Remarks on Moduli of Curves

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Abstract

We discuss some aspects of the theory of the moduli space of curves as well as some recent research.

1 Introduction

Rather than trying to provide here a comprehensive introduction to moduli of curves, we have chosen to limit the discussion to certain aspects of the theory. We also survey some of the recent research directly related to the papers in this volume.

After introducing the mapping class group and the Torelli group, the moduli space of curves is constructed as an analytic orbifold. We discuss the Deligne-Mumford-Knudsen compactification and the fact that there exist global smooth covers of it. Next we recall the definition of the tautological classes and the results on the stability of the homology of the mapping class group as well as Mumford's conjecture. Sections 10 and 13 discuss the Witten conjecture that was proved by Kontsevich and its generalization to moduli spaces of stable maps. We conclude with a discussion of complete subvarieties of moduli space and of recent results regarding its intersection theory.

For more information on the moduli space of curves, the reader could consult, e.g., the books [62, 63, 52, 45], the collections [10, 17], and the survey papers [44, 40].

2 Mapping Class Groups

Fix a closed connected oriented surface S_g of genus g and a sequence of distinct points $x_0,x_1,\dots$ on S_g and let us write $S_{g,n}$ for $S_g - \{x_1,\dots,x_n\}$ and $\pi_{g,n}$ for $\pi_1(S_{g,n},x_0)$. If the subscript n is omitted, it is assumed to be zero. We stick to this notation throughout this introduction.

We begin with noting that in the absence of any punctures we have a natural isomorphism $H_2(\pi_g;\mathbb{Z}) \cong H_2(S_g,\mathbb{Z})$, so that the orientation of S_g defines a distinguished generator of $H_2(\pi_g;\mathbb{Z})$. For positive n, the simple positively oriented loops around x_i make up a distinguished conjugacy class B_i in $\pi_{g,n}$, $i=1,\dots,n$. There is a standard way to present the group $\pi_{g,n}$ with generators $\alpha_{\pm 1},\dots,\alpha_{\pm g},\beta_1,\dots,\beta_n$, subject to the relation

$$(\alpha_1,\alpha_{-1})\cdots(\alpha_g,\alpha_{-g})\beta_n\cdots\beta_1 = 1,$$

where in case $n=0$, the generators $\alpha_{\pm 1},\dots,\alpha_{\pm g}$ have been chosen compatibly with the orientation, and $\beta_i \in B_i$ in case $n>0$. In the latter case, $\pi_{g,n}$ is just a free group on $2g+n-1$ generators and it is the data of $B_1,\dots,B_n$ that give this group its extra structure. The inclusion $S_{g,n+1} \subset S_{g,n}$ induces a surjective homomorphism $\pi_{g,n+1} \to \pi_{g,n}$ on fundamental groups and we can arrange that the generators of $\pi_{g,n}$ are the images of their namesakes in $\pi_{g,n+1}$.

One defines the *n-pointed mapping class group of genus g*, here denoted by Γ_g^n, as the connected component group of the group of orientation preserving self-homeomorphisms of S_g that fix $x_1,\dots,x_n$. This group acts by outer automorphisms (that is, the action is given up to inner automorphisms) on the fundamental group $\pi_{g,n}$. It is a classical result that this action is faithful. According to Nielsen and Zieschang [76] the image can be characterized as follows: for $n=0$, $\Gamma_g = \Gamma_g^0$ maps onto the group of outer automorphisms of π_g which act trivially on $H_2(\pi_g;\mathbb{Z})$, and for $n>0$, Γ_g^n maps onto the group of outer automorphisms of $\pi_{g,n}$ which preserve each conjugacy class B_i, $i=1,\dots,n$. So this yields a description of the mapping class group purely in terms of group theory. Forgetting x_{n+1} defines an obvious homomorphism $\Gamma_g^{n+1} \to \Gamma_g^n$. Since any orientation preserving self-homeomorphism of S_g that fixes $x_1,\dots,x_n$ is isotopic to one that also fixes x_{n+1}, this homomorphism is surjective. An arc in $S_{g,n}$ connecting x_0 and x_{n+1} allows us to identify Γ_g^{n+1} with the group of automorphisms of $\pi_1(S_{g,n},x_{n+1}) \cong \pi_{g,n}$ which preserve the conjugacy classes $B_1,\dots,B_n$. With this identification, the kernel of $\Gamma_g^{n+1} \to \Gamma_g^n$ is the group of inner automorphisms of $\pi_{g,n}$. If we take $g=n=1$, then we get the familiar identification of Γ_1^1 as the group of orientation preserving automorphisms of $\pi_{1,0} \cong \mathbb{Z}^2$: $\Gamma_1^1 \cong \mathrm{SL}(2,\mathbb{Z})$.

A special set of elements of the mapping class group Γ_g^n are the Dehn twists. A *Dehn twist* is given by a regularly embedded circle $\delta \subset S_{g,n}$ and then represented by a homeomorphism which on a closed neighborhood of that circle (with oriented parameterization by $(\phi,t) \in S^1 \times [-1,1]$) equals $(\phi,t) \mapsto (\phi+\pi(t+1),t)$ and is the identity outside that neighborhood. The corresponding element of Γ_g^n, denoted by τ_δ, only depends on the isotopy class of δ. The Dehn twist τ_δ is the identity element of Γ_g^n if and only if δ bounds a disk on S_g which contains at most one x_i, $i = 1, \dots, n$; for that reason such a circle is called *trivial* also. It is known that finitely many Dehn twists generate the whole mapping class group. For the unpunctured case $n = 0$ a relatively simple finite presentation of Γ_g (with Dehn twist generators) has been given by Wajnryb [72], and for the general case one was recently obtained by Gervais [29].

Although mapping class groups have been studied since their introduction some seventy years ago, they are still mysterious in many ways. In certain regards they behave as if they were arithmetic groups, but as Ivanov has shown, they are, apart from a few exceptions, not isomorphic to such a group. Following Hain and Morita, a mapping class group can however be naturally embedded in a proarithmetic group (its proarithmetic hull), and the latter is at present much better accessible. We discuss this briefly in section 9.

3 The Torelli Group

We here focus on the unpunctured case: $n = 0$. The homology group $H_1(S_g; \mathbb{Z})$ is then of rank $2g$ and the orientation equips it with a unimodular symplectic form. Let us write V_g for $H_1(S_g; \mathbb{Z})$ and $\omega_g \in \wedge^2 V_g$ for the 'inverse' symplectic form. So if $a_i \in V_g$ is the class of the generator α_i of π_g, then $a_{\pm 1}, \dots, a_{\pm g}$ is a basis for V_g and $\omega_g = a_1 \wedge a_{-1} + \cdots + a_g \wedge a_{-g}$. It is clear that the mapping class group Γ_g acts on V_g and leaves ω_g invariant. The image of this representation is in fact the full integral symplectic group $\mathrm{Sp}(V_g)$. Its kernel is called the *Torelli group of genus* g, denoted T_g. This group is trivial for genus zero and one; so that for instance $\Gamma_1 \cong \mathrm{Sp}(V_1) \cong \mathrm{SL}(2,\mathbb{Z})$. But for $g \geq 2$ the Torelli group contains the Dehn twists around circles which separate S_g into two connected components and for $g \geq 3$, also the elements of the form $\tau_\delta \tau_{\delta'}^{-1}$, where δ and δ' are disjoint circles on S_g, which together separate S_g into two pieces. According to Powell these elements generate T_g. The Dehn twists around separating circles generate a subgroup K_g of T_g that is clearly normal in Γ_g. We have $K_2 = T_2$. In fact, G. Mess showed that K_2 is the free group on the separating Dehn twists and that these generators are in bijective correspondence with the symplectic splittings of V_2 into two copies of V_1, hence infinite in number. The situation is quite different when $g \geq 3$. Dennis Johnson, who in the early eighties began a systematic study of the Torelli group, showed that T_g is then finitely generated and exhibited a remarkable epimorphism of T_g onto the lattice $\wedge_o^3 V_g := \wedge^3 V_g / (V_g \wedge \omega_g)$ with kernel K_g. An explicit (but perhaps not very insightful) way to describe this epimorphism is to say what it does to an element $\tau_\delta \tau_{\delta'}^{-1}$ as above: let $S \subset S_g$ be a connected component of $S_g - \delta - \delta'$ and orient δ as boundary component of S, so that it determines a class $d_S \in V_g$. The image V_S of $H_1(S; \mathbb{Z})$ in V_g is a sublattice on which the symplectic form is degenerate with kernel spanned by d_S. The form on $V_S/\mathbb{Z}d_S$

is unimodular and thus defines an element $\omega_S \in \wedge^2(V_S/\mathbb{Z}d_S)$. We can regard $d_S \wedge \omega_S$ as an element of $\wedge^3 V_g$. If S' is the other component of $S_g - \delta - \delta'$, then $d_{S'} = -d_S$ and $d_S \wedge (\omega_S + \omega_{S'}) = d_S \wedge \omega_g$. So the image of $d_S \wedge \omega_S$ in $\wedge^3_o V_g$ only depends on the ordered pair (δ,δ'). This is the image of $\tau_\delta \tau_{\delta'}^{-1}$ under Johnson's homomorphism. A more natural description will be given in section 9.

4 Moduli Spaces of Curves

Let us now assume that g and n are such that the Euler characteristic $2-2g-n$ of $S_{g,n}$ is negative, in other words, exclude the cases of genus zero with at most two punctures and genus one without punctures. Then the set of complex structures on S_g compatible with the given orientation and given up to isotopy relative $x_1,\dots,x_n$ is in a natural way a complex manifold of complex dimension $3g-3+n$. This manifold, which we shall denote $\mathcal{T}_{g,n}$, is called the *n-pointed Teichmüller space of genus g*. It is known that $\mathcal{T}_{g,n}$ is contractible and isomorphic to a bounded domain. Notice that there is an evident action of the mapping class group Γ_g^n on $\mathcal{T}_{g,n}$. This action is faithful and properly discrete, and so the orbit space has the structure of an *analytic orbifold*. As such it is denoted by $\mathcal{M}_{g,n}$. From the definition it is clear that the points of $\mathcal{M}_{g,n}$ are in bijective correspondence with isomorphism classes of n-pointed closed Riemann surfaces of genus g.

There is an evident forgetful map $\mathcal{T}_{g,n+1} \to \mathcal{T}_{g,n}$. This map is an analytic submersion that is equivariant over $\Gamma_g^{n+1} \to \Gamma_g^n$, and hence determines a morphism of orbifolds $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. The latter is an analytic submersion (in the orbifold sense) and the fiber over a point $p \in \mathcal{M}_{g,n}$ is the quotient of $S_{g,n}$ equipped with a complex structure defining p, modulo its (finite) group of complex automorphisms. So we might think of this morphism as the universal family of n-pointed closed Riemann surfaces of genus g.

Since $\mathcal{T}_{g,n}$ appears as a contractible universal covering of $\mathcal{M}_{g,n}$ (in the sense of orbifolds) with Galois group Γ_g^n, the rational cohomology of $\mathcal{M}_{g,n}$ is naturally isomorphic to the rational cohomology of Γ_g^n.

But $\mathcal{M}_{g,n}$ has more structure. Recall that a closed Riemann surface is in a natural way a smooth complex projective algebraic curve. So we may regard $\mathcal{M}_{g,n}$ as a moduli space of such curves. This interpretation leads to an algebraization of $\mathcal{M}_{g,n}$. Better yet: geometric invariant theory enables us to characterize $\mathcal{M}_{g,n}$ as a quasi-projective variety with the orbifold structure lifting to the structure as a stack over $\mathrm{Spec}(\mathbb{Z})$. From now on, we consider $\mathcal{M}_{g,n}$ as endowed with this structure.

5 Deligne-Mumford-Knudsen Completion

Deligne, Mumford and Knudsen [14, 55] discovered that there is a natural completion of $\mathcal{M}_{g,n}$ by allowing curves to degenerate in a mild way and that this completion has itself the interpretation of a moduli stack. The central notion here is that of *stable n-pointed curve of genus g*. This consists of a complete connected curve C of arithmetic genus g whose singularities are ordinary double points and n distinct points $x_1,\dots,x_n$ on the smooth part of C subject to the condition that the group $\mathrm{Aut}(C;x_1,\dots,x_n)$ of automorphisms of C fixing these points is finite. The last condition amounts to requiring that every connected component of $C_{\mathrm{reg}}-\{x_1,\dots,x_n\}$ has negative Euler characteristic: no component is a smooth rational curve with at most two points removed or a smooth curve of genus one.

The local deformation theory of such curves is as nice as it could possibly be. For instance, small deformations of stable n-pointed curves of genus g are again of that type. More is true: such a curve $(C;x_1,\dots,x_n)$ has a *universal* deformation with smooth base S of dimension $3g-3+n$. So this is given by a curve over S: $\mathcal{C}\to S$ with n disjoint sections $s_1,\dots,s_n$ giving each fiber the structure of stable n-pointed curve of genus g, together with an identification of the closed fiber with $(C;x_1,\dots,x_n)$. The discriminant of the morphism $\mathcal{C}\to S$ is quite simple: for every singular point p of C the locus in S parameterizing the curves where p persists as a singularity is a smooth hypersurface D_p in S and their union D is a normal crossing divisor. The group $\mathrm{Aut}(C;x_1,\dots x_n)$ acts naturally on the whole system.

What Deligne, Mumford and Knudsen prove is that there is a moduli stack of stable n-pointed curves of genus g, $\overline{\mathcal{M}}_{g,n}$, and that it is projective, irreducible, defined over $\mathrm{Spec}(\mathbb{Z})$ and contains $\mathcal{M}_{g,n}$ as an open-dense subscheme. Notice that $\overline{\mathcal{M}}_{g,n}$ is locally given by a universal deformation as above. In particular, the underlying variety is at the point defined by $(C;x_1,\dots,x_n)$ isomorphic to the quotient $\mathrm{Aut}(C;x_1,\dots x_n)\backslash S$. It is clear from this local picture that the Deligne-Mumford boundary $\Delta=\overline{\mathcal{M}}_{g,n}-\mathcal{M}_{g,n}$ is a normal crossing divisor (in the sense of stacks).

The generic points of this divisor parameterize n-pointed curves with a single singular point. The case where the curve is irreducible accounts for one such point; the corresponding irreducible component of Δ is usually denoted Δ_0. Otherwise the curve is a one-point union of two smooth connected projective curves, say of genera g_1 and g_2 (with $g_1+g_2=g$, of course) with an ensuing decomposition of $x_1,\dots x_n$ given by a partition $I_1\sqcup I_2$ of $\{1,\dots,n\}$. If $g_k=0$, then the corresponding part I_k must contain at least two elements. The set of unordered pairs $\{(g_1,I_1),(g_2,I_2)\}$, subject to this condition effectively indexes the irreducible components $\neq\Delta_0$ of the boundary divisor Δ.

The normal crossing structure of Δ defines a natural decomposition of Δ into (connected) strata. In characteristic zero a stratum parameterizes the stable pointed curves of a fixed topological type. If we remove from an n-pointed curve its singular points and the n given points, then we get a (possibly disconnected) smooth curve, hence a stratum parameterizes such curves. This can be expressed in a characteristic free manner and thus it is not difficult to see that any stratum S is a smooth stack that admits a product of moduli stacks $\prod_j \mathcal{M}_{g_j,n_j}$ as a finite cover. The closure of S is then covered by $\prod_j \overline{\mathcal{M}}_{g_j,n_j}$.

6 Covers of Moduli Stacks

The moduli stack $\overline{\mathcal{M}}_{g,n}$ admits many coverings. Any subgroup Γ of the mapping class group Γ_g^n of finite index (more precisely, a conjugacy class of those) defines a finite flat morphism $\mathcal{M}_{g,n}^{\Gamma} \to \mathcal{M}_{g,n}$ of stacks and then we can take the normalization $\overline{\mathcal{M}}_{g,n}^{\Gamma} \to \overline{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ in $\mathcal{M}_{g,n}^{\Gamma}$. In characteristic zero there is a modular interpretation of $\mathcal{M}_{g,n}^{\Gamma}$: it is the moduli space of smooth projective n-pointed curves $(C; x_1, \dots, x_n)$ of genus g endowed with an isomorphism of the fundamental group of $C - \{x_1, \dots, x_n\}$ (relative some base point) with $\pi_{g,n}$, given up an automorphism of $\pi_{g,n}$ mapping to Γ. But it is not clear whether such an interpretation is possible for its completion $\overline{\mathcal{M}}_{g,n}^{\Gamma}$.

Subgroups of Γ_g^n that are of particular interest are the so-called congruence subgroups. They are defined as follows: let $\pi \subset \pi_{g,n}$ be a normal subgroup of finite index that is also invariant under every automorphism that preserves the distinguished conjugacy classes $B_1, \dots, B_n$ (see section 2). There is an evident homomorphism from Γ_g^n to the outer automorphism group of the finite group $\pi_{g,n}/\pi$. Subgroups of Γ_g^n that contain the kernel of such a homomorphism are called *congruence subgroups*. They are obviously of finite index. For the case $g = n = 1$, this yields the familiar notion of a congruence subgroup of $\Gamma_1^1 \cong \mathrm{SL}(2,\mathbb{Z})$: this is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ that contains all the matrices in $\mathrm{SL}(2,\mathbb{Z})$ congruent modulo d to the identity, for some integer d. It is well-known that there exist subgroups of finite index of $\mathrm{SL}(2,\mathbb{Z})$ that are not congruence subgroups. Ivanov has raised the question whether the situation is different for the mapping class groups Γ_g, $g \geq 2$ [47].

For some time it was not known whether for a suitable choice of Γ, the variety underlying $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ is smooth over a given base field and this led to the foundation of an elaborate intersection theory for smooth stacks, needed to define the appropriate Chow groups. It is now known that such Γ exist and that we can take Γ to be a congruence subgroup (Looijenga [57], Pikaart-De Jong [69], Boggi-Pikaart [8]). This means that the stack $\overline{\mathcal{M}}_{g,n}$ is obtained as the orbit space of a smooth variety with respect to a finite group action. The kth Chow group of $\overline{\mathcal{M}}_{g,n}$, $\mathrm{CH}^k(\overline{\mathcal{M}}_{g,n})$, is then definable as the invariant part of the kth Chow group of this smooth variety. (Here $\mathrm{CH}^k(\text{—})$ stands for the Chow group in codimension k defined by rational equivalence, tensorized with $\mathbb{Q}$.)

7 Tautological Classes

If $(\tilde{C};\tilde{x}_1,\dots,\tilde{x}_{n+1})$ is a stable $(n+1)$-pointed curve, then forgetting the last point $\tilde{x}_{n+1}$ yields a stable n-pointed curve unless the component of $\tilde{C}_{\rm reg}-\{\tilde{x}_1,\dots,\tilde{x}_{n+1}\}$ punctured by $\tilde{x}_{n+1}$ is a thrice punctured $\mathbb{P}^1$. But then contraction of the irreducible component of $\tilde{C}$ containing $\tilde{x}_{n+1}$ produces a stable n-pointed curve. In either case, the result is a morphism from $(\tilde{C};\tilde{x}_1,\dots,\tilde{x}_n)$ onto a stable n-pointed curve $(C;x_1,\dots,x_n)$. The image x of $\tilde{x}_{n+1}$ can be any point of C and it is easy to see that the $(n+1)$-pointed curve can be recovered up to canonical isomorphism from the system $(C;x_1,\dots,x_n;x)$.

This also works in families, so that we have a forgetful morphism $\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ which may be thought of as the universal stable n-pointed curve of genus g. It comes in particular with n disjoint sections $s_1,\dots,s_n$.

The functor that associates to every stable n-pointed curve its cotangent line at the ith point $(i\in\{1,\dots,n\})$ is realized on the universal example $\overline{\mathcal{M}}_{g,n}$ as a line bundle (in the sense of stacks). This line bundle can be gotten more directly as the pull-back of the relative dualizing sheaf $\omega_{g,n}$ of the universal family $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ along the ith section: $s_i^*\omega_{g,n}$. The dependence on n is not entirely obvious. To see this, notice that the morphism $\tilde{C}\to C$ above defines a homomorphism $T^*_{x_i}C\to T^*_{\tilde{x}_i}\tilde{C}$. This is an isomorphism unless $\tilde{x}_i$ lies on a component that gets contracted. Which is the case precisely when $(\tilde{C};\tilde{x}_1,\dots,\tilde{x}_{n+1})$ defines a point on the irreducible component $\Delta_{i,n+1}$ of Δ defined by the pair $\{(g_1,I_1),(g_2,I_2)\}$ with $g_2=0$ and $I_2=\{\tilde{x}_i,\tilde{x}_{n+1}\}$. So the line bundle $\tilde{s}_i^*\omega_{g,n+1}$ on $\overline{\mathcal{M}}_{g,n+1}$ is the pull-back of $s_i^*\omega_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$ twisted by $\sum_{i=1}^n\Delta_{i,n+1}$.

We can now tell what the basic classes on $\overline{\mathcal{M}}_{g,n}$ are:

(i) the *Witten* classes

$$\psi_i := c_1(s_i^*\omega_{g,n})\in \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n}),\quad i=1,\dots,n,$$

(ii) the *Mumford* classes (à la Arbarello-Cornalba [1])

$$\kappa_r := \pi_!(c_1(\omega_{g,n})^{r+1})\in \mathrm{CH}^r(\overline{\mathcal{M}}_{g,n}),\quad r=1,2,\dots,$$

The *tautological subalgebra* $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,n})$ of $\mathrm{CH}^\bullet(\overline{\mathcal{M}}_{g,n})$ is defined as follows: recall from Section 5 that the closure $\overline{S}$ of every stratum S is finitely covered by a product $\hat{S}\cong\prod_j\mathcal{M}_{g_j,n_j}$. The basic classes of the factors generate a subalgebra of $\mathrm{CH}^\bullet(\hat{S})$ whose direct image in $\mathrm{CH}^\bullet(\overline{\mathcal{M}}_{g,n})$ we denote by $A^\bullet(S)$. Then $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,n})$ is defined as the algebra generated by the $A^\bullet(S)$, where S runs over all the strata, $\mathcal{M}_{g,n}$ included. We use the same terminology (and similar notation) for its restriction $\mathcal{R}^\bullet(\mathcal{M}_{g,n})$ to $\mathcal{M}_{g,n}$. The latter is of course already generated by the ψ_i's and the κ_r's.

The tautological algebras are respected by the obvious morphisms between moduli stacks of pointed curves, such as the pull-back and the push-forward along the finite morphisms $\hat{S} \to \overline{\mathcal{M}}_{g,n}$ (with $\hat{S}$ as above) and the projection $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. It is possible to characterize $\mathcal{R}^\bullet$ in this way as the smallest bivariant subfunctor of $\mathrm{CH}^\bullet$ restricted to an appropriate category of moduli spaces of pointed curves that contains the fundamental classes of the Deligne-Mumford moduli stacks.

8 Stability

In section 4 we observed that a mapping class group and the corresponding moduli space have the same rational cohomology. So any homological property of Γ_g^n has immediate relevance for $\mathcal{M}_{g,n}$. Unfortunately, our knowledge of the homology of Γ_g^n is still rather limited. A central result is the stability theorem, due to Harer [42], that says that the homology group $H_k(\Gamma_g^n;\mathbb{Z})$ is independent of g, if g is sufficiently large (according to Ivanov, $g \geq 2k$ will do, but probably we may take $g \geq \frac{3}{2}k$). A more precise statement says how the isomorphism $H_k(\Gamma_g^n;\mathbb{Z}) \cong H_k(\Gamma_{g+1}^n;\mathbb{Z})$ is defined. There is no obvious map between the groups in question, but there is a homological correspondence defined as follows. Choose a separating circle $\delta \subset S_{g+1}$ which splits S_{g+1} into a surface of genus g and a surface S of genus one with the former containing the points labeled $x_1,\dots,x_n$, and choose an orientation preserving homeomorphism of $S_g - \{x_{n+1}\}$ onto this component in such a way that the points $x_1,\dots,x_n$ of S_g retain their name. If $\Gamma_{g+1,S}^n$ stands for the group of mapping classes of S_{g+1} relative to $S \cup \{x_1,\dots,x_n\}$, then we have a natural monomorphism $\Gamma_{g+1,S}^n \to \Gamma_{g+1}^n$ and a composite epimorphism $\Gamma_{g+1,S}^n \to \Gamma_g^{n+1} \to \Gamma_g^n$. The stability theorem states that these two homomorphisms induce isomorphisms on integral homology in degree k if g is sufficiently large. We can slightly generalize the above construction to define a homomorphism

$$H_k(\Gamma_g^n;\mathbb{Z}) \otimes H_{k'}(\Gamma_{g'}^{n'};\mathbb{Z}) \to H_{k+k'}(\Gamma_{g+g'}^{n+n'};\mathbb{Z}),$$

provided that g and g' are sufficiently large: choose here the separating circle δ on $S_{g+g'}$ such that one piece has genus g with punctures $x_1,\dots,x_n$ and the other has genus g' with punctures $x_{n+1},\dots,x_{n+m}$ and let the group of mapping classes of $S_{g+g'}$ relative to $\delta \cup \{x_1,\dots,x_{n+n'}\}$ take the role of $\Gamma_{g+1,S}^n$.

The stable homology of the mapping class groups $\{\Gamma_g^n\}_{g=1}^\infty$ can be realized as the homology of a group Γ_∞^n that is defined in much the same way as Γ_g^n: replace S_g by a surface of infinite genus (but beware that such surfaces are not all mutually homeomorphic) and allow only self-homeomorphisms that are the identity outside a compact subset. In particular, we get a product $H_\bullet(\Gamma_\infty;\mathbb{Z}) \otimes H_\bullet(\Gamma_\infty;\mathbb{Z}) \to H_\bullet(\Gamma_\infty;\mathbb{Z})$. After tensoring with $\mathbb{Q}$, this product and the standard coproduct on $H_\bullet(\Gamma_\infty;\mathbb{Q})$ turn $H_\bullet(\Gamma_\infty;\mathbb{Q})$ into a graded-bicommutative Hopf algebra. Of course, the same applies to its dual $H^\bullet(\Gamma_\infty;\mathbb{Q})$.

According to a structure theorem such an algebra is as a graded algebra freely generated by its primitive subspace.

This construction can be imitated in the moduli context. Identifying the last point of an $(n+1)$-pointed smooth genus g curve with the origin of an elliptic curve produces a stable n-pointed genus $g+1$ curve. This defines a morphism $f : \mathcal{M}_{g,n+1} \times \mathcal{M}_{1,1} \to \overline{\mathcal{M}}_{g+1,n}$ whose image is an open subset of a boundary divisor. This morphism has a normal (line) bundle in the orbifold sense. Let E_f be the complement of the zero section of this normal bundle. Although there is no obvious map $E_f \to \mathcal{M}_{g+1,n}$, a tubular neighborhood theorem asserts that there is a natural homotopy class of such maps. So if we choose $p \in \mathcal{M}_{1,1}$, and let $E_f(p)$ be the restriction of E_f to $\mathcal{M}_{g,n+1} \times \{p\}$, then we have a well-defined homomorphism $H_k(E_f(p);\mathbb{Q}) \to H_k(\mathcal{M}_{g+1,n};\mathbb{Q})$. On the other hand, projection induces a homomorphism $H_k(E_f(p);\mathbb{Q}) \to H_k(\mathcal{M}_{g,n+1};\mathbb{Q}) \to H_k(\mathcal{M}_{g,n};\mathbb{Q})$. These two homomorphisms are the geometric incarnations of the stability maps and hence they are isomorphisms in the stable range. In a similar fashion we get a natural homomorphism $H_k(\mathcal{M}_{g,n};\mathbb{Q}) \otimes H_{k'}(\mathcal{M}_{g',n'};\mathbb{Q}) \to H_{k+k'}(\mathcal{M}_{g+g',n+n'};\mathbb{Q})$ (g and g' sufficiently large). An important feature of these homomorphisms is that they are in a sense 'motivic': they respect all the extra structure that homology groups of algebraic varieties carry, such as a mixed Hodge structure. In particular, it follows that the stable cohomology $H^\bullet(\Gamma_\infty^n;\mathbb{Q})$ has a natural mixed Hodge structure that is preserved by the coproduct (which is dual to the product defined above). It was shown by Pikaart [68] that this mixed Hodge structure is actually not mixed at all: $H^k(\Gamma_\infty^n;\mathbb{Q})$ is pure of weight k.

The tautological class κ_r introduced in 7 is, when regarded as an element of $H^{2r}(\mathcal{M}_g;\mathbb{Q})$, stable for g sufficiently large. It is not hard to prove that the corresponding element of the stable cohomology Hopf algebra is primitive. Miller [60] and Morita [61] have shown that it is nonzero and so $H^\bullet(\Gamma_\infty;\mathbb{Q})$ contains the polynomial algebra $\mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\dots]$. Mumford wrote in [64] that it seems reasonable to guess that $H^\bullet(\Gamma_\infty;\mathbb{Q})$ is no bigger than this; this 'reasonable guess' now goes under the name of Mumford's conjecture.

9 A Proarithmetic Hull of the Mapping Class Group

The lower central series of a group π is defined inductively by $\pi^{(0)} = \pi$ and $\pi^{(k+1)} = (\pi,\pi^{(k)})$. So $\pi/\pi^{(k+1)}$ is a nilpotent group. We take $\pi = \pi_g$ and note that the mapping class group Γ_g acts in $\pi_g/\pi_g^{(k+1)}$. If $\Gamma_g(k)$ denotes the image of this action, then it is clear that $\Gamma_g(0) = \mathrm{Sp}(V_g)$. It is not hard to see that $\Gamma_g(k+1)$ is an extension of $\Gamma_g(k)$ by a lattice. For $k=0$, this lattice turns out to be just $\wedge_o^3 V_g$, and as one may expect, the resulting map $T_g \to \wedge_o^3 V_g$ is just the Johnson homomorphism. For higher values k, these lattices are not so easy to describe, but the least one can say is that they are obtained in a functorial manner from the symplectic lattice V_g. Things simplify a great deal if we tensor the lattice with $\mathbb{C}$: then it turns out that the resulting vector space is obtained in a functorial manner from the symplectic vector space $V_g \otimes \mathbb{C}$ (a fact that is not obvious a priori). In particular, the $\mathrm{Sp}(V_g)$ action on this vector space is algebraic in the sense that it extends to an action of the algebraic group $\mathrm{Sp}(V_g)(\mathbb{C}) = \mathrm{Sp}(V_g \otimes \mathbb{C})$. With induction one can now construct a sequence of extensions of algebraic groups by vector groups which contains $\Gamma_g(0) \leftarrow \Gamma_g(1) \leftarrow \Gamma_g(2) \leftarrow \dots$ as a sequence of arithmetic groups. We now form the 'proarithmetic hull' of Γ_g, $\Gamma_g \to \Gamma_g(\infty) := \lim_k \Gamma_g(k)$. This map is injective, so that we may regard this as a kind of arithmetic completion of Γ_g. We are interested in the induced map on rational cohomology $H^\bullet(\Gamma_g(\infty);\mathbb{Q}) \to H^\bullet(\Gamma_g;\mathbb{Q})$. Results of Borel imply that the rational cohomology group $H^k(\Gamma_g(\infty);\mathbb{Q})$ stabilizes as $g \to \infty$ in a way that is compatible with the stabilization maps for $H^k(\Gamma_g;\mathbb{Q})$. In particular, for g sufficiently large, the image of $H^k(\Gamma_g(\infty);\mathbb{Q}) \to H^k(\Gamma_g;\mathbb{Q})$ is independent of g. Kawazumi-Morita [50] and Hain-Looijenga [40] proved that this stable image is precisely the tautological subalgebra $\mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\dots]$. (The stable cohomology $\lim_{g\to\infty} H^\bullet(\Gamma_g(\infty);\mathbb{Q})$ is however much bigger.) This indicates that a construction of a stable class not in this algebra must be rather sophisticated. It would be interesting to see whether a similar result holds if the central lower series of π_g is replaced by the direct system of its finite index subgroups. (This completes Γ_g by the system of its congruence subgroups; the result is an 'adelization' of Γ_g.)

10 The Witten Conjecture

Given a positive integer n and an n-tuple of nonnegative integers $(k_1,k_2,\dots,k_n)$, then for every genus g we can form the integral

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

This is of course zero unless $\sum_i k_i = 3g-3+n$ and if that equality is satisfied, we can regard it as an intersection number of tautological classes. Such a number need not be integral though, because $\overline{\mathcal{M}}_{g,n}$ is not smooth. For instance, $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$. Witten [73] stated in 1989 a conjecture that predicted their values. He phrased his conjecture in terms of a generating function. In this context, the basic classes are the multiples $(2k+1)!!\psi_i^k$ (where the double factorial stands for the product of the odd positive integers $\le$ its argument) and therefore we find it convenient to introduce the *Witten numbers*

$$[\tau_{k_1} \cdots \tau_{k_n}]_g := (2k_1+1)!! \cdots (2k_n+1)!! \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

It is clear that this number is invariant under permutation of the indices. The suffix g is redundant, in the sense that the number can be nonzero for only one possible value of g. But we keep it so that we can define

$$F_g := \sum_n \frac{1}{n!} \sum_{k_1,\ldots,k_n} [\tau_{k_1} \cdots \tau_{k_n}]_g t_{k_1} t_{k_2} \cdots t_{k_n} \in \mathbb{Q}[[t_0,t_1,t_2,\ldots]].$$

This is a symmetric function in its variables. Note that if we give t_i degree $i-1$, then F_g is homogeneous of degree $3(g-1)$. We shall not state the original conjecture (that says that $Z := \exp(\sum_g F_g) \in \mathbb{Q}[[t_0,t_1,t_2,\ldots]]$ satisfies a certain KdV-hierarchy), but give an equivalent conjecture, due to Dijkgraaf-Verlinde-Verlinde, instead. It says that Z satisfies a certain system of differential equations (known as the *Virasoro relations*). In terms of the individual F_g's these amount to:

$$\frac{\partial F_g}{\partial t_0} = \sum_{m\geq 1} (2m+1) t_m \frac{\partial F_g}{\partial t_{m-1}} + \tfrac{1}{2}\delta_{0,g} t_0^2, \qquad (\ell_{-1})$$

$$\frac{\partial F_g}{\partial t_1} = \sum_{m\geq 0} (2m+1) t_m \frac{\partial F_g}{\partial t_m} + \tfrac{1}{8}\delta_{1,g}, \qquad (\ell_0)$$

$$\frac{\partial F_g}{\partial t_{k+1}} = \sum_{m\geq 0} (2m+1) t_m \frac{\partial F_g}{\partial t_{m+k}} + \tfrac{1}{2} \sum_{m'+m''=k-1} \Big(\frac{\partial^2 F_{g-1}}{\partial t_{m'} \partial t_{m''}} + \sum_{g'+g''=g} \frac{\partial F_{g'}}{\partial t_{m'}} \frac{\partial F_{g''}}{\partial t_{m''}} \Big). \qquad (\ell_{k\geq 1})$$

The last term of (ℓ_{-1}) resp. (ℓ_0) comes from $[\tau_0^3]_0 = 1$ resp. $[\tau_1]_1 = \frac{1}{8}$. By comparing coefficients we obtain a set of relations among the Witten numbers that allows us to calculate them recursively: equation (ℓ_k) gives $[\tau_{k_1} \cdots \tau_{k_n} \tau_{k+1}]_g$ in terms of Witten numbers involving smaller (g,n) (for the lexicographical ordering). Notice that the first two equations involve each F_g alone. They give $[\tau_{k_1} \cdots \tau_{k_n} \tau_0]_g$ and $[\tau_{k_1} \cdots \tau_{k_n} \tau_1]_g$ in terms of Witten numbers on $\overline{\mathcal{M}}_{g,n}$. These relations can be easily accounted for by means of simple intersection calculus. For $k \geq 1$, equation (ℓ_k) expresses $[\tau_{k_1} \cdots \tau_{k_n} \tau_{k+1}]_g$ in terms of Witten numbers of $\overline{\mathcal{M}}_{g,n}$ and its boundary divisors (i.e., of $\overline{\mathcal{M}}_{g-1,n+2}$ and $\overline{\mathcal{M}}_{g',n'+1} \times \overline{\mathcal{M}}_{g'',n''+1}$ with $g'+g''=g$ and $n'+n''=n$). These equations have been proved by Kontsevich [53], using a combinatorial substitute for the varieties $\overline{\mathcal{M}}_{g,n}$. It is desirable to find a purely algebro-geometric proof of these identities, because such a proof has a fair chance of generalizing to Gromov-Witten invariants (unlike the combinatorial approach).

11 Complete Subvarieties of Moduli Spaces

The moduli spaces $\mathcal{M}_{g,n}$ are not projective, with the exception of the point $\mathcal{M}_{0,3}$. This seems intuitively clear; one can deduce it from the fact that the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ in the Deligne-Mumford compactification is non-empty.

It is clear that the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$ are all affine. To see this in a uniform way, note that the ample divisor $\kappa_1 = 12\lambda_1 - \delta + \psi$ can be written as a sum of boundary divisors in these cases, cf. [9, 73].

Let therefore $g \geq 2$; we first consider the case $n = 0$. It is well-known that $\mathcal{M}_2$ is affine; more generally, the moduli spaces $\mathcal{H}_g$ of hyperelliptic curves of genus g are affine. In characteristic $\neq 2$, we can see this by writing a hyperelliptic curve as a double cover of $\mathbb{P}^1$ branched in $2g+2$ distinct points; a description of $\mathcal{H}_g$ as the quotient of the complement of a hypersurface in $\mathbb{A}^{2g-1}$ by the action of the symmetric group $\mathbb{S}_{2g+2}$ results. In characteristic 2 one obtains the result starting with the observation that every hyperelliptic curve of genus g can be written in the form

$$y^2+(1+a_1x+a_2x^2+\cdots+a_gx^g)y = x^{2g+1}+b_{g-1}x^{2g-1}+b_{g-2}x^{2g-3}+\cdots+b_2x^5+b_1x^3.$$

Igusa [46] has given a description of $\mathcal{M}_2$ over $\mathbb{Z}$; in particular, in characteristics $\neq 2{,}5$ it is the quotient of $\mathbb{A}^3$ by a diagonal action of $\mathbb{Z}/5\mathbb{Z}$ with a unique fixed point.

For all $g \geq 3$, the moduli space $\mathcal{M}_g$ is not affine; a well-known consequence of the existence of the Satake compactification of $\mathcal{M}_g$ in which the boundary has codimension two. In particular there exist complete curves passing through any finite number of points of $\mathcal{M}_g$.

Such complete curves are not explicit, however. So for some time the problem of constructing explicit complete curves was studied. A nice solution to this problem was found by González-Díez and Harvey [38]. For all $g \geq 4$, they construct explicit complete curves in $\mathcal{M}_g$ in the following way. Take a genus 2 curve C mapping onto an elliptic curve E. Let a be a point of E different from the origin. The inverse image in $C \times C$ of the translated diagonal $\Delta_a = \{(e,e+a) : e \in E\}$ is a complete curve of pairs of distinct points. By going over to a finite cover of this curve, we obtain a complete curve of pairs of distinct points of C together with a square root of the corresponding divisor class of degree 2. That determines a complete curve of double covers of C ramified in two distinct points: a complete one-dimensional family of smooth curves of genus 4. One checks that these curves vary in moduli and obtains a complete curve in $\mathcal{M}_4$. Similarly, one finds complete curves in $\mathcal{M}_g$: start with a translate of the diagonal embedding of E in E^{2g-6} that avoids all diagonals, take its inverse image in C^{2g-6}, and form a family of double covers of C ramified in $2g-6$ points.

In genus 3, this construction doesn't work. The genus 3 problem was solved by Zaal. Starting with a complete family of curves of genus 4 with a nonzero point of order two in the Jacobian, one obtains a complete family of 3-dimensional Prym varieties. Zaal showed that suitable choices guarantee that all these Pryms are Jacobians of smooth curves [74]. (For a very different solution, see part II of [38].)

What about complete subvarieties of $\mathcal{M}_g$ of higher dimension? We cannot use the Satake compactification (it appears); almost all results rely on a variant of the classical Kodaira construction. Kodaira observed that one may construct an explicit complete curve in $\mathcal{M}_6$ by starting with a genus 3 curve C that is a double unramified cover of a genus 2 curve. This gives a complete curve of pairs of distinct points of C; one proceeds as above and obtains the result. The construction can be repeated, since one finds in fact a complete one-dimensional family of curves of genus 6 with a pair of distinct points, the ramification points coming from the double cover of C. The monodromy problems arising in the choice of a square root can always be resolved, so this leads to a complete surface in $\mathcal{M}_{12}$, a complete threefold in $\mathcal{M}_{24}$, etc. This can be improved upon by starting with the complete curve in $\mathcal{M}_4$ of [38]: one finds a complete surface in $\mathcal{M}_8$, a complete threefold in $\mathcal{M}_{16}$, etc. Another variant is to use triple covers ramified in one point: if the covered curve has genus h, the cover has genus $3h-1$. Asymptotically this doesn't lead to better results, but since one can start with a complete curve in $\mathcal{M}_3$, one obtains another construction of a complete surface in $\mathcal{M}_8$.

A new development occurs here through the recent work of Zaal [75]. Using the important work of Keel [51], Zaal constructs in characteristic $p > 0$ a complete surface in $\mathcal{M}_{3,2}$. (At the moment it is not clear yet whether it is also possible to do this in characteristic 0.) This leads to a complete surface in $\mathcal{M}_6$ in characteristic $p > 2$ via double covers. One also finds a complete threefold in $\mathcal{M}_{12}$, etc.

Observe that all known complete subvarieties of $\mathcal{M}_g$ of dimension > 1 lie in the locus of curves that admit a map onto a curve of lower (but positive) genus. (Every component of this locus has codimension $\geq g-1$.) In particular, we don't know whether a complete surface in $\mathcal{M}_g$ could contain a general point. Perhaps it is more important to study complete subvarieties passing through a general point than arbitrary ones, cf. [45], p. 55. In [65] Nicorestianu shows that the base of a complete, generically non-degenerate 2-dimensional family of smooth curves (of genus $g \geq 4$) is necessarily a surface of general type (in characteristic 0).

A celebrated result is Diaz's upper bound $g-2$ for the dimension of a complete subvariety of $\mathcal{M}_g$, see [15]. Looijenga's result on the tautological ring of $\mathcal{M}_g$ gives a different proof, valid also in positive characteristic [58]. Note that this bound is known to be sharp only for $g = 2$ and 3; since we don't know whether $\mathcal{M}_4$ contains a complete surface, it might be argued that we don't understand curves of genus 4.

The class $\lambda_g \lambda_{g-1}$ in the Chow ring of $\overline{\mathcal{M}}_g$ vanishes on the boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ ([23], see also [24]). Therefore it (or a positive multiple of it) is a candidate for the class of a complete subvariety of $\mathcal{M}_g$ of dimension $g-2$, if that exists. One might also phrase the existence of this class as the absence of an intersection-theoretical obstruction for the existence of a complete subvariety of dimension $g-2$. Compare the discussion in [40], §5. In genus 4 the class $\lambda_4 \lambda_3$ probably is the only candidate *in cohomology* for the class of a complete surface. This would follow from the calculation of the codimension 2 Chow group of $\overline{\mathcal{M}}_4$ [22] if $H^4(\overline{\mathcal{M}}_4)$ is generated by tautological classes (cf. [2], discussed in section 12, and [18], where Edidin shows that $H^4(\overline{\mathcal{M}}_g)$ is generated by tautological classes in the stable range). In [24] it is pointed out that the structure of the tautological ring of $\mathcal{M}_g$ (known for $g \leq 15$) suggests that there are no constraints on complete subvarieties of dimension $\leq g/3$, while there are many constraints on complete subvarieties of dimension $g-2$; so that it might be a better idea to look for the former rather than the latter. Zaal's construction of complete surfaces in $\mathcal{M}_6$ in characteristic $p > 2$ might be considered as evidence for this idea. (On p. 57 of

[45] it is stated that the maximal dimension of a complete subvariety of $\mathcal{M}_6$ (over $\mathbb{C}$) is known to be at least 2, but this appears to be a typo.)

Diaz's original motivation [16] for finding an upper bound for the dimension of a complete subvariety of $\mathcal{M}_g$ was the implication that a family of curves whose image in moduli has larger dimension, necessarily degenerates—for many types of questions, this knowledge can be of great help. In the same spirit, he shows that a complete subvariety of $\overline{\mathcal{M}}_g$ of dimension $\geq 2g-2$ necessarily meets Δ_0, the divisor of irreducible singular curves and their degenerations ([16], p. 80, Corollary; $2g-2$ is certainly what is intended). In other words, one has the upper bound $2g-3$ for the dimension of a complete subvariety of the moduli space $\widetilde{\mathcal{M}}_g = \overline{\mathcal{M}}_g - \Delta_0$ of curves of compact type. This bound is a direct corollary of the bound for $\mathcal{M}_g$, hence it holds in all characteristics as well. The surprise is that in positive characteristic the bound $2g-3$ for $\widetilde{\mathcal{M}}_g$ is known to be sharp. One obtains this result from a consideration of the locus $V_0(\overline{\mathcal{M}}_g) = V_0(\widetilde{\mathcal{M}}_g)$ of stable curves with p-rank 0. In [26] it is shown that it is pure of codimension g. The role that the class $\lambda_g\lambda_{g-1}$ played in relation to $\mathcal{M}_g$ is now played by λ_g: it vanishes on Δ_0 and has the right codimension. Van der Geer [28] (this volume) explicitly determined the class of $V_0(\mathcal{A}_g)$; it is a multiple of λ_g, hence the same holds for the class of $V_0(\widetilde{\mathcal{M}}_g)$.

The fact that the bound $2g-3$ for $\widetilde{\mathcal{M}}_g$ is sharp in positive characteristic (more precisely, that the known maximal complete subvariety of $\widetilde{\mathcal{M}}_g$ occurs only in characteristic $p>0$) as well as Keel's result [51] that the relative dualizing sheaf of $\overline{\mathcal{M}}_{g,1}$ over $\overline{\mathcal{M}}_g$ is semi-ample in positive characteristic, but not in characteristic 0, lead to the idea that the maximal dimension of a complete subvariety of $\mathcal{M}_g$ or of $\widetilde{\mathcal{M}}_g$ or of $\mathcal{A}_g$ may well depend on the characteristic. This is poignantly expressed by a conjecture of Oort (conjecture 2.3 G in [66]) that $\mathcal{A}_3$ over $\mathbb{C}$ does *not* contain a complete threefold.

Equivalently, $\widetilde{\mathcal{M}}_3$ over $\mathbb{C}$ would not contain a complete threefold. Even the answer to the following question appears to be unknown (cf. [65] for $g=3$):

Question 11.1. Does the moduli space $\widetilde{\mathcal{H}}_g \otimes \mathbb{C}$ of complex hyperelliptic curves of compact type of genus $g \geq 3$ contain a complete surface?

It is easy to see that it contains a complete curve. The existence of a complete surface in $\widetilde{\mathcal{H}}_3 \otimes \mathbb{C}$ is a necessary condition for the existence of a complete threefold in $\widetilde{\mathcal{M}}_3 \otimes \mathbb{C}$. The question can be formulated in terms of genus 0 curves, so it should be more approachable.

Finally a brief discussion of $\mathcal{M}_{g,n}$ in case $n>0$ (and $g \geq 2$). There are obvious relations to the case $n=0$, for different values of the genus. As mentioned above, $\mathcal{M}_{3,2}$ contains a complete surface in positive characteristic [75], while this is not known in characteristic 0. (The construction of the complete surface works for all $g \geq 3$.) Since the projection $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is projective, while the projections $\mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$ are affine, the Diaz-bound for $\mathcal{M}_{g,1}$ is $g-1$, while for $n>1$ it is at most $g-1$. The only existence result we know is that $\mathcal{M}_{g,n}$ is never affine for $n>0$ and $g \geq 2$. For $n>1$ there are fibers of $\mathcal{M}_{g,n}$ over $\mathcal{M}_g$ that contain a complete curve: take a curve C of genus g mapping onto an elliptic curve E, and proceed as in [38], discussed above. Note also that $C \times C - \Delta$ always contains complete curves: the difference map $(p,q) \mapsto p - q$ to the surface $C - C$ in the jacobian contracts the diagonal (Van Geemen). The following question seems relevant:

Question 11.2. For which smooth curves C of genus $g \geq 2$ does the complement in $C \times C \times C$ of all diagonals contain a complete curve?

We end with a conjecture:

Conjecture 11.3. *(Looijenga) $\mathcal{M}_g$ can be covered with $g-1$ affine opens.*

Harer's bound $4g-5$ for the cohomological dimension of $\mathcal{M}_g$ [43] would be one of several consequences of this result.

12 Intersection Theory

Here we discuss some of the developments regarding the Chow, tautological, and cohomology rings of the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ since the survey [40] was written. Those directly related to Gromov-Witten theory will be reviewed in section 13.

A great deal of progress has been made in genus 1. Getzler [30, 31, 32] has calculated the $\mathbb{S}_n$-equivariant Serre polynomials of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$. Hence the $\mathbb{S}_n$-representations $H^{p,q}(\overline{\mathcal{M}}_{1,n})$ are known. In particular, $H^{0,11}(\overline{\mathcal{M}}_{1,11})$ is one-dimensional, which was known before via Eichler-Shimura theory, cf. [13, 70, 71]; the representation is the alternating one. (The corresponding 2-dimensional Hodge structure of weight 11 is associated to the discriminant cusp form Δ.) It follows that $\overline{\mathcal{M}}_{1,n}$ is not unirational for all $n \geq 11$. With a beautiful construction, Belorousski [5] has shown that $\overline{\mathcal{M}}_{1,10}$ is rational, so that $\overline{\mathcal{M}}_{1,n}$ is unirational for all $n \leq 10$. In fact, using analogous constructions he shows that the Chow ring of $\overline{\mathcal{M}}_{1,n}$ (with $\mathbb{Q}$-coefficients as always) is generated by boundary cycles for $n \leq 10$. By induction this implies that the Chow ring of $\overline{\mathcal{M}}_{1,n}$ equals the tautological ring for $n \leq 10$. This cannot hold (over $\mathbb{C}$) for any $n \geq 11$ by (a suitable extension of) Jannsen's result ([48], Thm. 3.6, Rem. 3.11).

A crucial case is $n = 4$. Getzler's calculation implies that the $\mathbb{S}_4$-invariant part of $H^4(\overline{\mathcal{M}}_{1,4})$ is 7-dimensional. But there are 9 invariant boundary cycles, with only one WDVV-relation (i.e., coming from $\overline{\mathcal{M}}_{0,4}$) between them. Hence there is here a new, genus 1, relation. Getzler computes it in [33]. (He also announces there a proof that the even-dimensional homology of $\overline{\mathcal{M}}_{1,n}$ is spanned by boundary cycles, and that all relations among these cycles come from this genus 1 relation and the genus 0 relations.) In [67], Pandharipande uses the Hurwitz scheme to construct Getzler's relation algebraically, and Belorousski uses this to analyze the Chow rings of $\overline{\mathcal{M}}_{1,n}$ for low n in detail. E.g., he shows that the tautological ring (or equivalently, the ring generated by boundary cycles) is multiplicatively generated by divisors for $n \leq 5$, while for $n \geq 6$ it is generated in codimensions one and two.

For $n \leq 5$, he also obtains explicit presentations of $A^\bullet(\overline{\mathcal{M}}_{1,n})$. Returning to $n = 4$, by identifying the 4 points in 2 pairs, one obtains a map $\overline{\mathcal{M}}_{1,4} \to \overline{\mathcal{M}}_3$. Getzler's relation pushes forward to a relation in $A^4(\overline{\mathcal{M}}_3)$. Belorousski and Pandharipande verified that the obtained relation equals (modulo genus 0 relations) the non-trivial relation found in [21], Lemma 4.4, from associativity(!) considerations.[1]

In genus 2, far less is known. Mumford [64] determined the Chow ring of $\overline{\mathcal{M}}_2$, and it is not hard to determine $A^\bullet(\overline{\mathcal{M}}_{2,1})$ from his results. The Chow, tautological, and cohomology rings coincide here. With a delicate calculation using lots of ingredients, Getzler [34] computes the $\mathbb{S}_n$-equivariant Serre polynomials of $\overline{\mathcal{M}}_{2,n}$ for $n = 2$ and 3. He also computes the cohomology ring $H^\bullet(\overline{\mathcal{M}}_{2,2})$ and announces the result for $n = 3$. In particular, $h^4(\overline{\mathcal{M}}_{2,3}) = 44$. This result is the starting point for [6]. As Belorousski and Pandharipande point out, there are 47 *descendent stratum classes* in $A^2(\overline{\mathcal{M}}_{2,3})$. (It is not hard to see that Getzler's topological recursion relations [34] are algebraic, so these 47 classes span $\mathcal{R}^2(\overline{\mathcal{M}}_{2,3})$.) Exactly 2 relations come from genus 0, none from genus 1, so there must exist a new, genus 2 relation in homology. Belorousski and Pandharipande construct such a relation algebraically using admissible double covers. Bini, Gaiffi and Polito [7] have computed the generating function for the Euler characteristic of $\overline{\mathcal{M}}_{2,n}$.

If one believes the conjecture [24] (this volume) that the tautological ring $\mathcal{R}^\bullet(\mathcal{M}_g)$ satisfies Poincaré duality, then it is quite reasonable to believe that the same holds for $\mathcal{R}^\bullet(\widetilde{\mathcal{M}}_g)$, especially because of the role that the classes $\lambda_g\lambda_{g-1}$ resp. λ_g play in these cases (cf. §11 and [40]). These classes also tell us what in this respect the correct moduli spaces of pointed curves should be: $\widehat{\mathcal{M}}_{g,n} = \pi^{-1}(\mathcal{M}_g) \subset \overline{\mathcal{M}}_{g,n}$ resp. $\widetilde{\mathcal{M}}_{g,n}$. The first thing to observe is that the tautological rings of these spaces are one-dimensional in codimension $g-2+n$ resp. $2g-3+n$ and vanish in higher codimensions (this follows quite easily from [58] and [23], cf. [40], p. 108). The assumption that the tautological rings of these moduli spaces satisfy Poincaré duality can be used to predict (but not prove) relations of the type we saw above for $\overline{\mathcal{M}}_{1,4}$ and $\overline{\mathcal{M}}_{2,3}$. (Both for $\widetilde{\mathcal{M}}_{1,4} = \widehat{\mathcal{M}}_{1,4}$ and for $\widehat{\mathcal{M}}_{2,3}$ the $\mathbb{S}_n$-invariant part of $\mathcal{R}^1$ is 3-dimensional, while there are 4 invariant generators in degree 2, which is assumed to be dual to degree 1.)

In their recent paper [2], Arbarello and Cornalba show how one can in principle compute the low degree cohomology groups $H^k(\overline{\mathcal{M}}_{g,n})$ for k fixed and *arbitrary* g and n. Their elegant method proceeds as follows. *If* the boundary divisor $\partial\mathcal{M}_{g,n}$ were ample, $H^k(\overline{\mathcal{M}}_{g,n})$ would inject for low k into $H^k(\partial\mathcal{M}_{g,n})$. It hardly ever is ample, but Harer's calculation [43] of the virtual cohomological dimension of $\mathcal{M}_{g,n}$ implies that $H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n})$ is injective for $k \leq d(g,n)$ (where $d(g,n) = 2g-3+n$ for $g,n > 0$, while $d(g,0) = d(g,1) = 2g-2$ and $d(0,n) = n-4$). Mixed Hodge theory shows that the map $H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(N)$

[1] While doing the calculation, they discovered that some of the genus 0 relations in [21] are stated incorrectly. The correct relations are $[(7)] = 3[(6)]$ and $\delta_1[(e)]_Q = [(6)]_Q = \frac{2}{3}[(7)]_Q$ in codimension 4 and $[(c)] + [(e)] = 2[(d)]$, $[(c)] = 3[(b)]$, $\delta_1[(2)]_Q = [(b)]_Q = \frac{4}{3}[(c)]_Q = \delta_0[(6)]_Q$, $\lambda[(6)]_Q = \frac{1}{9}[(c)]_Q$, $\delta_1[(6)]_Q = -\frac{1}{9}[(c)]_Q - \frac{2}{3}[(f)]_Q$ in codimension 5 (cf. Thm. 3.1, p. 385, p. 400, Lemma 4.5, p. 403, Table 8).

is then injective as well, where N is the normalization of $\partial\mathcal{M}_{g,n}$. Now one uses the structure of N and a double induction on g and n to compute some of the low degree cohomology groups. E.g., for odd k, if one shows $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for all (g,n) with $d(g,n) < k$, then $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for *all* (g,n). So the proof that $H^1(\overline{\mathcal{M}}_{g,n}) = 0$ is reduced to checking it for the point $\overline{\mathcal{M}}_{0,3}$ and the projective lines $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{1,1}$! By checking more *seed cases*, Arbarello and Cornalba prove that $H^3(\overline{\mathcal{M}}_{g,n})$ and $H^5(\overline{\mathcal{M}}_{g,n})$ vanish for all (g,n) (this uses results of Getzler and Looijenga).

For low even k, one would like to show that $H^k(\overline{\mathcal{M}}_{g,n})$ is generated by tautological classes. At present this is known for $k = 2$. Arguing by induction, one assumes that $H^2(\overline{\mathcal{M}}_{h,m})$ is tautological for the moduli spaces $\overline{\mathcal{M}}_{h,m}$ appearing in $\partial\mathcal{M}_{g,n}$. Writing $N = \coprod_i X_i$, one knows that $f = \oplus_i f_i : H^2(\overline{\mathcal{M}}_{g,n}) \to \oplus_i H^2(X_i)$ is injective when $d(g,n) \geq 2$. Now on the one hand one knows exactly what happens to the tautological classes under f, which provides a lower bound for $\mathrm{Im}(f)$. But on the other hand, any collection of classes $(f_i(\alpha))_i \in \oplus_i H^2(X_i)$ satisfies obvious compatibility relations on the "intersections" of the X_i. Since by the induction hypothesis the $H^2(X_i)$ are tautological, the upper bound for $\mathrm{Im}(f)$ that this gives can be described exactly. The beautiful idea is that the lower and upper bound coincide, essentially. The low genus cases have to be treated carefully because the tautological classes are not independent. In this way one obtains a different proof of Harer's result [41] that $H^2(\overline{\mathcal{M}}_{g,n})$ is tautological.

13 Stable Maps and the Virasoro Conjecture

A general method to define invariants of a space X, that was developed through the work of Donaldson, Gromov, Witten, Kontsevich, and many others, is to consider an auxiliary space, e.g., a space of maps of curves to X, and then to compute a (well-defined) "natural" integral on that auxiliary space. In algebraic geometry, a breakthrough occurred through Kontsevich's construction of the space of stable maps. In this section, we briefly review the basic definitions and formulate some of the most important results. Then we discuss the Virasoro conjecture. We will show how this theory has repercussions for the study of $\overline{\mathcal{M}}_{g,n}$ itself—somewhat contrary to its original motivation.

Let X be a nonsingular complex projective variety, and let β be a class in $H_2(X,\mathbb{Z})$. One can consider the moduli stack $\mathcal{M}_{g,n}(X,\beta)$ classifying n-pointed smooth curves of genus g with a map $f : C \to X$ satisfying $f_*([C]) = \beta$. The *expected dimension* of this stack is

$$3g - 3 + n + \chi(f^*T_X) = 3g - 3 + n + (\dim X)(1 - g) - K_X \cdot \beta.$$

One seeks to compactify this space in a natural way; as Kontsevich [54] showed, this can be done using *stable maps*. A map $f : C \to X$ from a reduced, connected, nodal, n-pointed curve C of genus g to X, with $f_*([C]) = \beta$, is stable if each nonsingular rational component of C that is mapped to a point contains at least three special (nodal or marked) points, and each component of genus 1 that is mapped to a point contains at least one special point. (Equivalently, the map has finitely many automorphisms.)

Fulton and Pandharipande [27] explain in detail how a projective coarse moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps can be constructed. When X is a point (hence $\beta = 0$), one recovers the Deligne-Mumford-Knudsen moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g. In general, however, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is reducible, singular, nonreduced, and has components whose dimension is not the expected one. That one can nevertheless do intersection theory on this space is something of a miracle; it is possible thanks to the construction of the virtual fundamental class [56, 4, 3]. This cycle $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ lives in the expected dimension and satisfies the axioms of Gromov-Witten theory given by Kontsevich and Manin.

Natural cohomology classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ arise in two ways. Via the n evaluation morphisms $e_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ sending a stable map to the image of the i-th marked point, one can pull back cohomology classes from X. (Fix an additive homogeneous basis $1 = T_0,T_1,\dots,T_m$ of $H^\bullet(X) \otimes \mathbb{Q}$.) One also has the first Chern classes of the n cotangent line bundles $\mathcal{L}_i = s_i^* \omega_{\mathcal{U}/\overline{\mathcal{M}}}$, where $\mathcal{U}$ is the universal curve and s_i the section corresponding to the i-th marked point. Define classes

$$\tau_k^j = \tau_k^j(i) = e_i^*(T_j) \cup c_1(\mathcal{L}_i)^k \,.$$

Gromov-Witten invariants, and their descendents (i.e., τ_k^j with $k > 0$ occur), are defined in the algebraic context by integrating these classes against the virtual fundamental class:

$$\langle \tau_{k_1}^{j_1} \cdots \tau_{k_n}^{j_n} \rangle_{g,n,\beta} = \langle \tau_{k_1}^{j_1}(1) \cdots \tau_{k_n}^{j_n}(n) \rangle_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \tau_{k_i}^{j_i}(i)$$

(some care is required when X has odd-dimensional cohomology classes). More generally, one has the Gromov-Witten classes (or "full system of Gromov-Witten invariants"):

$$I_{g,n}^{\beta}(T_{j_1},\dots,T_{j_n}) = \pi_*([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \cap e_1^*(T_{j_1}) \cap \cdots \cap e_n^*(T_{j_n})) \in H^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

with $\pi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ the forgetful map $(2g - 2 + n > 0)$.

There are several ways to produce relations between Gromov-Witten invariants:

a. Under the natural map $\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ the virtual fundamental class pulls back to the virtual fundamental class. Also, $c_1(\mathcal{L}_i) = \pi^* c_1(\mathcal{L}_i) + \Delta_{i,n+1}$, where $\Delta_{i,n+1}$ is the divisor 'where the i-th and $(n+1)$-st point have come together'. This leads to generalized string and dilaton equations, expressing GW invariants involving a τ_0^0 resp. a τ_1^0 in terms of simpler GW invariants, and a divisor equation, for GW invariants involving a τ_0^D, where D is a divisor class.

b. Relations between the classes of the strata (and their descendents) occurring in the topological stratification of $\overline{\mathcal{M}}_{g,n}$ yield relations between GW invariants, via the 'splitting axiom'. Examples of such relations were given in section 12. One also has the so-called topological recursion relations (TRR) that in genus 0 and 1 express the class of a cotangent line in boundary divisor classes. In higher genus, this is not possible. Getzler [34] conjectures that monomials of degree g in cotangent line classes can be expressed in terms of boundary classes, and proves this (explicitly) for genus 2.

c. In case X admits a torus action satisfying certain conditions, one can attempt to compute GW invariants using Bott localization. Ellingsrud and Strømme [20] introduced Bott's formula to enumerative geometry. Subsequently Kontsevich [54] used localization to compute GW invariants in genus 0. In higher genus, one needs a localization formula for the virtual fundamental class; this was accomplished by Graber and Pandharipande [39]. One should note that the use of this method is not restricted to the calculation of GW invariants on X with a torus action: certain GW invariants can be expressed in terms of an ambient projective space. Compare the work of Kontsevich [54] expressing the number of rational curves on a quintic threefold as a sum over trees—the first step in Givental's solution [37] of the mirror conjecture.

We end with the Virasoro conjecture of Eguchi, Hori and Xiong [19]. A proof of it will lead to a wealth of relations between GW invariants and their descendents (although the extent to which it determines all such invariants is not clear at the moment). Just as in section 10, one organizes the GW invariants and their descendents for a fixed X into a generating function, the so-called full gravitational potential function. Eguchi, Hori and Xiong conjecture that its exponential is annihilated by certain formal differential operators that form a representation of the affine Virasoro algebra. (The initial form of the conjecture was for X with only (p,p) cohomology; the extension to general X is due to Katz. See, e.g., [11, 36].) There is considerable evidence for the Virasoro conjecture, but we will not discuss this here (see e.g. [35]). Instead, we mention the work of Getzler and Pandharipande [36] who investigate the implications of the Virasoro conjecture in the case $\beta = 0$. There is a natural isomorphism $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$ under which the virtual fundamental class is identified with the top Chern class of the exterior tensor product of the dual of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$ and the tangent bundle of X. Getzler and Pandharipande show that for $X = \mathbb{P}^2$ this case of the Virasoro conjecture implies the conjectured proportionality formulas [24] (this volume) for the tautological ring of $\mathcal{M}_g$, while for $X = \mathbb{P}^1$ it implies the beautiful identities

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{i=1}^{n} \psi_i^{a_i} = \binom{2g-3+n}{a_1\, a_2\, \cdots\, a_n} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}$$

for a_i with sum $2g-3+n$. In [25] the integral on the right side is computed using virtual localization [39] and more classical techniques.

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