cases (4.1).

Theorem 4.2. *For the tautological rings of the moduli spaces of curves, we have*

(i) $\ell(R^k(\overline{M}_{g,n})) \leqslant \min\left(k+1, 3g-2+n-k, \left\lfloor \frac{2g-1+n}{2} \right\rfloor\right)$,

(ii) $\ell(R^k(M^c_{g,n})) \leqslant \min\,(k+1, 2g-2+n-k)$,

(iii) $\ell(R^k(M^{rt}_{g,n})) \leqslant \min\,(k+1, g-1+n-k)$.

The length bounds in all cases are consistent with the conjectures of Poincaré duality for these tautological rings, see [18, 21, 38, 53]. For (i), the bound is invariant under

$$k \longleftrightarrow 3g-3+n-k\,.$$

For (ii), the bound is invariant under

$$k \longleftrightarrow 2g-3+n-k$$

which is consistent with a socle in degree $2g-3+n$. For (iii), the bound is invariant under

$$k \longleftrightarrow g-2+n-k$$

which is consistent with a socle in degree $g-2+n$ (for $g>0$).

In genus g, the bound $\lfloor \frac{2g-1+n}{2} \rfloor$ in case (i) improves upon the trivial length bound n only for $n \geqslant 2g$.

4.2. Induction

The proof of Theorem 4.2 relies heavily on a simple length property of induced representations of symmetric groups.

Let V_1 and V_2 be representations of Σ_{n_1} and Σ_{n_2} of lengths ℓ_1 and ℓ_2 respectively. For $n = n_1 + n_2$, we view

$$\Sigma_{n_1} \times \Sigma_{n_2} \subset \Sigma_n$$

in the natural way.

Proposition 4.3. *We have* $\ell\left(\mathrm{Ind}_{\Sigma_{n_1}\times\Sigma_{n_2}}^{\Sigma_n} V_1 \otimes V_2\right) = \ell_1 + \ell_2\,.$

Proof. To prove the result, we may certainly assume V_i is an irreducible representation V_{λ_i}. In fact, we will exchange V_{λ_i} for a representation more closely related to induction.

For $\alpha = (\alpha_1, \ldots, \alpha_k)$ a partition of m, let U_α be the representation of Σ_m induced from the trivial representation of the Young subgroup

$$\Sigma_\alpha = \Sigma_{\alpha_1} \times \cdots \times \Sigma_{\alpha_k}$$

of Σ_m. Young's rule expresses U_λ in terms of irreducible representations V_μ for which μ precedes λ in the lexicographic ordering,

$$U_\lambda \cong V_\lambda \oplus \bigoplus_{\mu > \lambda} K_{\mu\lambda} V_\mu\,. \tag{4.4}$$

The Kostka number $K_{\mu\lambda}$ does not vanish if and only if μ dominates λ, which implies $\ell(\mu) \leqslant \ell(\lambda)$. Therefore $V_\lambda - U_\lambda$ can be written in the representation ring as a $\mathbb{Z}$-linear combination of U_μ for which $\mu > \lambda$ and $\ell(\mu) \leqslant \ell(\lambda)$.

If we let $V_1 = U_{\lambda_1}$ and $V_2 = U_{\lambda_2}$, the induction of representations is easy to calculate,

$$\mathrm{Ind}_{\Sigma_{n_1}\times\Sigma_{n_2}}^{\Sigma_{n_1+n_2}} U_{\lambda_1} \otimes U_{\lambda_2} = \mathrm{Ind}_{\Sigma_{\lambda_1}\times\Sigma_{\lambda_2}}^{\Sigma_{n_1+n_2}} 1 \cong \mathrm{Ind}_{\Sigma_{\lambda_1+\lambda_2}}^{\Sigma_{n_1+n_2}} 1 = U_{\lambda_1+\lambda_2}\,,$$

where $\lambda_1 + \lambda_2$ is the partition of n consisting of the parts of λ_1 and λ_2, reordered. Hence, the Proposition is proven for U_{λ_1} and U_{λ_2}. By the decomposition (4.4), the Proposition follows for V_{λ_1} and V_{λ_2}. □

4.3. Proof of Theorem 4.2

4.3.1. Genus 0 The genus 0 result

$$\ell(R^k(\overline{M}_{0,n})) \leqslant \min(k+1, n-k-2)$$

plays an important role in the rest of the proof of Theorem 4.2 and will be proven first.

By Keel [43], we have isomorphisms

$$A^k(\overline{M}_{0,n}) = R^k(\overline{M}_{0,n}) = H^{2k}(\overline{M}_{0,n}).$$

Poincaré duality then implies

$$\ell(R^k(\overline{M}_{0,n})) = \ell(R^{n-3-k}(\overline{M}_{0,n})) \,. \tag{4.5}$$

The vector space $R^k(\overline{M}_{0,n})$ is generated by strata classes of curves with k nodes (and hence with $k+1$ components). The bound

$$\ell(R^k(\overline{M}_{0,n})) \leqslant k+1$$

follows by a repeated application of Proposition 4.3 or by observing

$$\ell(U_\lambda) = \ell(\lambda),$$

for any partition λ (here with at most $k+1$ parts). The bound

$$\ell(R^k(\overline{M}_{0,n})) \leqslant n-k-2$$

is obtained by using (4.5). □

4.3.2. Rational tails We prove next a length bound for the subrepresentations of $R^k(\overline{M}_{g,n})$ generated by the decorated strata classes of curves with rational tails, in other words, by the classes

$$\xi_{A*}\Big(\prod_{v\in V(A)} \theta_v\Big)$$

as in Theorem 1.4, where A is a stable graph of genus $g \geqslant 1$ with n legs and exactly one vertex of genus g (and all other vertices of genus 0).

Let α be such a class of codimension k and graph A. Let v be the vertex of genus g. We may assume the decoration is supported only at v, so

$$\alpha = \xi_{A*}\theta_v \,.$$

The graph A arises by attaching t rational trees T_i to v. Let T_i have (before attaching) k_i edges and m_i legs. Let $m \geqslant t$ denote the valence of v. After attaching the trees, v

has $m-t$ legs. Of the $m-t$ legs, let s_0 come without ψ class, s_1 come with ψ^1, s_2 come with ψ^2, and so forth, so

$$\sum_j s_j = m - t\,.$$

After discarding the s_j which vanish, we obtain a partition of $m-t$ with p positive parts. The $m-t$ legs contribute a class ζ_v of degree $\sum_j js_j$ to θ_v. Let η_v be the product of the κ classes at the vertex and the ψ classes at the t legs at which the trees are attached, so

$$\theta_v = \zeta_v \eta_v\,.$$

Let e be the degree of η_v. We clearly have

$$k = \sum_i k_i + t + e + \sum_j js_j \qquad \text{and} \qquad n = m + \sum_i m_i - 2t.$$

At the vertex v, we obtain a representation of Σ_{m-t} of length p. By the result for genus 0, each tree T_i generates a subrepresentation V_i of $R^{k_i}(\overline{M}_{0,m_i})$ with

$$\ell(V_i) \leqslant \min(k_i+1, m_i-k_i-2).$$

One of the m_i legs is used in attaching T_i to v. After restricting V_i to Σ_{m_i-1}, the length can only decrease. Applying Proposition 4.3, we see α generates a subrepresentation V of $R^k(\overline{M}_{g,n})$ with

$$\ell(V) \leqslant p + \sum_i \min(k_i+1, m_i-k_i-2).$$

Taking the first terms in the minimum expressions, we have

$$\ell(V) \leqslant p + \sum_i (k_i+1) = p + k - e - \sum_j js_j\,. \tag{4.6}$$

Since

$$\sum_j js_j = \sum_{j\geqslant 1:\, s_j>0} js_j \geqslant \sum_{j\geqslant 1:\, s_j>0} 1 = p - 1 + \delta_{s_0,0}\,,$$

we see

$$p - \sum_j js_j \leqslant 1 - \delta_{s_0,0} \leqslant 1\,.$$

After substituting in (4.6) and using $e \geqslant 0$, we conclude

$$\ell(V) \leqslant k+1. \tag{4.7}$$

Taking the second terms in the minimum expressions, we have

$$\begin{aligned}\ell(V) &\leqslant p + \sum_i (m_i - k_i - 2)\\ &= p + n - m + t + e + \sum_j js_j - k\\ &= n - k + (p - m + t) + e + \sum_j js_j\,.\end{aligned}$$

The bound

$$e + \sum_j j s_j \leqslant g - 1 \tag{4.8}$$

will be assumed here. Further, $p \leqslant m - t$, so

$$\ell(V) \leqslant g - 1 + n - k. \tag{4.9}$$

Combining the bounds (4.7) and (4.9) yields

$$\ell(V) \leqslant \min(k + 1, g - 1 + n - k)$$

for any subrepresentation V of $R^k(\overline{M}_{g,n})$ generated by a class supported on a stratum of curves with rational tails and respecting the bound (4.8).

If the bound (4.8) is violated, then θ_v can be expressed as a tautological boundary class at the vertex v by Proposition 2 of [23]. The boundary class will include terms which are supported on strata of curves with rational tails (to which the argument can be applied again) and terms which are not supported on rational tails strata (to which the argument can not be applied). Statement (iii) of Theorem 4.2 is an immediate consequence. □

4.3.3. Compact type We prove here a length bound for the subrepresentations of $R^k(\overline{M}_{g,n})$ generated by the decorated strata classes of curves of compact type with $g \geqslant 1$ and $n \geqslant 1$.

Let β be such a class of codimension k and graph B. Let B have b vertices v_a of positive genus g_a and valence $n_a \geqslant 1$. The graph B arises by attaching t rational trees T_i to the vertices v_a. Let T_i have (before attaching) k_i edges and m_i legs of which $l_i \geqslant 1$ legs are used in the attachment. We also allow the degenerate case in which T_i represents a single node, then

$$k_i = -1 \quad \text{and} \quad l_i = m_i = 2\,.$$

At the vertex v_a, the attachment of the trees uses $\widehat{n}_a$ of the n_a legs. Just as in the case of curves with rational tails, we keep track of the powers of ψ classes attached to the remaining $n_a - \widehat{n}_a$ legs. Let $s_{a,j}$ of those legs come with ψ^j. We obtain a partition of $n_a - \widehat{n}_a$ with length p_a. Finally, let e_a be the degree of the product of the κ classes at v_a and the ψ classes at the $\widehat{n}_a$ attachment legs. We have

$$g = \sum_a g_a\,, \qquad n = \sum_i m_i + \sum_a n_a - 2\sum_i l_i\,,$$

$$k = \sum_i k_i + \sum_i l_i + \sum_a e_a + \sum_{a,j} j s_{a,j}\,,$$

$$\sum_i l_i = \sum_a \widehat{n}_a = t + b - 1\,.$$

The last equality follows since B is a tree.

Applying Proposition 4.3, we see that β generates a subrepresentation V of $R^k(\overline{M}_{g,n})$ with

$$\ell(V) \leqslant \sum_a p_a + \sum_i \min(k_i+1, m_i-k_i-2).$$

Taking the first terms of the minimum expressions,

$$\begin{aligned}\ell(V) &\leqslant \sum_a p_a + \sum_i (k_i+1)\\ &= \sum_a p_a + t + k - \sum_i l_i - \sum_a e_a - \sum_{a,j} j s_{a,j}\\ &= k+1-b+\sum_a p_a - \sum_a e_a - \sum_{a,j} j s_{a,j}\,.\end{aligned}$$

As in the rational tails case,

$$\sum_j j s_{a,j} \geqslant p_a - 1, \qquad \text{so} \qquad \sum_{a,j} j s_{a,j} \geqslant \sum_a p_a - b\,.$$

Therefore

$$\ell(V) \leqslant k+1-\sum_a e_a \leqslant k+1\,.$$

Taking the second terms of the minimum expressions,

$$\begin{aligned}\ell(V) &\leqslant \sum_a p_a + \sum_i (m_i-k_i-2)\\ &= \sum_a p_a + n - \sum_a n_a + 3\sum_i l_i + \sum_a e_a + \sum_{a,j} j s_{a,j} - k - 2t\\ &= n-k+2b-2+\sum_a (p_a - n_a + \widehat{n}_a) + \sum_a e_a + \sum_{a,j} j s_{a,j}\\ &\leqslant n-k+2b-2+g-b\\ &= n-k+g+b-2\\ &\leqslant n-k+2g-2.\end{aligned}$$

We have used the bound (4.8) at every vertex v_a. Therefore,

$$\ell(V) \leqslant \min(k+1, 2g-2+n-k) \leqslant \left\lfloor \tfrac{2g-1+n}{2} \right\rfloor$$

for any subrepresentation V of $R^k(\overline{M}_{g,n})$ generated by a class supported on a stratum of curves of compact type satisfying the bound (4.8) everywhere. As before, Statement (ii) of Theorem 4.2 is an immediate consequence. □

4.3.4. Stable curves Finally, we prove Statement (i) by induction on the genus g. Statement (i) has been proven already for genus 0. We assume (i) is true for genus at most $g-1$.

We have three bounds to establish to prove (i). The first bound to prove is

$$\ell(R^k(\overline{M}_{g,n})) \leqslant k+1\,. \tag{4.10}$$

The bound holds for the subspace of $R^k(\overline{M}_{g,n})$ generated by decorated strata classes of compact type on $\overline{M}_{g,n}$ satisfying (4.8) at all vertices by the results of Section 4.3.3. If (4.8) is violated, we use Proposition 2 of [23] to express the vertex term via tautological boundary classes and repeat. If a class not of compact type arises, the class must occur as a push forward from $R^*(\overline{M}_{g-1,n+2})$. Here, we use the induction hypothesis, and conclude the bound (4.10) for all of $R^k(\overline{M}_{g,n})$. In fact, for the subspace in $R^k(\overline{M}_{g,n})$ generated by decorated strata not of compact type, the bound $\ell \leqslant k+1$ holds by induction.

The second bound in Statement (i) of Theorem 4.2 is

$$\ell(R^k(\overline{M}_{g,n})) \leqslant 3g-2+n-k\,.$$

Since $2g-2+n-k < 3g-2+n-k$, the bound holds for the subspace of $R^k(\overline{M}_{g,n})$ generated by decorated strata classes of compact type on $\overline{M}_{g,n}$ satisfying (4.8) at all vertices by the results of Section 4.3.3. We conclude as above. For a class not of compact type, hence pushed forward from $R^{k-1}(\overline{M}_{g-1,n+2})$, we have by induction

$$\ell \leqslant 3(g-1)-2+(n+2)-(k-1) = 3g-2+n-k$$

for the generated subspace.

The third and last bound to consider is

$$\ell(R^*(\overline{M}_{g,n})) \leqslant \left\lfloor \tfrac{2g-1+n}{2} \right\rfloor\,.$$

The result holds for the subspace of $R^*(\overline{M}_{g,n})$ generated by decorated strata classes of compact type on $\overline{M}_{g,n}$ satisfying (4.8) at all vertices. We conclude as above using the induction to control classes associated to decorated strata not of compact type. □

4.4. Sharpness

In low genera, the length bounds of Theorem 4.2 are often sharp. In fact, we have not yet seen a failure of sharpness in genus 0 or 1. In genus 2, the first failure occurs in $R^2(\overline{M}_{2,3})$. A discussion of the data is given here for $g \leqslant 2$.

In genus 0, we have $R^*(\overline{M}_{0,n}) = H^*(\overline{M}_{0,n}, \mathbb{Q})$. Using the calculation of the Σ_n-representation on the latter space [29], the bound

$$\ell(R^k(\overline{M}_{0,n})) \leqslant \min(k+1, n-k-2)$$

has been verified to be sharp for $3 \leqslant n \leqslant 20$. However, the behavior is somewhat subtle. For $6 \leqslant n \leqslant 20$, the representation $[n-k-1, 2, 1^{k-1}]$ of length $k+1$ occurs for $1 \leqslant k \leqslant \lfloor \frac{n-3}{2} \rfloor$. For $10 \leqslant n \leqslant 20$, the representation $[n-k, 1^k]$ of length $k+1$ occurs for $0 \leqslant k \leqslant \lfloor \frac{n-3}{2} \rfloor$.

Consider next genus 1. Without too much difficulty, $R^*(\overline{M}_{1,n})$ can be shown to be Gorenstein (with socle in degree n) for $1 \leqslant n \leqslant 5$. Yang [66] has calculated the ranks of the intersection pairing on $R^*(\overline{M}_{1,n})$ for $n \leqslant 5$ and found them to coincide with the Betti numbers of $\overline{M}_{1,n}$ computed by Getzler [32]. Using Getzler's calculations of the cohomology groups as Σ_n-representations, the length bounds for $R^k(\overline{M}_{1,n})$ are seen to be sharp for $n \leqslant 5$.

Getzler has claimed $R^*(\overline{M}_{1,n})$ surjects onto the even cohomology on page 973 of [30] (though a proof has not yet been written). Assuming the surjection, we can use the Σ_n-equivariant calculation of the Betti numbers to check whether the length bounds for $R^k(\overline{M}_{1,n})$ are sharp for larger n. We have verified the sharpness for $n \leqslant 14$.

Tavakol [59] has proven $R^*(M^c_{1,n})$ is Gorenstein with socle in degree $n-1$. Yang [66] has calculated the ranks of the intersection pairing for $n \leqslant 6$. Using the results for $R^*(\overline{M}_{1,n})$, we have verified the length bounds for $R^k(M^c_{1,n})$ are sharp for $n \leqslant 5$. Probably, the Σ_n-action on $R^k(M^c_{1,n})$ can be analyzed more directly.

Finally, consider genus 2. The length bounds are easily checked to be sharp for $n = 2$. In fact,

$$R^*(M^{rt}_{2,2}),\ R^*(M^c_{2,2}),\ \text{and}\ R^*(\overline{M}_{2,2})$$

all are Gorenstein, with socles in degrees 2, 3, and 5 respectively.

The case $n = 3$ is more interesting. By Theorem 4.2, $R^2(\overline{M}_{2,3})$ and $R^4(\overline{M}_{2,3})$ have length at most 3. According to Getzler [31],

$$H^4(\overline{M}_{2,3}) \cong H^8(\overline{M}_{2,3})$$

has length 2. Yang [66] has shown that all the cohomology of $\overline{M}_{2,3}$ is tautological. The Gorenstein conjecture for $R^*(\overline{M}_{2,3})$ then implies both $R^2(\overline{M}_{2,3})$ and $R^4(\overline{M}_{2,3})$ have length 2. By analyzing separately the strata of compact type, the strata for which the dual graph has one loop, and the strata for which the dual graph has two loops, we can indeed prove the length 2 restriction. The length bound for $R^2(M^c_{2,3})$ is not sharp either.

The failure of sharpness signals unexpected symmetries among the tautological classes. Such symmetries can come from combinatorial symmetries of the strata or from unexpected relations. For the failure in $R^2(\overline{M}_{2,3})$, the origin is combinatorial symmetries in the strata. The nontrivial relation of [7] is not required. Also, relations can exist without the failure of sharpness: Getzler's relation [30] in $R^2(\overline{M}_{1,4})$ causes no problems.

By Tavakol [60], $R^*(M^{rt}_{2,n})$ is Gorenstein with socle in degree n. Using the Gorenstein property, we have verified the length bounds are sharp for $n \leqslant 4$ in the rational tails case.

5. Boundary geometry

5.1. Diagonal classes

We have seen the existence of non-tautological cohomology for $\overline{M}_{1,11}$,

$$H^{11,0}(\overline{M}_{1,11}, \mathbb{C}) = \mathbb{C}, \quad H^{0,11}(\overline{M}_{1,11}, \mathbb{C}) = \mathbb{C}\,.$$

As a result, the diagonal

$$\Delta_{11} \subset \overline{M}_{1,11} \times \overline{M}_{1,11}$$

has Künneth components which are not in $RH^*(\overline{M}_{1,11})$. Let

$$\iota : \overline{M}_{1,11} \times \overline{M}_{1,11} \to \overline{M}_{2,20}$$

be the gluing map. A natural question raised in [36] is whether

$$\iota_*[\Delta] \notin RH^*(\overline{M}_{2,20}) \; ? \tag{5.1}$$

Via a detailed analysis of the action of the ψ classes on $H^*(\overline{M}_{1,12}, \mathbb{C})$, the following result was proven in [36].

Theorem 5.2. *Let $\Delta_{12} \subset \overline{M}_{1,12} \times \overline{M}_{1,12}$ be the diagonal. After push-forward via the gluing map*

$$\iota : \overline{M}_{1,12} \times \overline{M}_{1,12} \to \overline{M}_{2,22} \; ,$$

we obtain a non-tautological class

$$\iota_*[\Delta] \notin RH^*(\overline{M}_{2,22}) \; .$$

While we are still unable to resolve (5.1), we give a simple new proof of Theorem 5.2 which has the advantage of producing new non-tautological classes in $H^*(\overline{M}_{2,21}, \mathbb{Q})$.

5.2. Left and right diagonals

We will study curves of genus 2 with 21 markings

$$[C, p_1, \ldots, p_{10}, q_1, \ldots, q_{10}, r] \in \overline{M}_{2,21} \; .$$

Consider the product $\overline{M}_{1,12} \times \overline{M}_{1,11}$ with the markings of the first factor given by $\{p_1, \ldots, p_{10}, r, \star\}$ and the markings of the second factor given by $\{q_1, \ldots, q_{10}, \bullet\}$. Define the *left diagonal*

$$\Delta_L \subset \overline{M}_{1,12} \times \overline{M}_{1,11}$$

to be the inverse image of the diagonal[13]

$$\Delta_{11} \subset \overline{M}_{1,11} \times \overline{M}_{1,11}$$

under the map forgetting the marking r,

$$\pi : \overline{M}_{1,12} \times \overline{M}_{1,11} \to \overline{M}_{1,11} \times \overline{M}_{1,11} \; .$$

The cycle Δ_L has dimension 12.

For the right diagonal, consider the product $\overline{M}_{1,11} \times \overline{M}_{1,12}$ with the markings of the first factor given by $\{p_1, \ldots, p_{10}, \star\}$ and the markings of the second factor given by $\{q_1, \ldots, q_{10}, r, \bullet\}$. Define

$$\Delta_R \subset \overline{M}_{1,11} \times \overline{M}_{1,12}$$

[13] The diagonal in $\overline{M}_{1,11} \times \overline{M}_{1,11}$ is defined by the bijection $p_i \leftrightarrow q_i$ and $\star \leftrightarrow \bullet$.

to be the inverse image of the diagonal Δ_{11} under the map forgetting the marking r,

$$\pi : \overline{M}_{1,11} \times \overline{M}_{1,12} \to \overline{M}_{1,11} \times \overline{M}_{1,11} \ ,$$

as before.

Our main result concerns the push-forwards $\iota_{L*}[\Delta_L]$ and $\iota_{R*}[\Delta_R]$ under the boundary gluing maps,

$$\iota_L : \overline{M}_{1,12} \times \overline{M}_{1,11} \to \overline{M}_{2,21} \ ,$$

$$\iota_R : \overline{M}_{1,11} \times \overline{M}_{1,12} \to \overline{M}_{2,21} \ ,$$

defined by connecting the markings $\{\star, \bullet\}$.

Theorem 5.3. *The push-forwards are non-tautological,*

$$\iota_{L*}[\Delta_L], \ \iota_{R*}[\Delta_R] \notin RH^*(\overline{M}_{2,21}) \ .$$

We view the markings of $\overline{M}_{2,22}$ as given by $\{p_1, \ldots, p_{10}, r, q_1, \ldots, q_{10}, s\}$ and the diagonal Δ_{12} as defined by the bijection

$$p_i \leftrightarrow q_i, \quad r \leftrightarrow s \ .$$

The cycle $\iota(\Delta_{12}) \subset \overline{M}_{2,22}$ maps birationally to $\iota_L(\Delta_L) \subset \overline{M}_{2,21}$ under the map

$$\pi : \overline{M}_{2,22} \to \overline{M}_{2,21}$$

forgetting s. Hence,

$$\pi_* \iota_* [\Delta_{12}] = \iota_{L*}[\Delta_L] \ .$$

In particular, Theorem 5.3 implies Theorem 5.2 since tautological classes are closed under π-push-forward.

5.3. Proof of Theorem 5.3

We first compute the class

$$\iota_L^* \iota_{L*}[\overline{M}_{1,12} \times \overline{M}_{1,11}] \in H^*(\overline{M}_{1,12} \times \overline{M}_{1,11}, \mathbb{Q}) \ .$$

The rules for such self-intersections are given in [36]. Since ι_L is an injection,

$$\iota_L^* \iota_{L*}[\overline{M}_{1,12} \times \overline{M}_{1,11}] = -\psi_\star - \psi_\bullet \ .$$

As a consequence, we find

$$\iota_L^* \iota_{L*}[\Delta_L] = (-\psi_\star - \psi_\bullet) \cdot [\Delta_L] \ . \tag{5.4}$$

If $\iota_{L*}[\Delta_L] \in RH^*(\overline{M}_{2,21})$, then the Künneth components of $\iota_L^* \iota_{L*}[\Delta_L]$ must be tautological cohomology by property (iii) of Section 1.4. Let

$$\pi : \overline{M}_{1,12} \times \overline{M}_{1,11} \to \overline{M}_{1,11} \times \overline{M}_{1,11} \tag{5.5}$$

be the map forgetting r in the first factor. If the Künneth components of $\iota_L^* \iota_{L*}[\Delta_L]$ are tautological in cohomology, then the Künneth components of

$$\pi_* \iota_L^* \iota_{L*}[\Delta_L] \in H^*(\overline{M}_{1,11} \times \overline{M}_{1,11}, \mathbb{Q})$$

must also be tautological in cohomology. We compute

$$\pi_*\Big((-\psi_\star - \psi_\bullet)\cdot[\Delta_L]\Big) = -\pi_*(\psi_\star\cdot[\Delta_L]) - \psi_\bullet\cdot(\pi_*[\Delta_L]) = -\pi_*(\psi_\star\cdot[\Delta_L])$$

since π restricted to Δ_L has fiber dimension 1.

The class $\psi_\star \in R^1(\overline{M}_{1,12})$ has a well-known boundary expression,

$$\psi_\star = \frac{1}{12}[\delta_{irr}] + \sum_{S\subset\{p_1,\ldots,p_{10},r\},\ S\neq\emptyset} [\delta_S] \ .$$

Here, δ_{irr} is the 'irreducible' boundary divisor (parameterizing nodal rational curves with 12 markings), and δ_S is the 'reducible' boundary divisor generically parameterizing 1-nodal curves

$$\mathbb{P}^1 \cup E$$

with marking $S\cup\{\star\}$ on $\mathbb{P}^1$ and the rest on E. The intersections

$$\delta_{irr}\times\overline{M}_{1,11}\ \cap\Delta_L, \quad \delta_S\times\overline{M}_{1,11}\ \cap\Delta_L$$

all have fiber dimension 1 with respect to (5.5) except when $S=\{r\}$. Hence,

$$-\pi_*(\psi_\star\cdot[\Delta_L]) = -\pi_*(\delta_{\{r\}}\times\overline{M}_{1,11}\ \cap\Delta_L) \ .$$

Since π maps $\delta_{\{r\}}\times\overline{M}_{1,11}\ \cap\Delta_L$ birationally onto $\Delta_{11}\subset\overline{M}_{1,11}\times\overline{M}_{1,11}$, we conclude

$$\pi_*\iota_L^*\iota_{L*}[\Delta_L] = -[\Delta_{11}] \ \in H^*(\overline{M}_{1,11}\times\overline{M}_{1,11},\mathbb{Q}) \ .$$

Since $[\Delta_{11}]$ does not have a tautological Künneth decomposition, the argument is complete. The proof for Δ_R is identical. □

5.4. On $\overline{M}_{2,20}$

We finish with a remark about the relationship between question (5.1) and Theorem 5.3. Consider the map forgetting the marking labelled by r,

$$\pi:\overline{M}_{2,21}\to\overline{M}_{2,20} \ .$$

We easily see

$$\pi^*\left(\iota_*[\Delta_{11}]\right) = \iota_{L*}[\Delta_L] + \iota_{R*}[\Delta_R] \ .$$

The following push-forward relation holds by calculating the degree of the cotangent line $\mathbb{L}_r$ on the fibers of π,

$$\pi_*\Big(\psi_r\cdot(\iota_{L*}[\Delta_L]+\iota_{R*}[\Delta_R])\Big) = 22\cdot\iota_*[\Delta_{11}] \ .$$

As a consequence of the above two equations, we conclude the following result.

Proposition 5.6. *We have the equivalence:*

$$\iota_*[\Delta_{11}]\in RH^*(\overline{M}_{2,20}) \iff \iota_{L*}[\Delta_L]+\iota_{R*}[\Delta_R]\in RH^*(\overline{M}_{2,21}) \ .$$

5.5. Connection with representation theory

Consider again the cycle $\Delta_L \subset \overline{M}_{1,12} \times \overline{M}_{1,11}$. Let

$$\Gamma_L \in H^{11,0}(\overline{M}_{1,12}) \otimes H^{0,11}(\overline{M}_{1,11}) \oplus H^{0,11}(\overline{M}_{1,12}) \otimes H^{11,0}(\overline{M}_{1,11})$$

be the Künneth component of Δ_L. We can consider the Σ_{21}-submodule

$$V \subset H^{12,12}(\overline{M}_{2,21})$$

generated by $\iota_{L*}(\Gamma_L)$. The class Γ_R can be defined in the same manner, and then $\iota_{R*}(\Gamma_R) \in V$.

The class Γ_L is alternating for the symmetric group Σ_{10} permuting the points $p_1, \ldots, p_{10}$. Similarly, Γ_L is alternating for the Σ_{10} permuting the points $q_1, \ldots, q_{10}$. Let

$$\Sigma_{10} \times \Sigma_{10} \subset \Sigma_{21}$$

be the associated subgroup. We can then consider the Σ_{21}-module defined by

$$\widetilde{V} = \mathrm{Ind}_{\Sigma_{10}\times\Sigma_{10}}^{\Sigma_{21}}(\alpha \otimes \alpha),$$

where α is the alternating representation. In fact, the representation $\widetilde{V}$ decomposes as

$$[1^{21}] + \sum_{i=0}^{9} [3\, 2^i\, 1^{18-2i}] + 2 \sum_{j=1}^{10} [2^j\, 1^{21-2j}].$$

We can write the coefficient of $L \wedge^2 S[12]$ in $H^{12,12}(\overline{M}_{2,21})$ discussed in Section 3.6 as

$$\sum_{i=0,\,\text{even}}^{9} [3\, 2^i\, 1^{18-2i}] + \sum_{j=1}^{10} [2^j\, 1^{21-2j}].$$

We conjecture the canonical map

$$\widetilde{V} \to V$$

is simply a projection onto the above subspace of $H^{12,12}(\overline{M}_{2,21})$.

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