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Some complex system applications to social sciences

Road map

Agent based simulation

- A market partitioning example
- Another example on a different ‘grid’

Two meta-applications to complex systems

- Geometry impacts of the ‘phase space’
- Logical model building

The emphasis is on the interplay between sociological/ economical content and its formal representation.

EMERGENCE OF MARKET STRUCTURES

César García-Díaz (Universidad de los Andes, Colombia)
Arjen van Witteloostuijn (Tilburg University)
Gábor Péli (Utrecht University)

(The underlying paper just accepted by PLoS One)

INTRODUCTION

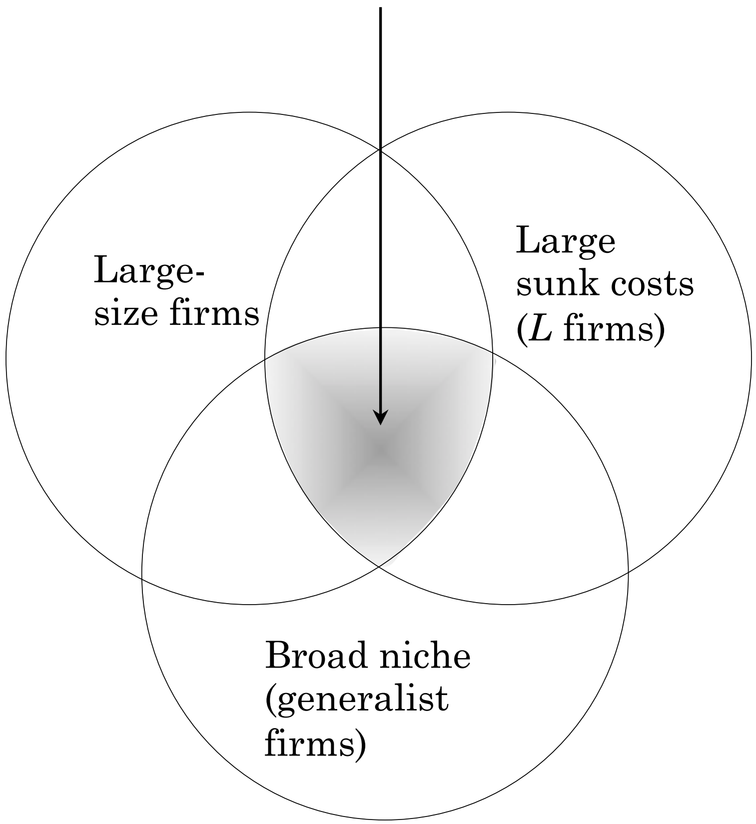
- Markets have emergent properties due to the interactions between firms.
- Aggregation of firm-level characteristics is not enough to predict market-level properties.
- Firm-level interaction and heterogeneity are important.

MICRO-LEVEL ADAPTATION, MACRO-LEVEL SELECTION, AND THE DYNAMICS OF MARKET PARTITIONING

- The emergent property we investigate is how ***discontinuitiues*** in markets' firm distributions develop.

- The discontinuities our model addresses are
  - in ***size*** (Are there characteristic firm sizes?),
  - in ***sunk costs*** (large vs. small sunk cost firms),
  - in firm ***niche width*** (Do they offer for a broad or narrow range of audience?)

A simulation finding (to be mentioned in advance):
Most large-sized firms of the model belong here



So the three typologies by and large coincide in the present model.

MICRO-LEVEL ADAPTATION, MACRO-LEVEL SELECTION, AND THE DYNAMICS OF MARKET PARTITIONING (cont)

- A key example is the formation of ***dual market structures***, with two typical firms.
- Markets with few large companies dominate the market's center while many smaller enterprises survive in the market's periphery.

- Dual markets have been investigated by both sociologists and economists.

- The dominant explanation in sociology is ***organizational ecology*** (OE)'s *resource-partitioning theory*:

  As market concentration rises, generalist firms increase their mortality rate, while that of specialist firms decline.

- In economics (***industrial organization***, IO), game-theoretical approaches explain how sunk endogenous costs investors (i.e. multi-product firms) target abundant demand spots in the market:

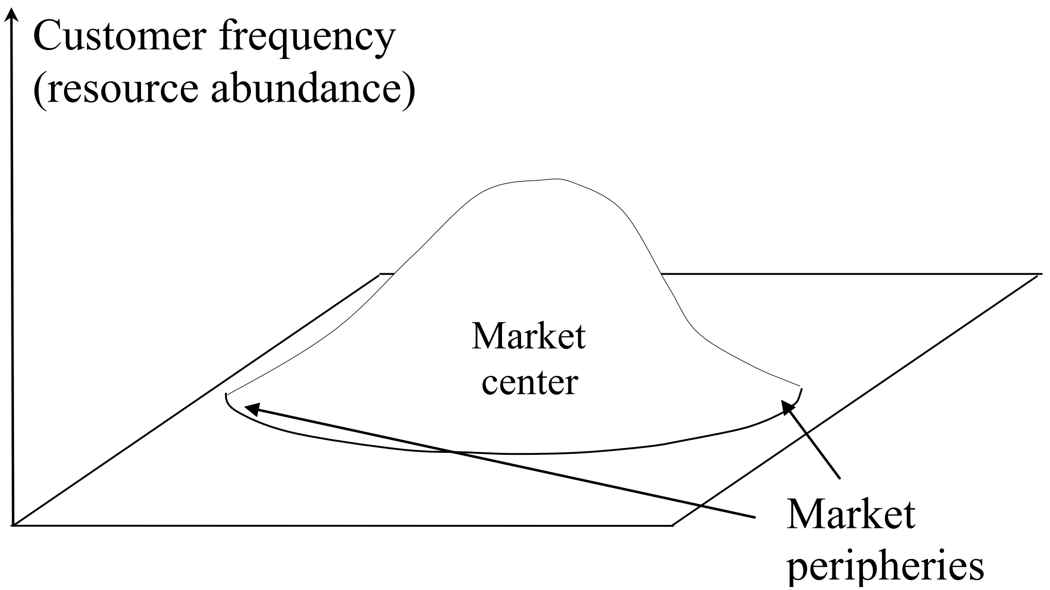
  Single-product specialists, characterized by low sunk cost investments, populate scarce demand areas.

- To tackle the above-mentioned issues, we build an ***agent-based computational model***.
- Agent-based simulations are typically about some jumps between observation levels:
- They are tools to investigate how lower level interactions cohere into observable outcomes at the higher level.
- The two levels in our research are the ***firm level*** and the ***population*** *(industry)* level.

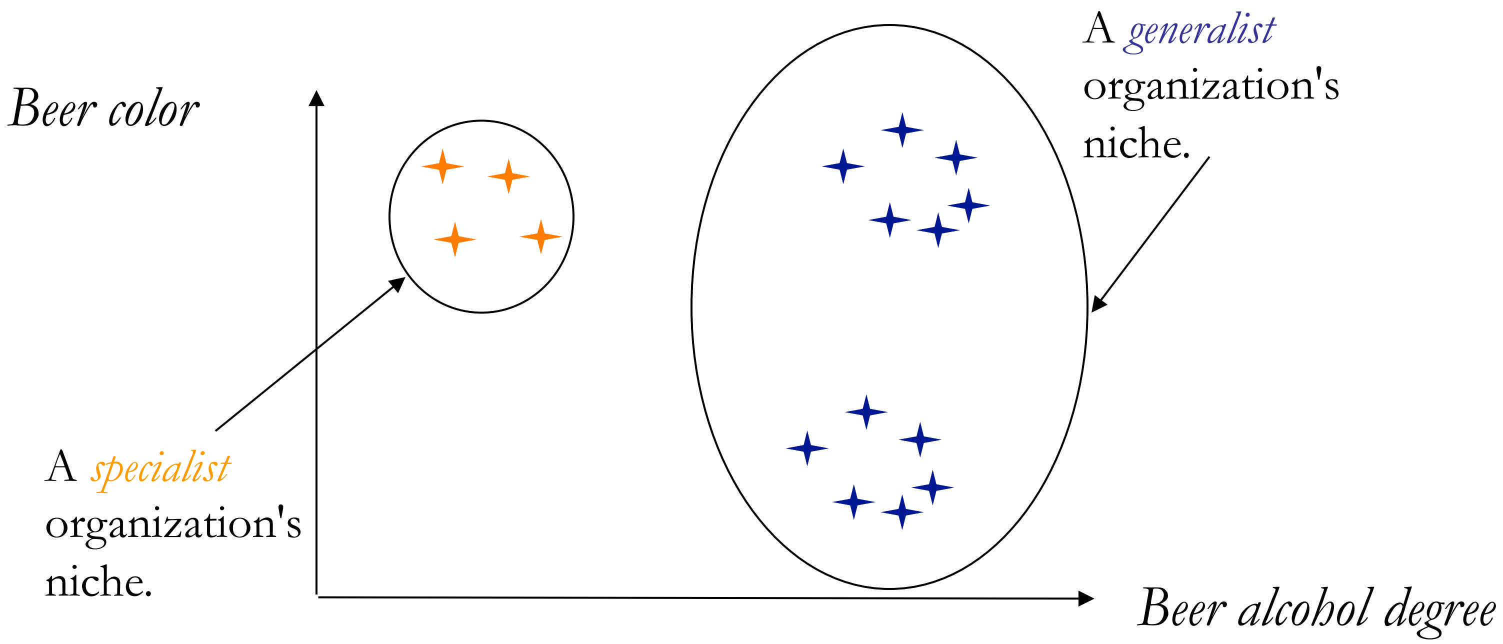
- We shortly introduce ***resource partitioning theory*** (Glenn Carroll, 1985) as our model uses several of its concepts.
- The theory was inspired by the following observation: ***In certain markets, a strong stream of specialist firm entry is observed when market concentration exceeds a certain threshold.***
- High market concentration is an entry barrier according to conventional economic wisdom.

From now on, we assume an *unimodal* distribution of resource (customers).

A resource landscape example with two resource dimensions:

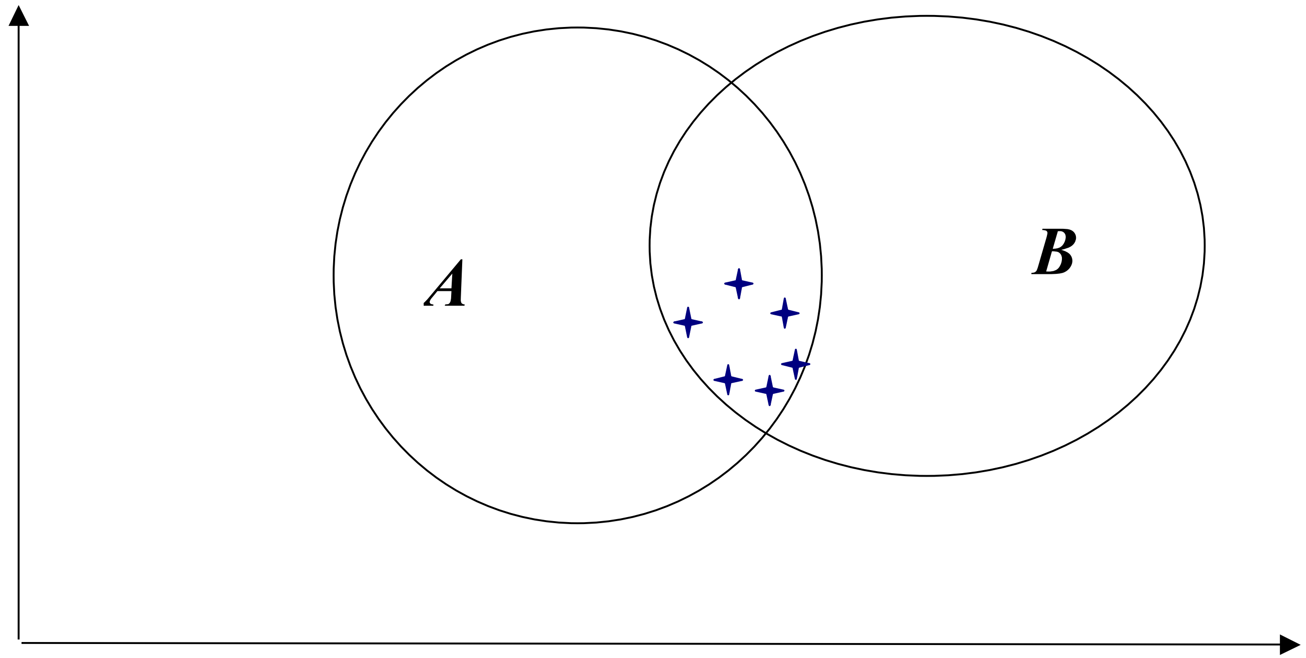


It is quite common that markets have some typical customer tastes. These tastes form the *market center*. Less frequent customer tastes form the *market peripheries*.



The *niche* of an organization is the zone in the resource (demand) space from where it draws resources (for which the firm has an offering).

Organizations with a *broad* niche are *generalists*.
Organizations with a *narrow* niche are *specialists*.



Niche overlap between organizations means *competition*: Organizations ***A*** and ***B*** go for the same type of resources (customer or voter taste, *etc*.).

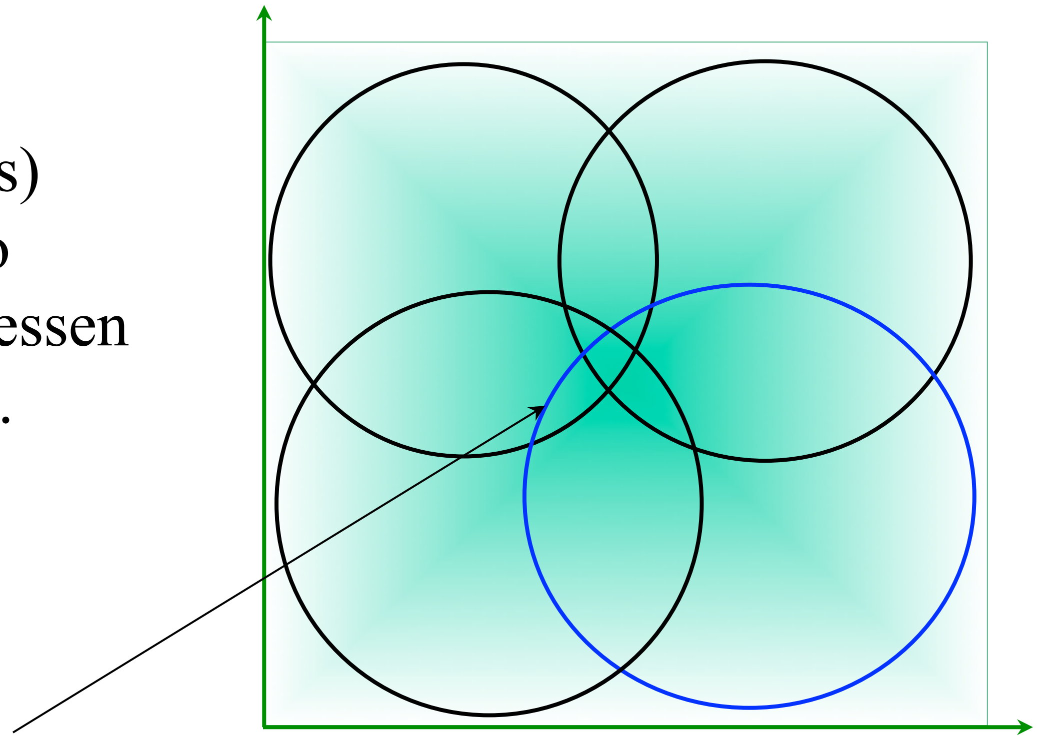
- Resource partitioning theory predicts that market resources (demand) can be—peacefully— partitioned between generalists and specialists as the market matures:
- One or a few generalists will occupy the resourceful market center.
- A large number of small specialists will populate the market peripheries. They are the “scavengers” of the resource space that go for thin market demand (*i.e.,* for customers with unfrequent tastes).

Illustration with two space dimensions

(Now, we look at the resource landspace from upwards.)

1. Early market phase:

Generalist niches (circles) overlap. But there is also niche *differentiation* to lessen the competitive pressure.



Market center
(the darker area denotes abundant demand)

Generalist consolidation

- Competition chills out after the stronger outcompetes the weak. Only a very few (or one) generalists are left in the market. Result: *increasing market concentration*.

- In lack of competitors, the surviving generalist(s) can now extend their niches. They occupy the very heart of the market center.

2. Mature market phase

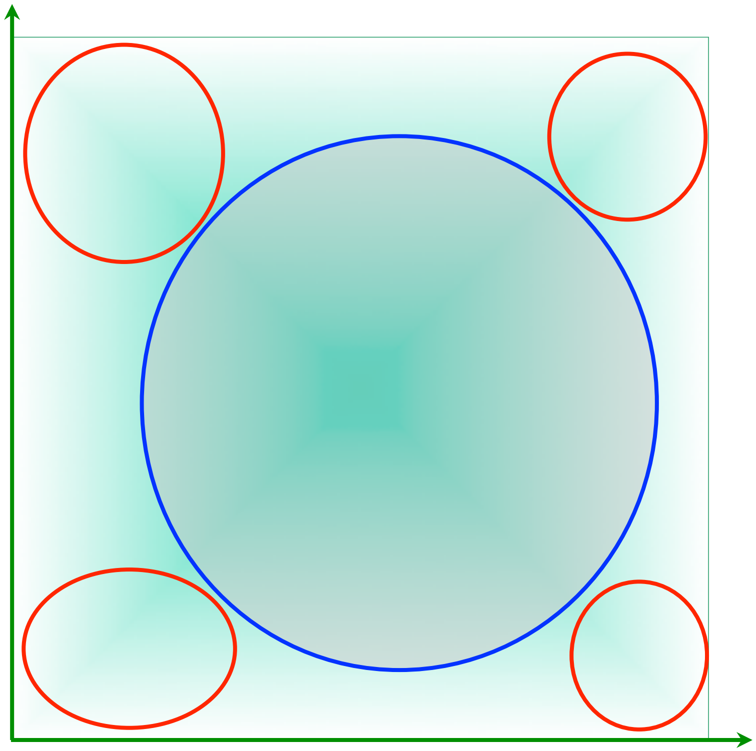
- It is not good for generalists to *overstretch* their niche. *Result*: they pull out from market peripheries as they move their niches towards the market center. Doing so, they abandon peripheral demand. This is called *resource release*.
- New entrant, small specialist organizations can populate the abandoned market peripheries.
- *Result*: generalists and specialists co-exist without significant competition (niche overlap).

The market resources become partitioned.

Mature market:

Resource partitioning between one (or a few) generalist at the center and specialists at the market peripheries.

(Small circles denote specialist niches.)



Address-type models in economics (Hotelling, 1929).

- Consider an *n*-dimensional space spun by products of *n* descriptors (*commodity space*, Lancaster, 1967).
- Each product is represented as a point in space. Each customer has a taste point (*ideal point*).
- Firms can *address* a customer group by positioning their offerings to their customer tastes.
- The distance (Euclidean or other) between the offering and the ideal point stands for the degree of customer dissatisfaction (qualities and prices considered constant).

Scale and scope economies/diseconomies

Production cost = fix costs + variable costs.

Scale economies: when the variable costs of *unit production* decrease with *production size* (e.g., software industry).

Scope economies: when *unit production* costs decrease when boundling the production/sale of an array of (typically similar) products.

Resource partitioning theory – and also our model – assume scale economies. It also assumes scope *dis*economies (at least beyond a certain product array breadth, *i.e,* beyond a certain *niche width*.)

THE MODEL; DEMAND REPRESENTATION

- We consider a one-dimensional resource space of N breadth (*i.e.*, with N taste positions).
- Demand is represented as a distribution of consumers along these taste positions.
- Demand distribution is *unimodal*: There are some mainstream taste positions.
- Each firm offers *one* product, for the sake of simplicity.
- Firms are rational profit maximizers.

THE MODEL: FIRM`S COST STRUCTURE (1)

- Two-piece cost function: One part related to production, the other one to niche-width costs (***scope diseconomies***):

$$C_t^i = C_{PROD,t}^i + C_{NW,t}^i$$

- Production level of firm i at time t (Q_{it}) is quantified with the standard *Cobb-Douglas function* of two factors: F (fixed costs) and V (variable costs). We assume $\alpha + \beta > 1$ in order to have ***scale economies***.

$$Q_{i,t} = F_i^{\alpha} V_{i,t}^{\ \beta}$$

THE MODEL: FIRM`S COST STRUCTURE (2)

- Given a fixed amount F, firms compute production quantities according to the following cost minimization problem (the W's refer to unit costs per factor F or V).

$$\begin{aligned} \min \quad & W_F F_i + W_V V_i \\ s.t. \quad & Q_{i,t} = F_i^{\alpha} V_{i,t}^{\ \beta} \end{aligned}$$

- Firms also face ***scope diseconomies***: Attending a more diverse (*i.e*. wider) audience increases costs. NWC denotes niche-width costs per unit of distance, w^u and w^l represent the upper and lower niche limits, respectively. Then, the overall niche-width cost of firm i at t is:

$$C_{NW,t}^{i} = NWC \left\| w_{i,t}^{u} - w_{i,t}^{l} \right\|$$

THE MODEL: FIRM`S COST STRUCTURE (3)

We consider two types of firms:

- **Small sunk cost firms (*S* firms)**: Their have scale economy advantages are small. But their demand requirements to cover total costs are also low.

- **Large sunk cost firms (*L* firms)**: Their have scale economy advantages are large. But they need a great amount of demand to cover their fixed (sunk) costs.

THE MODEL: CONSUMER BEHAVIOR (1)

- Consumers buy from the firm with the lower compound cost *U* of price *P* and the distance of their ideal taste point *k* from the firm's niche center location *nc*.
- Since the space has *N* positions, this distance is normed by dividing it with *N* - 1.

$$U^*_{k,t} = \min_{i \in S_{k,t}} \left\{ P^i_t + \gamma \frac{\left\| nc^i_t - k \right\|}{(N-1)} \right\}$$

- We assume a *Pmax* maximum price for the whole model.
- The highest price consumer *i* is willing to pay depends on the product distance from its idal point *k*.

$$P^{i*}_t \leq P_{\max} - \gamma \frac{\left\| nc^i_t - k \right\|}{(N-1)}$$

THE MODEL: FIRM EXPANSION

- Firms evaluate expanding to other niche positions according to a probability coefficient *ExpCoef* (that is, if we pick up a random number τ between 0 and 1, and $\tau < ExpCoef$, then the firm decides to expand).

- When a firm expands to a new niche position, it considers

  (i) the location of the **new niche center** relative to the targeted position),

  (ii) **the price** it offered at the previous iteration, discounted by the additional dissimilarity distance between the targeted position and the new niche center),

  (iii) **the expected quantity** the firm would get given the prices offered by *rivals* in that position in the previous iteration,

  (iv) the **additional niche spanning cost**.

- If the incremental profit (additional benefits – additional costs) is positive , the firm “invades” the targeted position.

EXPERIMENTAL DESIGN

We applied two statistical techniques to inspect model properties:

Survival analysis: This allows investigating mortality rates as a function of size, firm age, niche width, distance to market center, firm type (*L* or *S*), market concentration and market population (total number of firms).

Regression analysis: We took all the relevant model parameters as independent variables investigating their effects on

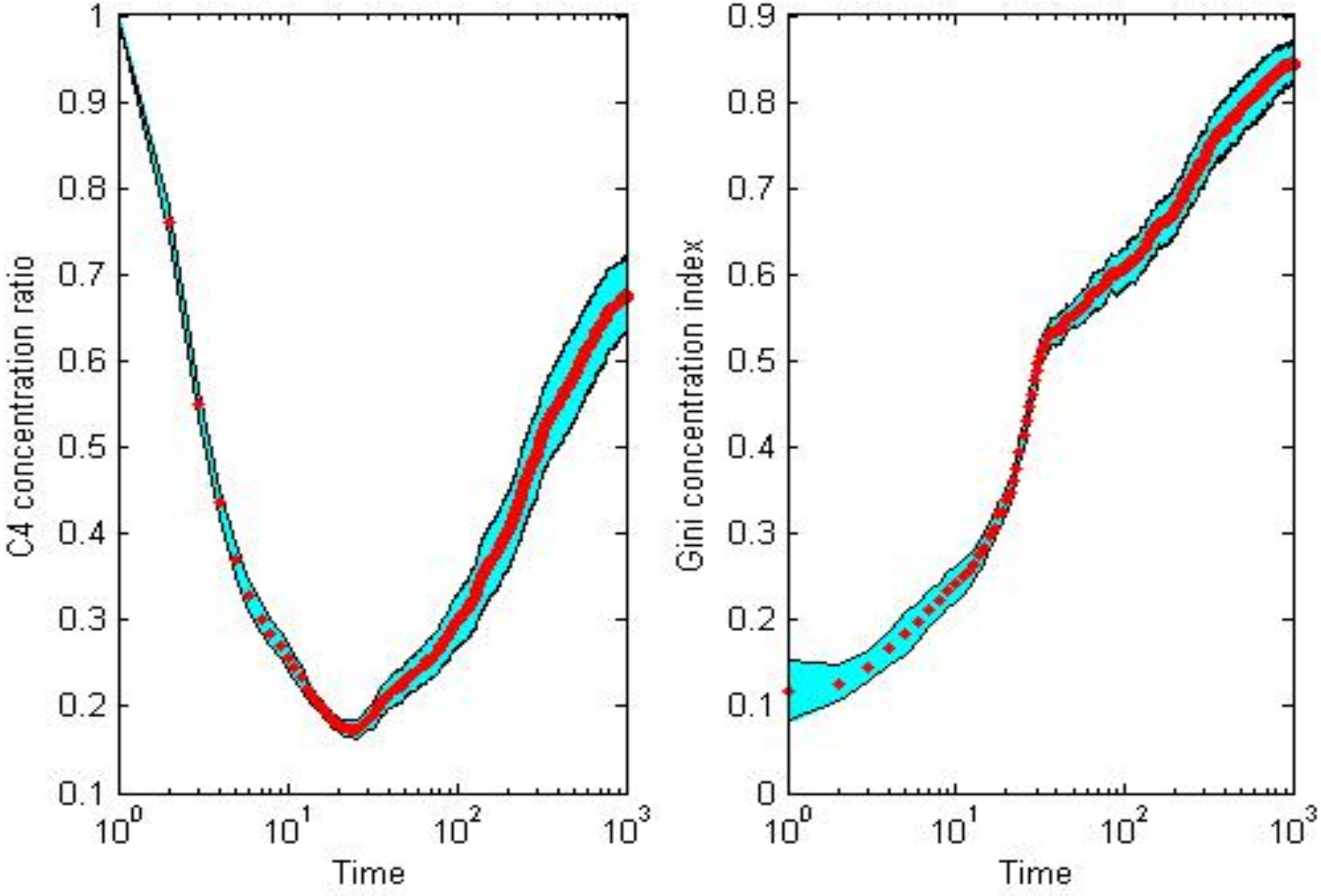
(i) market concentration,

(ii) *L* and *S* firm population,

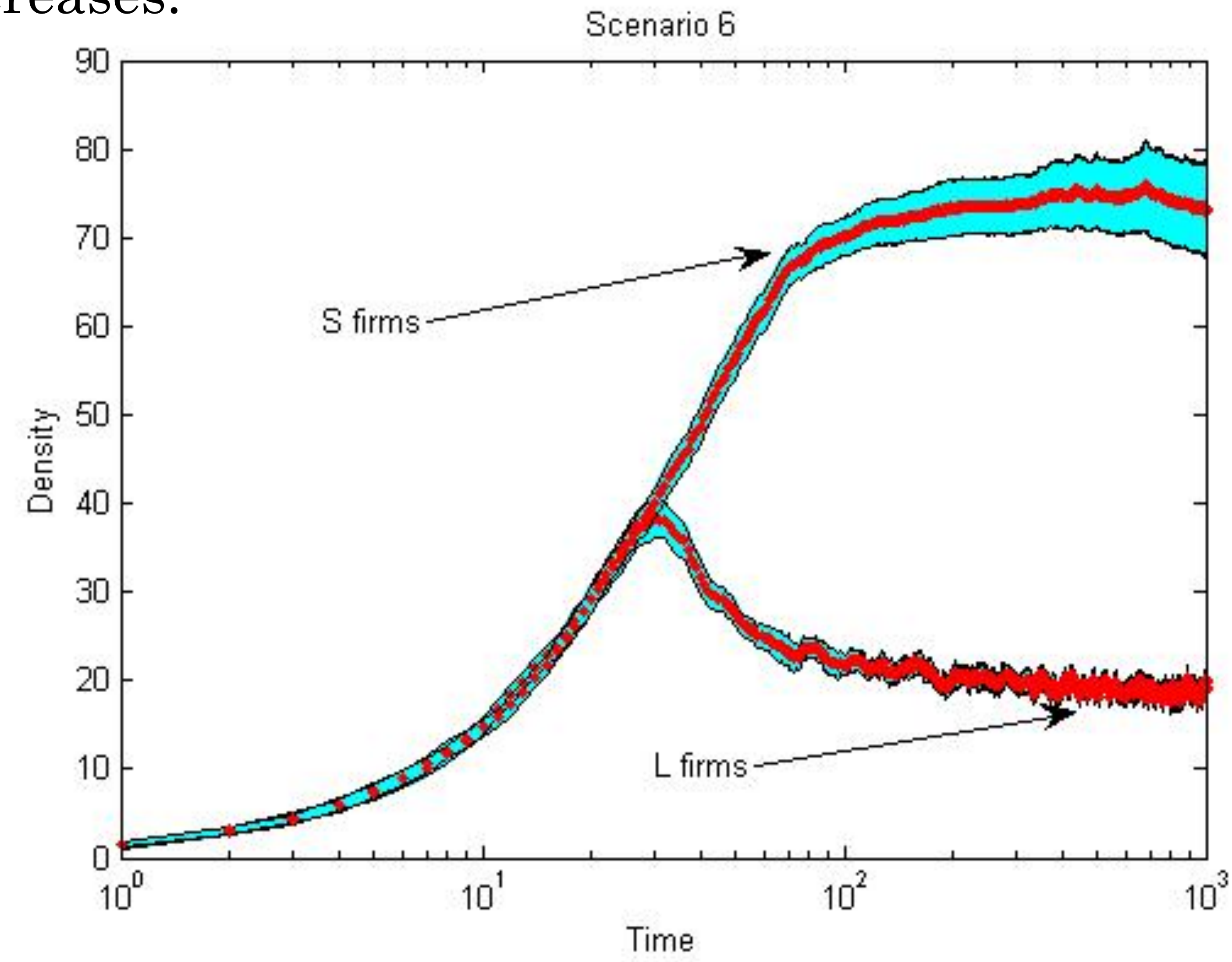
(iii) *resource release* (measured as the number of positions *L* firms abandon after moving to the peak demand positions).

RESULTS

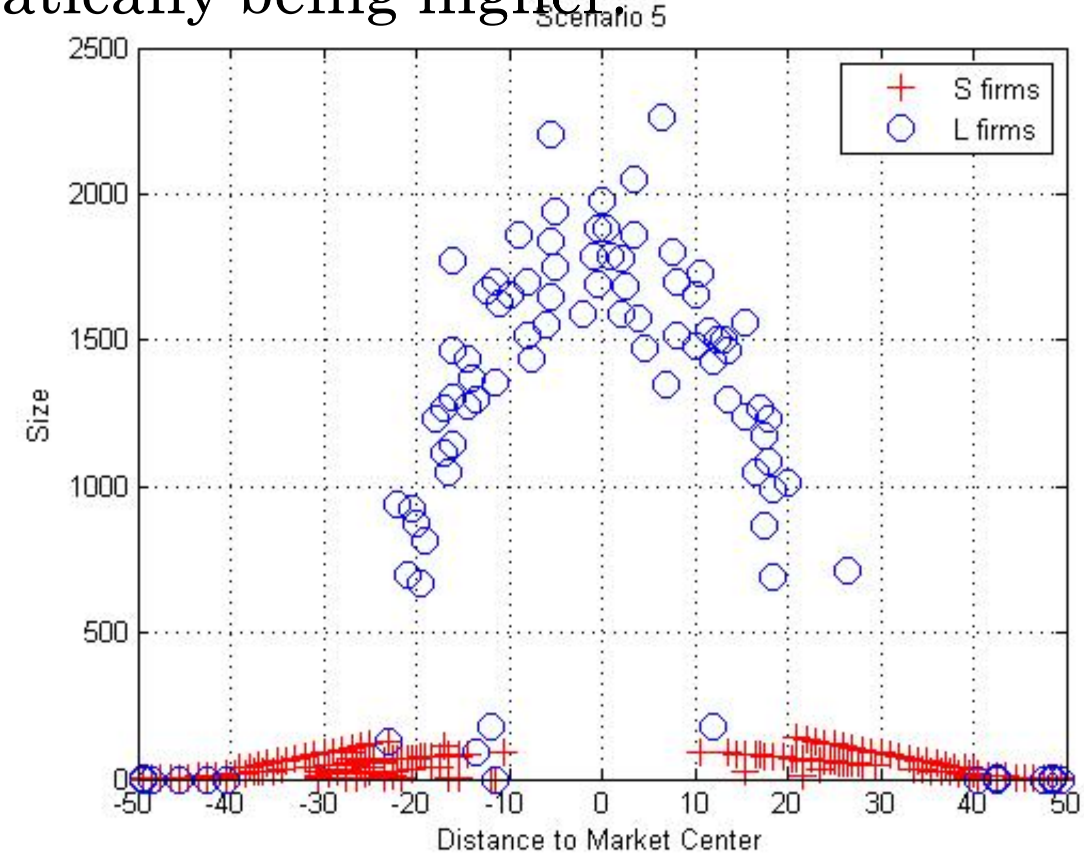
1. As the market gets crowded, market concentration increases



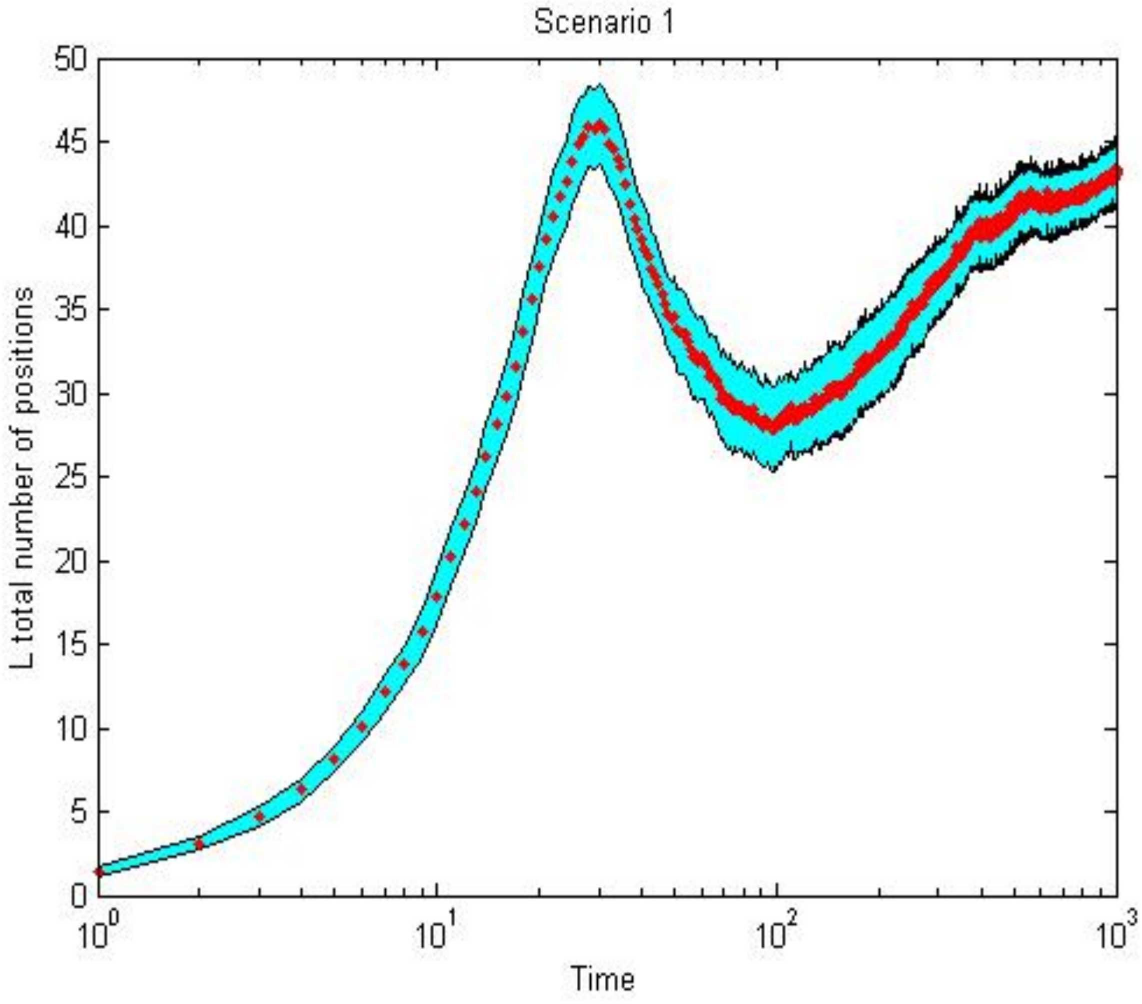
2. Large sunk cost (*L*) firm density first increases and then declines, while small sunk cost firm (*S*) density increases.



3. Broad-niche firms (typically L type) take over the market center. Narrow-niche firms (a mixture of L and S firms) locate at the market fringes, producing a dual market structure, with narrow-niche firms' density systematically being higher.



4. Scale-based competition in the market center may cause *resource release* at the market *semi*-periphery. But in the long run, large firms re-occupy (some of) this abandoned space.



5. Survival analyses show that *mortality risk* decreases with *firm size*, whilst increases with *market concentration* and with the *distance* to the market center.

That is:

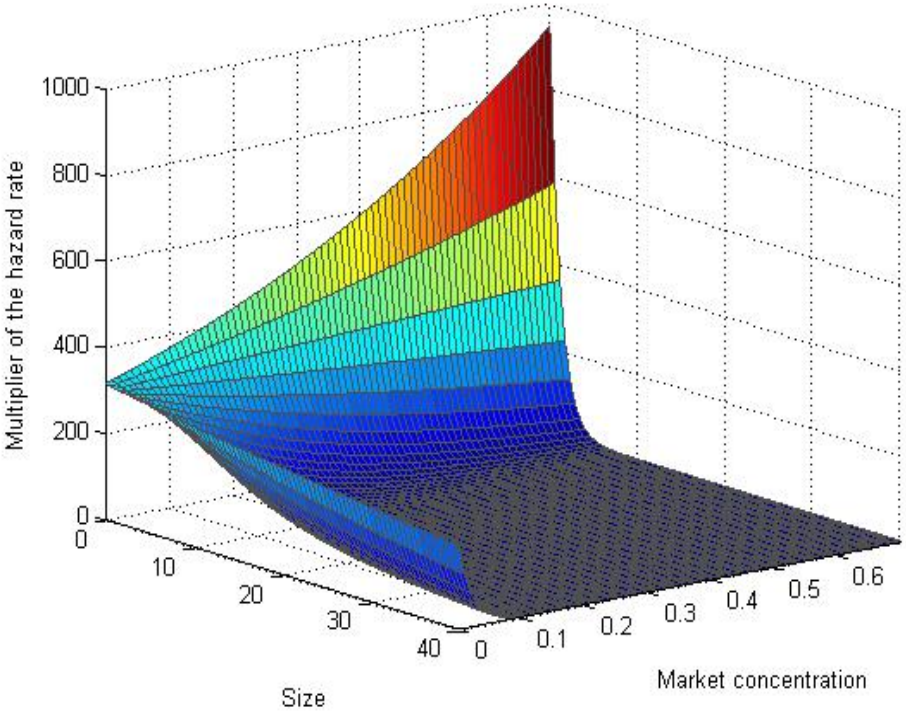
- In general, small firms have higher mortality than large firms.
- Both firm types have higher (average) mortality as market concentration increases (in line with conventional economic wisdom). We will see that this will not hold if we disaggregate L from S firms.
- Firms locating farther from the center have higher mort.

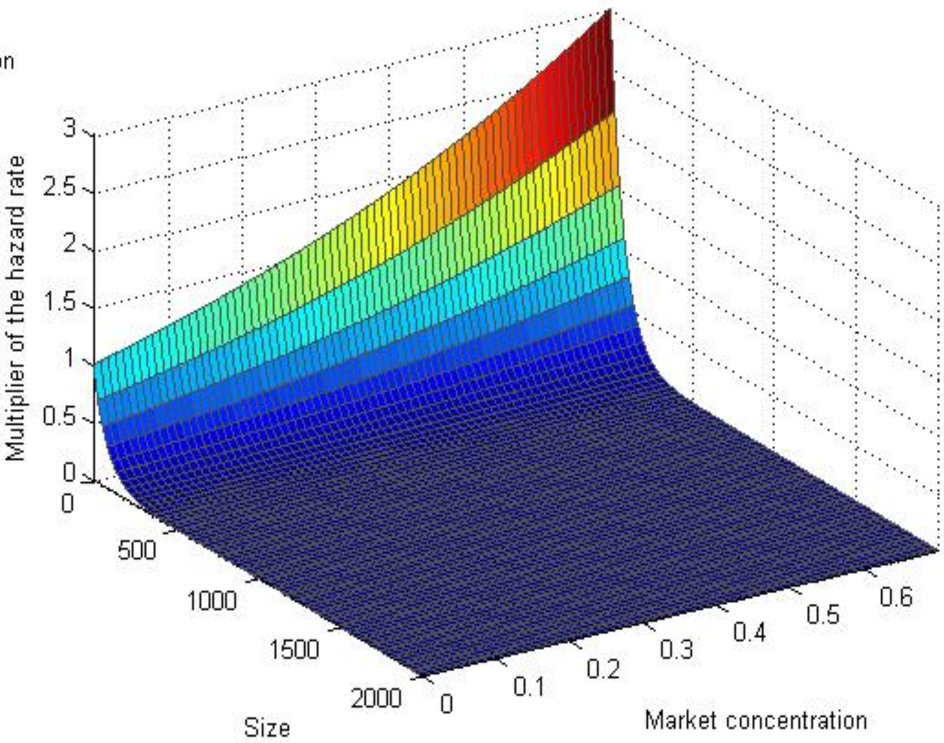
6.* As market concentration increases:

- either (a) large *S* firms' mortality hazard decreases, whilst that of *L* firms' does increase;
- or (b) *S* firms' mortality hazard increases at a slower pace than that of *L* firms;
- (c) the smallest *S* firms' mortality hazard always increases with concentration.

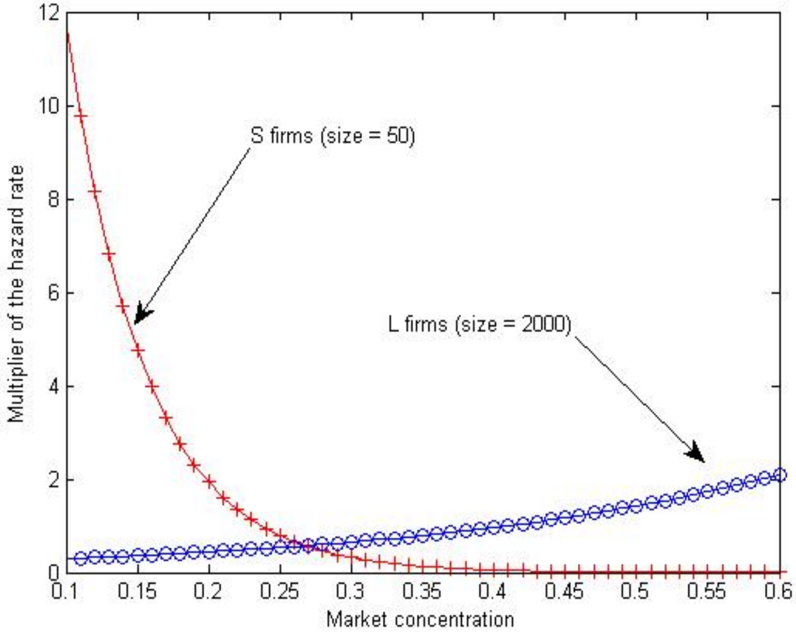
*We measured the effect of concentration relative to the minimum concentration value effect found in the simulation (this is known as the "multiplier of the rate"). See the next two slides.

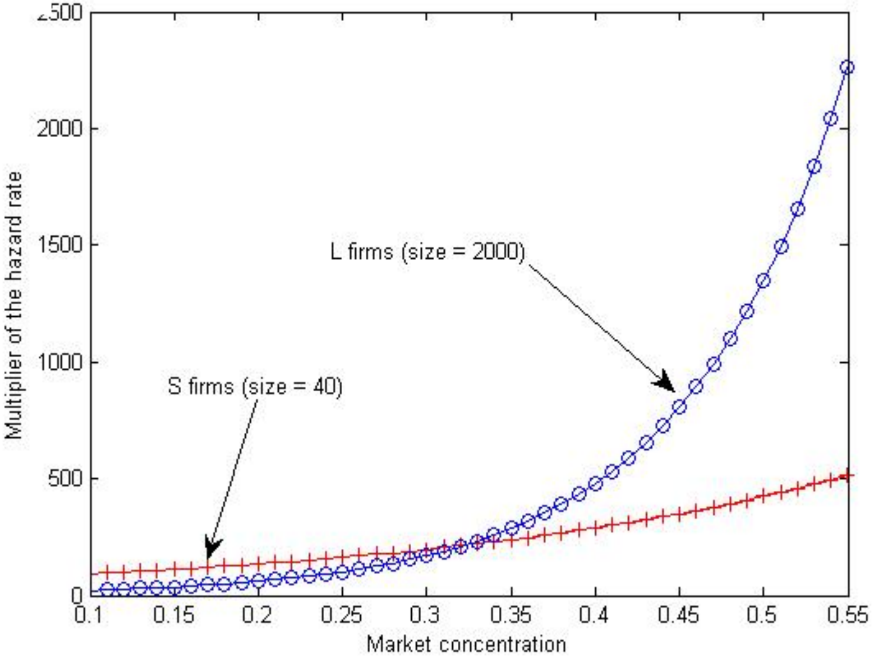
RESULT 6 (cont.)





RESULT 6 (cont.)





7. L firms' higher expansion probability and higher endowment allows them moving toward, and enduring competition in, the market center.

A higher L firm expansion probability decreases their resource release at the peripheries, while higher L firm endowment increases it.

Higher expansion capacity **reduces** resource release

Higher endowment (higher endurance) **increases** resource release.

Independent variables		Concentration (Gini) index	*L* density	*S* density	*L* space contraction
Q_S	Small sunk cost parameter	-.0175831*	-.1431061*	2.774796*	1.363785*
		(.0002214)	(.0045447)	(.0448818)	(.0217619)
γ	Product dissimilarity	-.0025169*	.0045806	.5720272*	.1815882*
		(.0001565)	(.0031975)	(.0340542)	(.0149023)
φ	Markup factor	.1163989*	-1.44986*	76.54684*	9.791984*
		(.0155546)	(.3160434)	(3.394657)	(1.464913)
ExpCoef	Expansion coefficient	3.238009*	8.51728*	-263.9791*	**-121.3129***
		(.1568174)	(3.169922)	(34.50324)	(14.7435)
E	Endowment	.00345*	1.250871*	.0937271	**.6340934***
		(.0002614)	(.0061466)	(.0563541)	(.0244667)
X	Entry rate	.0007879*	5.255838*	7.353022*	3.898299*
		(.0025293)	(.0516949)	(.5532305)	(.2395199)
Intercept		.8440625*	-9.539741*	-106.2712*	-39.89156*
		(.0233112)	(.4662876)	(5.094417)	(2.274647)
Number of observations		3223	3223	3223	3223
$F(6, 3126)$		1125.17	8786.44	801.88	784.76
R^2		0.6845	0.9545	0.5640	0.6199
Root MSE		0.07178	1.4674	15.711	6.8014

Robust standard errors in parenthesis; * $p < 0.05$.

APPLYING "*FRACTION DIMENSION*" AT STUDYING ATTRIBUTE SPACES

Intuition for the use of this concept:

- Not all geometrically possible space positions ('cells') are active in the attribute spaces standing for our markets.

- For example, there are taste combinations such that there is no existing product with this taste combination.

- We need a measure representing the *saturation level* of the attribute space segment with active cells (extant offerings).

Source:

García-Díaz C., A. van Witteloostuijn and G. Péli (2008). Market dimensionality and the proliferation of small-scale firms. *Advances in Complex Systems* 11(2): 231-247.

Fraction dimensionality as a patchiness measure

- We use the *similarity dimension* concept of Mandelbrot (1983) to define fraction dimensionality.
- Consider an n integer dimensional *frame space* with m scale elements along each dimension.
- Let H denote the count of 'filled' cells in space. Then, the *fraction dimension DIM* of the filled-up space is:

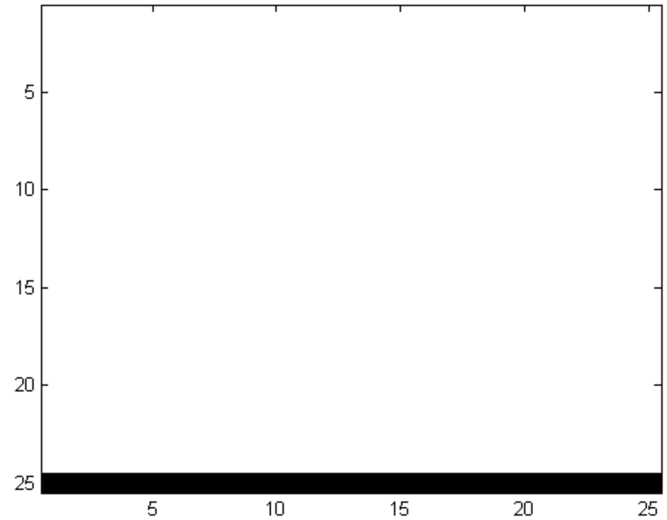
$$DIM = \frac{\ln H}{\ln m}$$

DIM is *not* sensitive to scale types (ratio, interval, ordering, nominal).

If the frame space is fully saturated, that is, $H = m^n$, then:

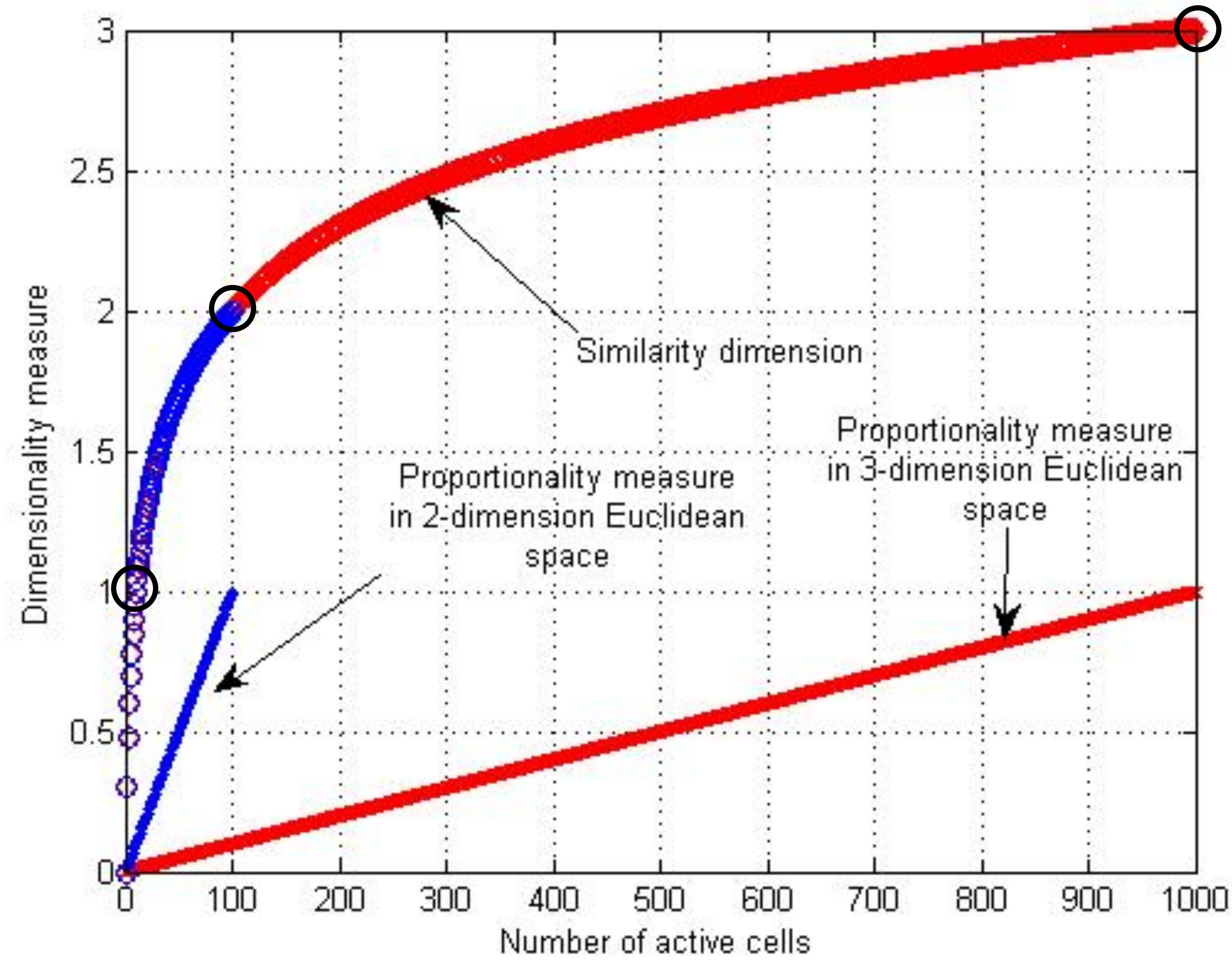
$$DIM = \frac{\ln m^n}{\ln m} = \frac{n \ln m}{\ln m} = n$$

If $H = m^k$ cells are active (for $1 \le k \le n$ integer values), then $DIM = k$.



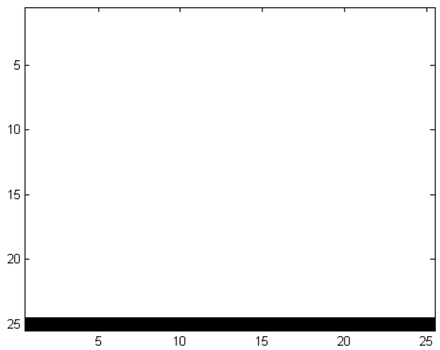
$m = H = 25$, $DIM = 1$

So fraction dimensionality gives integer dimensionality as special case for saturated (sub)spaces. A ‘simpler’ percentage measure of patchiness would not have this property.

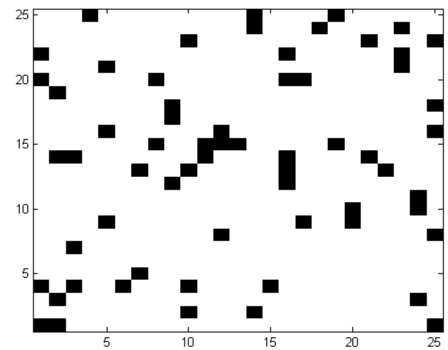


Dimensionality measure
Number of active cells
Similarity dimension
Proportionality measure
in 2-dimension Euclidean
space
Proportionality measure
in 3-dimension Euclidean
space
0
0.5
1
1.5
2
2.5
3
0
100
200
300
400
500
600
700
800
900
1000

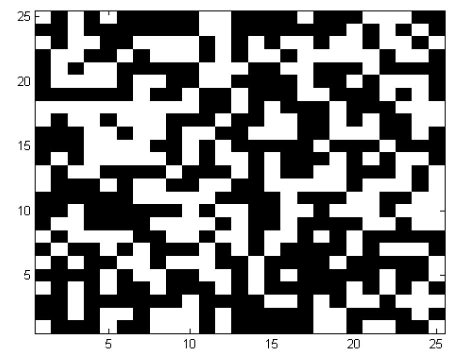
Examples (n = 2, m = 25)



m=25, *DIM*= 1



m=25, *DIM*= 1.29

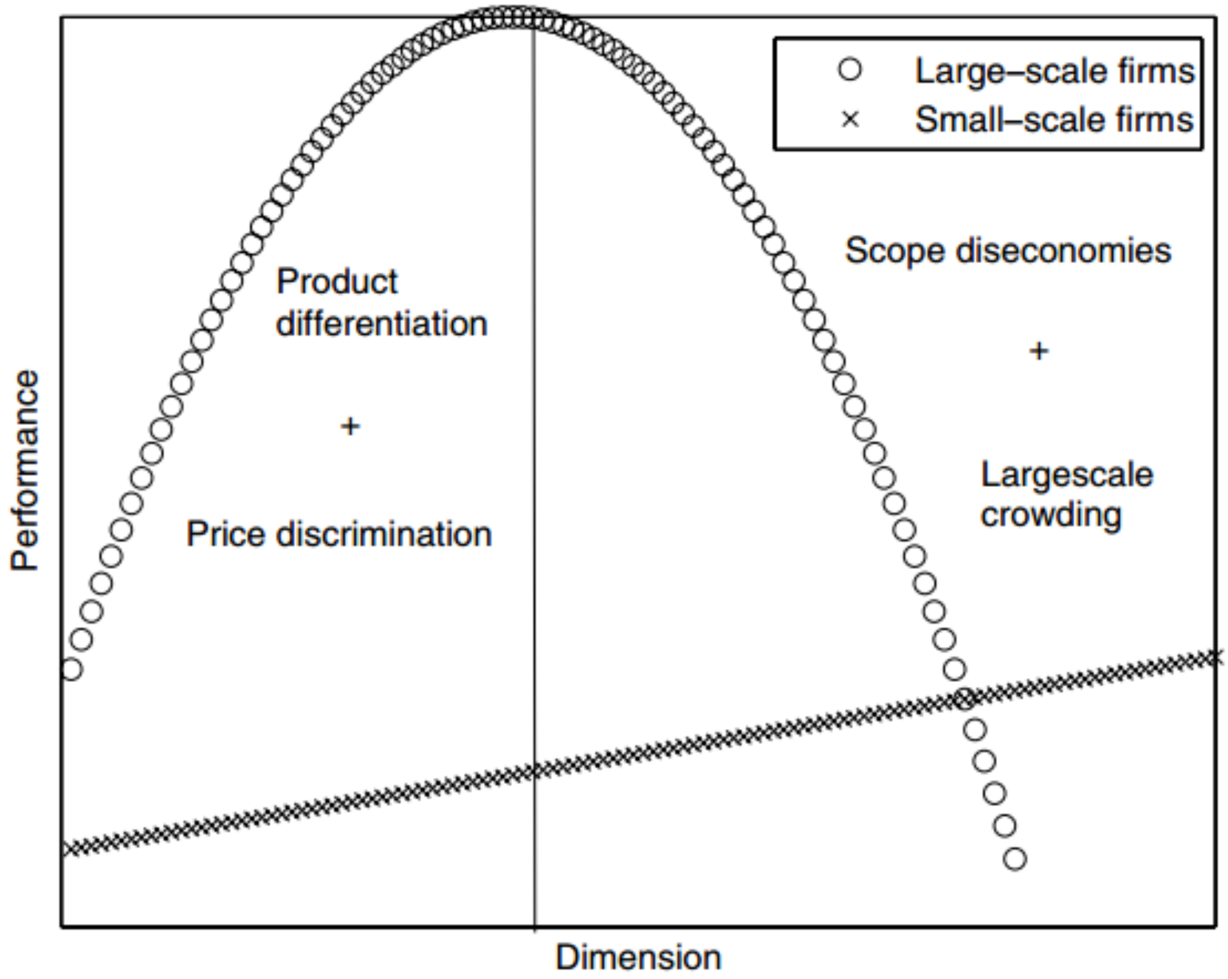


m=25, *DIM*= 1.81

INSIGHTS FROM A SIMULATION MODEL*

- A 2D frame space of 25 x 25 cells.
- Each cell can be activated.
- Once as cell is activated, it stays activated (so no once extant product variant gets ‘forgotten’).
- As before, two types of firms: *L* and *S*

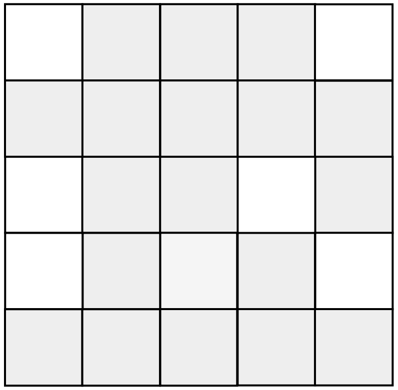
INSIGHTS FROM A SIMULATION MODEL*



Question

- How can we attach ‘meaning’ to the cells?
- What does it mean for a cell being activated?

- The *maximal space* is composed of those cells of the frame space that *may* stand for some known products.

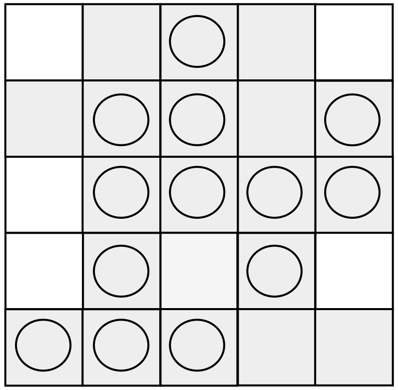


A cell belongs to the maximal space iff:

- It has positive *intrinsic* appeal for some audience members (a precondition for demand);
- No constraint bans an offering of the given feature combination (supply side).

Examples for excluded products:

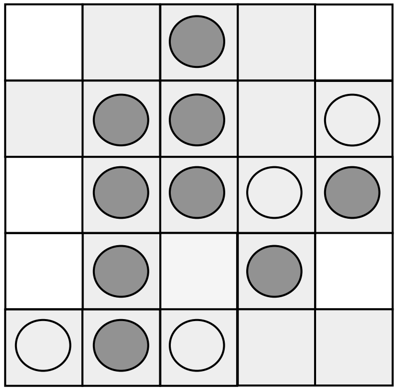
- Alcoholic drinks during the US Prohibition Period (institutional constraint);
- Extra strong jet engine with small size (technical constraint).



- Positions of the maximal space representing products known, meaningful for the audience constitute the *opened space*.

- Positions in the opened space were <u>once</u> activated by *engagement*.
    - So these positions then have gained positive *actual appeal* for some audience segments;
    - And these positions have become also known for the rest of the audience as holders of meaningful products.

- Positions of the opened space with positive demand (*actual appeal*) at least for some audience segments constitute the *active space.*



- *Actual appeal* requires *intrinsic appeal* and *engagement*.
  - Cells with demand are the active ones (dark circles);
  - Other cells of the opened space are passive (empty circles).

Impacts of n-space geometry

“How do the ‘stage’ influences the content of the ‘screenplay’ played on this stage?

How does the dimensionality of the phase space shapes the rules of interactions taking place in this place?

The *phase space* of physics/biology in social sciences:

Socio-demographic space (Blau-space) in sociology spun by social descriptors.

Commodity-space in economics (Lancaster, 1966) spun by *n* product characteristics.

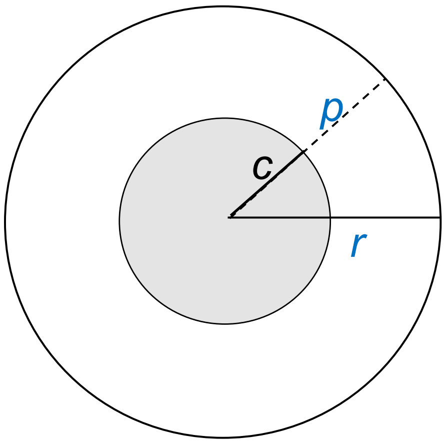
Political issue-space in political science (Downs, 1957) spun by positions on *n* political issues.

The focal *scarce resource* for organizations is people's demand for organizational services.

- Customers' purchasing power for firms;
- Demand for organizational services or membership for non-profit organizations;
- Electors' votes for political parties.

So, customers, members, clients, voters, *etc.*, can be seen as *resource carriers* for organizations.

Center - periphery structural changes with dimension change

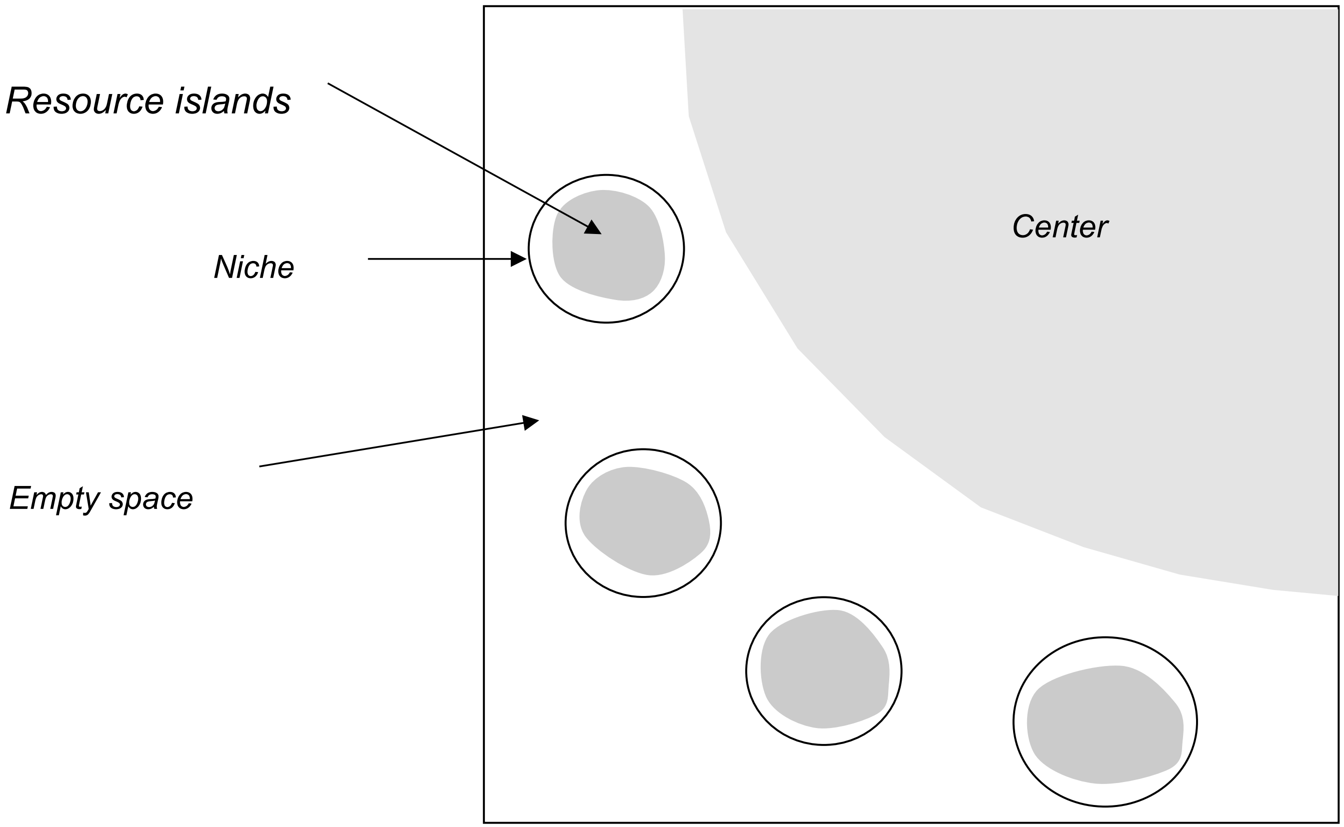


Market center (c) and periphery (p) $\boldsymbol{r = c + p}$

The ratio of the periphery to the volume of the entire market (*PVR*) converges to 1.

So the 'peel' of the *multi*-dimensional apple dominates over the 'flesh'.

$$PVR = \frac{\gamma_n \left(r^n - c^n\right)}{\gamma_n r^n} = 1 - \left(\frac{c}{r}\right)^n$$



A potential consequence
Bruggeman and Péli (2015)
Int. J. Modern Phys. C

THREE NICHE ASPECTS ADDRESSED:

(i) *Niche shape*. Rectangles or *n*-spheres?

(ii) *Optimal span*. Trade-off between breadth (cohesion) and affiliation chance.

(iii) *Niche positioning*. Overlap minimalization. Dense *vs*. simply building up arrangements.

Which Symmetry?

Two symmetric shapes in the ecological and sociological literature on niche:

- *n*-cubes (Levins 1968; McPherson et al.)

- *n*-spheres (Carroll 1985, Péli & Nooteboom 1999)

Proposition borrowed from marketing:

The shape depends on the pattern how people perceive (social, political, taste, etc.) distance.

The niche around ego is an *N*-cube

People evaluate social distance along each space dimension separately.

Perceived distance depends on the maximum misfit.

A corresponding association measure between nodes *a* and *b*, Freeman (1983):

$$A_{ab} = \max_{i = 1 \ldots n} | a_i - b_i |$$

Association is possible $A_{ab}. \leq \delta$

When misfits along dimensions do not add up in affiliates' evaluation, then niches are *n*-cubes.

NICHES AROUND FOCAL AGENTS ARE *N*-SPHERES

PERCEIVED COHESION DEPENDS ON THE OVERALL MATCH BETWEEN SOCIAL POSITIONS.

MISFITS ALONG DIMENSIONS "ADD UP" IN PEOPLE'S EVALUATION.

A corresponding association measure between nodes ***a*** and ***b*** is Euclidean distance:

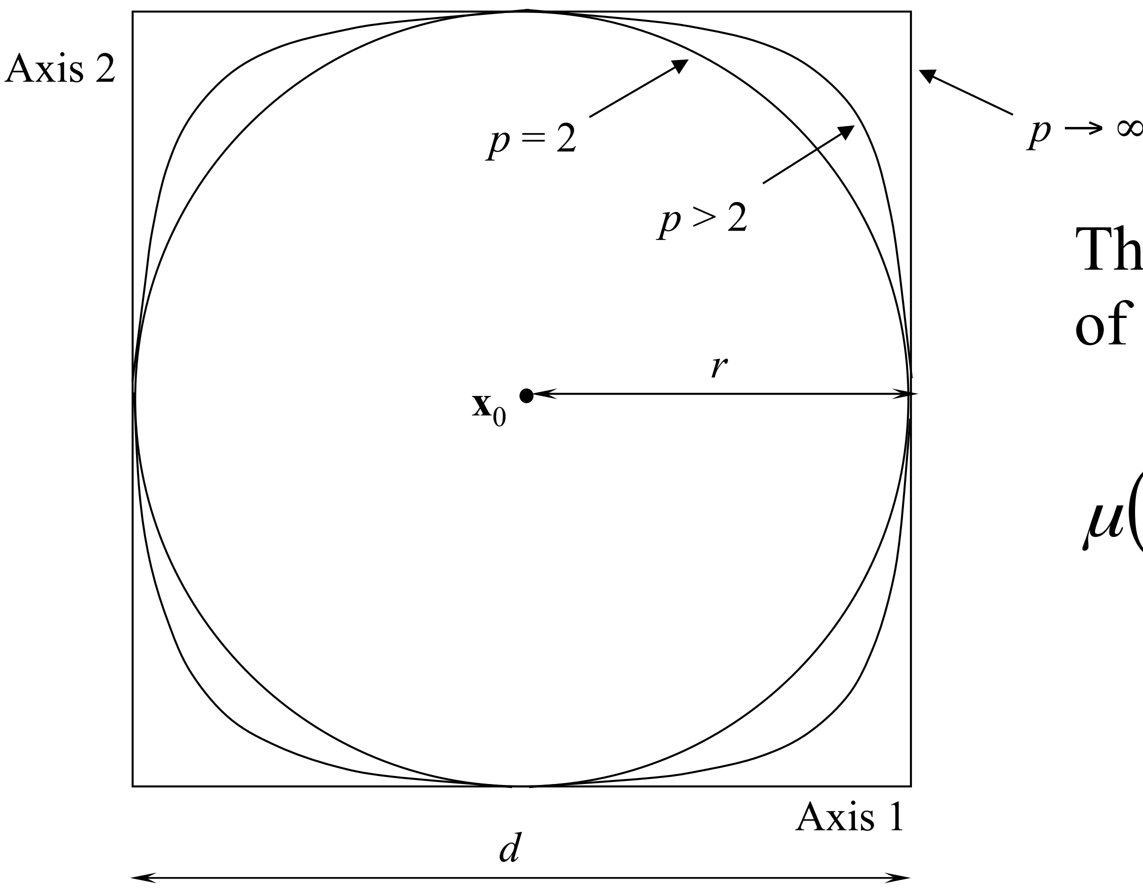
$$\sqrt{\sum_{i=1}^{n}(a_i - b_i)^2}$$

When misfits along social aspects do add up in affiliates' evaluation, then niches can be *n*-spheres …

… or something 'in between' spheres and cubes

Isosimilarity contours with three selected Minkowski-metrics. Two-dimensional visualization.

For all points **xj** along the contours: $\mu(\mathbf{x}_0, \mathbf{x}_j) = \text{constant.}$



The Minkowski-distance of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mu(\mathbf{x}_1, \mathbf{x}_2) = \sqrt[p]{\sum_{i=1}^{n} \left|x_{1i} - x_{2i}\right|^{p}}$$

(II) Optimal niche span

Consider the following *trade-off* between niche breadth and chance of selling :

The larger the niche breadth, lower the chance of buying from a certain taste group within the niche (keeping the price constant).

Is there a profit maximizing niche span?

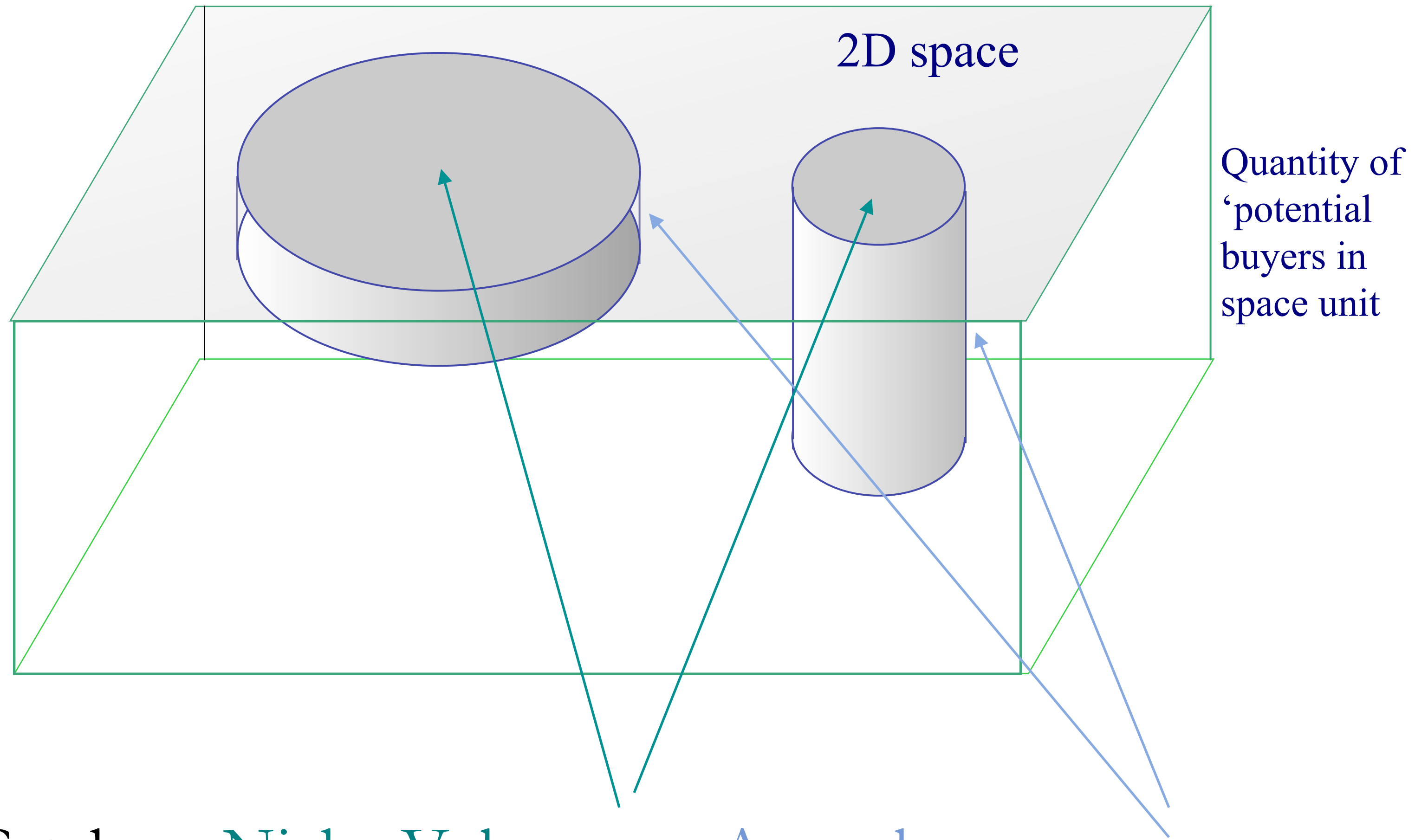
It depends on:

(1) How the selling probability diminish with niche diameter: On the shape of the 'appeal function $A(r)$;

(2) Space dimension n.

Péli, G. & A. van Witteloostuijn. 2009. "Optimal Monopoly Market Area Spanning in Multidimensional Commodity Spaces." *Managerial and Decision Economics* 30(1): 1-14.

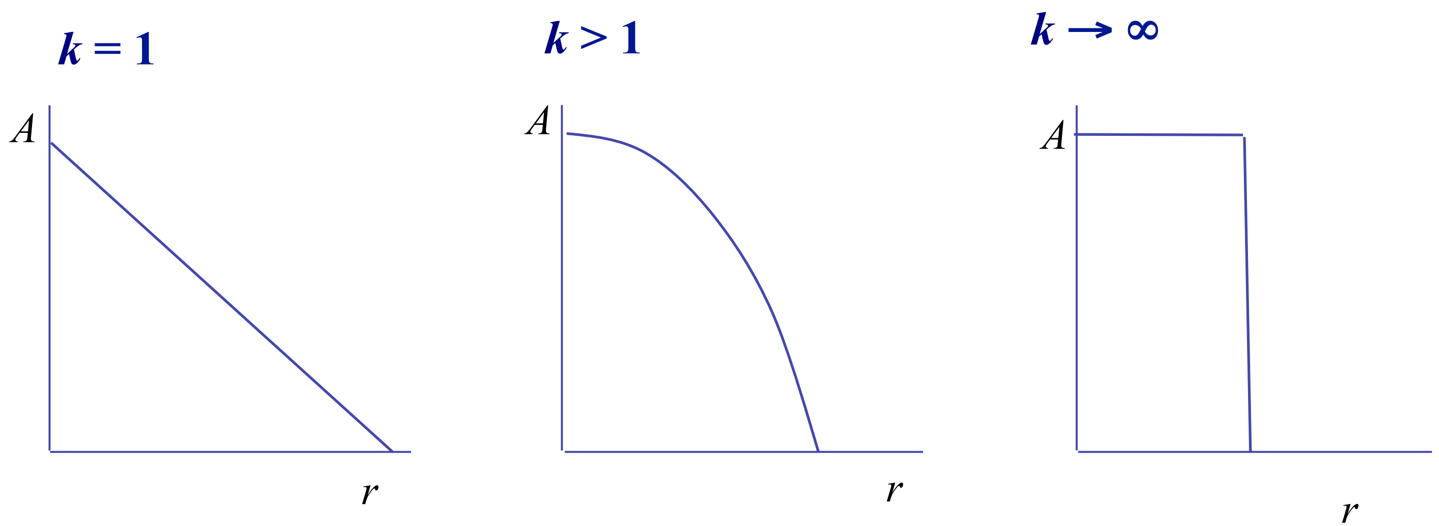
In-Breadth *vs.* In-Depth Niche Utilization



Catch = Niche Volume × Appeal

APPEAL FUNCTIONS

Power law: $A(r) = 1 - ar^k$ (*r* stands for niche radius)



Optimal radius: $$R_0 = \sqrt[k]{\frac{n}{a \cdot (n+k)}}$$

Niche Radius Optimum Growth with n (%).

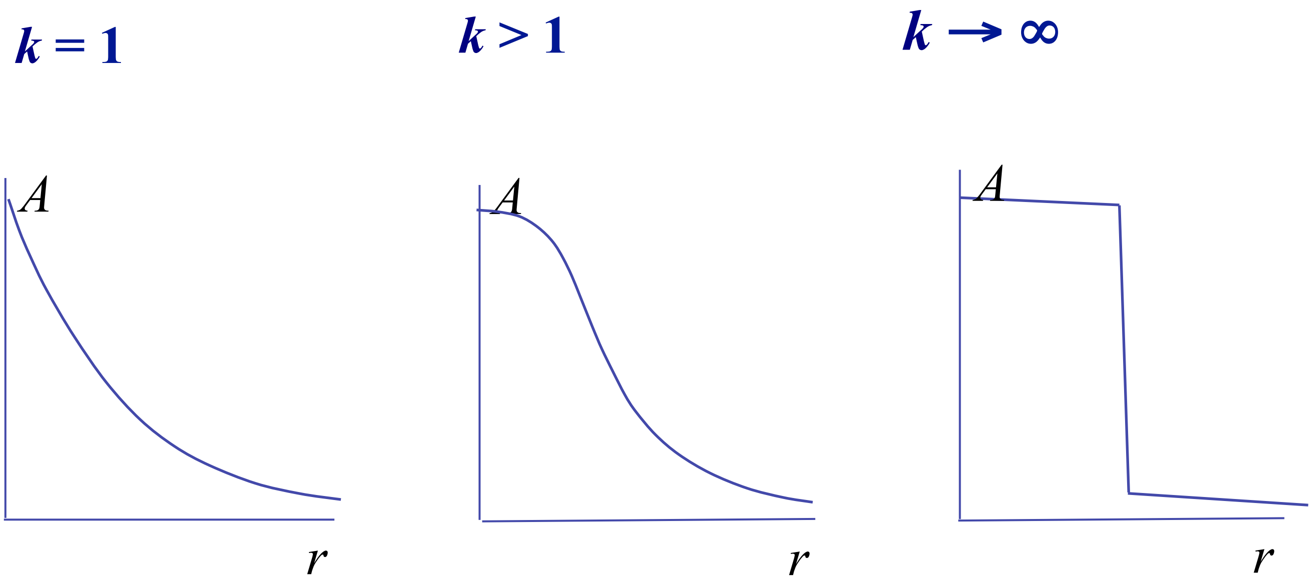
Power Law Appeal Functions

$n \rightarrow n+1$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$1 \rightarrow 2$	33.3	22.5	17.0	13.6
$2 \rightarrow 3$	12.5	9.5	7.7	6.5
$3 \rightarrow 4$	6.7	5.4	4.6	3.9
$4 \rightarrow 5$	4.2	3.5	3.0	2.7

General finding:

The optimal niche span tends to increase with n.

EXPONENTIAL TYPE $A(R)$ WITH ASYMPTOTIC DECREASE



$$A(r) = e^{-ar^k}$$

Optimal radius: $R_0 = \sqrt[k]{\frac{n}{ak}}$

No upper bound for the optimum.

Another asymptotic $A(R)$ class

$$A(r) = \frac{1}{a \cdot r^k + 1}$$

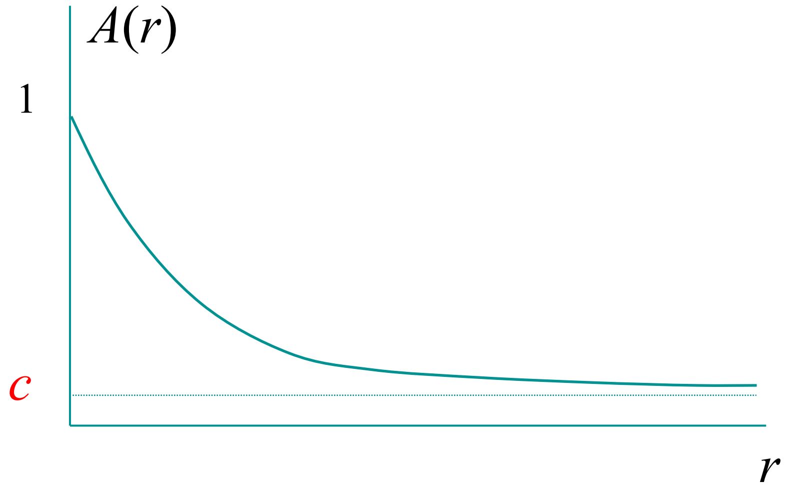
Optimal radius: $R_0 = \sqrt[k]{\frac{n}{a(k-n)}}$

Similar function graphs to the exponential type.

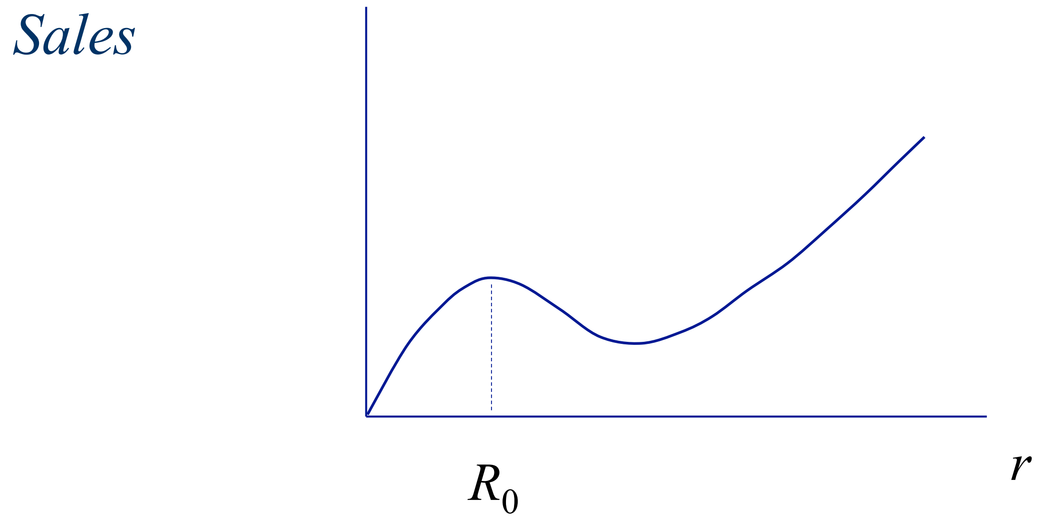
BUT

The optimum (R_0) increases until $n = k$;
then it disappears.
From then: the broader the niche is the better.

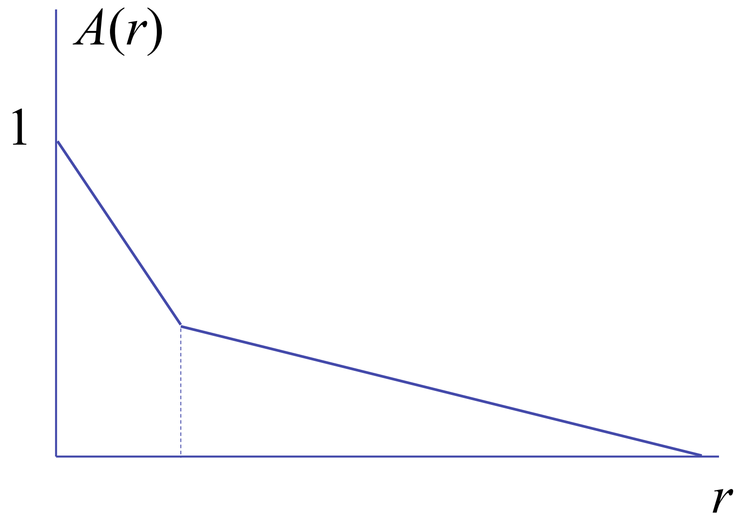
MORE THAN ONE "GOOD" SPAN

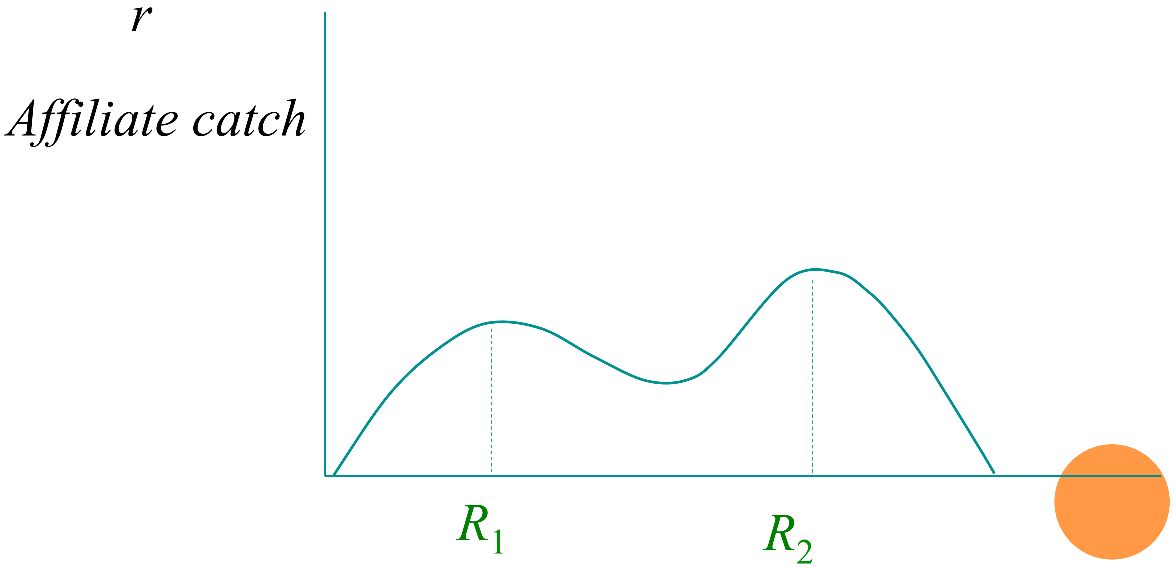


A fixed residual appeal at any niche span



Multiple maxima $\Rightarrow$ Dual niche structure





(iii) Niche positioning

Niche overlap (competition) evasion is the goal.

Dense niche arrangements without overlap

Cubic niches can always fill up the *n*-space without an overlap.

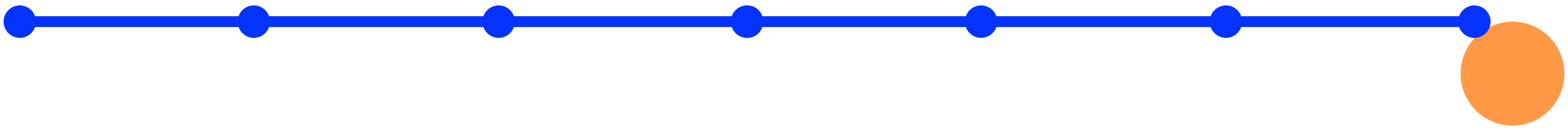
What about spherical niches?

Sphere packing problem in geometry:

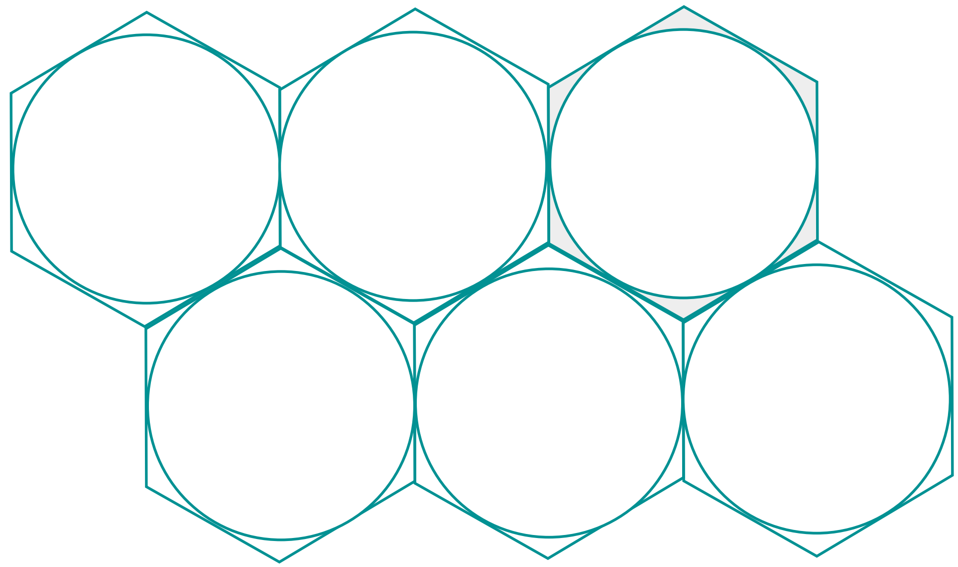
How can you pack up densely the *n*-space with spheres of equal radius without an overlap?

Packing density (Δ):
The ratio of the space occupied by the spheres to total space.

In one dimension the niches are line segments.
$\Delta = 1$ at the best arrangement:



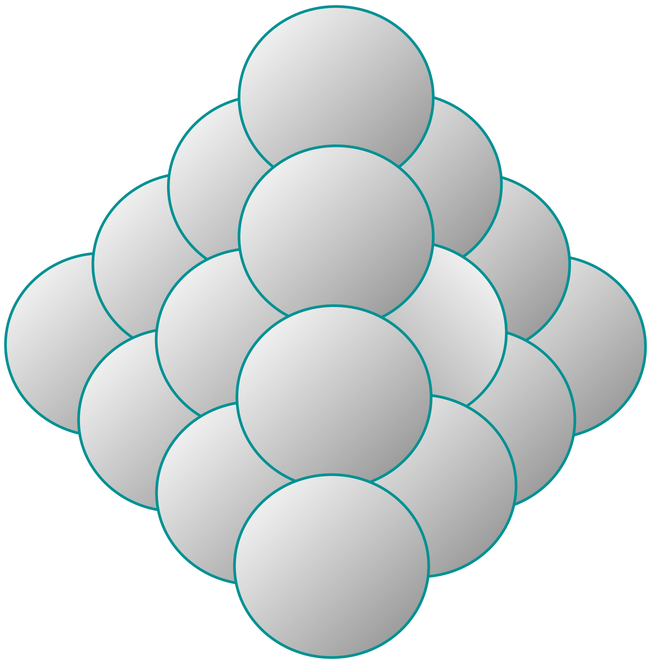
2-D space: Max. packing density is $\Delta = 0{,}91$ (hexagonal packing)



That is, about 9% of the space is empty around spheres.

3-D space: Max. packing density is Δ = 0,74

'Cannon-ball packing'



About 26% of the space is empty around spheres.

An what about *n* dimensions?

Upper bounds are known for the best packing.

- ♦ Packing density rapidly converges to zero with *n*.
- ♦ The number of immediate neighbors (*kissing number*, τ) goes to infinity with *n*.

Dim. n	Known thinnest covering Θ	Known densest packing Δ	Square lattice packing $\mathbf{Z}^n$ Δ	Kissing number, densest packing τ	Square lattice packing $\mathbf{Z}^n$ $\tau = 2n$
1	1	1	1	2	2
2	1.2092	0.90690	0.78540	6	4
3	1.4635	0.74048	0.52360	12	6
4	1.7655	0.61685	0.30843	24	8
5	2.1243	0.46526	0.16449	40	10
6	2.5511	0.37295	0.08075	72	12
7	3.0596	0.29530	0.03691	126	14
8	3.6658	0.25367	0.01585	240	16
9	4.3889	0.14577	0.00644	272	18
10	5.2517	0.09962	0.00249	372	20
11	6.2813	0.06624	0.00092	519.78	22
12	7.5101	0.04945	0.00033	756	24

DEEP HOLES, DEEP PROBLEMS

The *deep hole* is the point with the largest distance from the surrounding sphere centers (Conway and Sloane 1998).

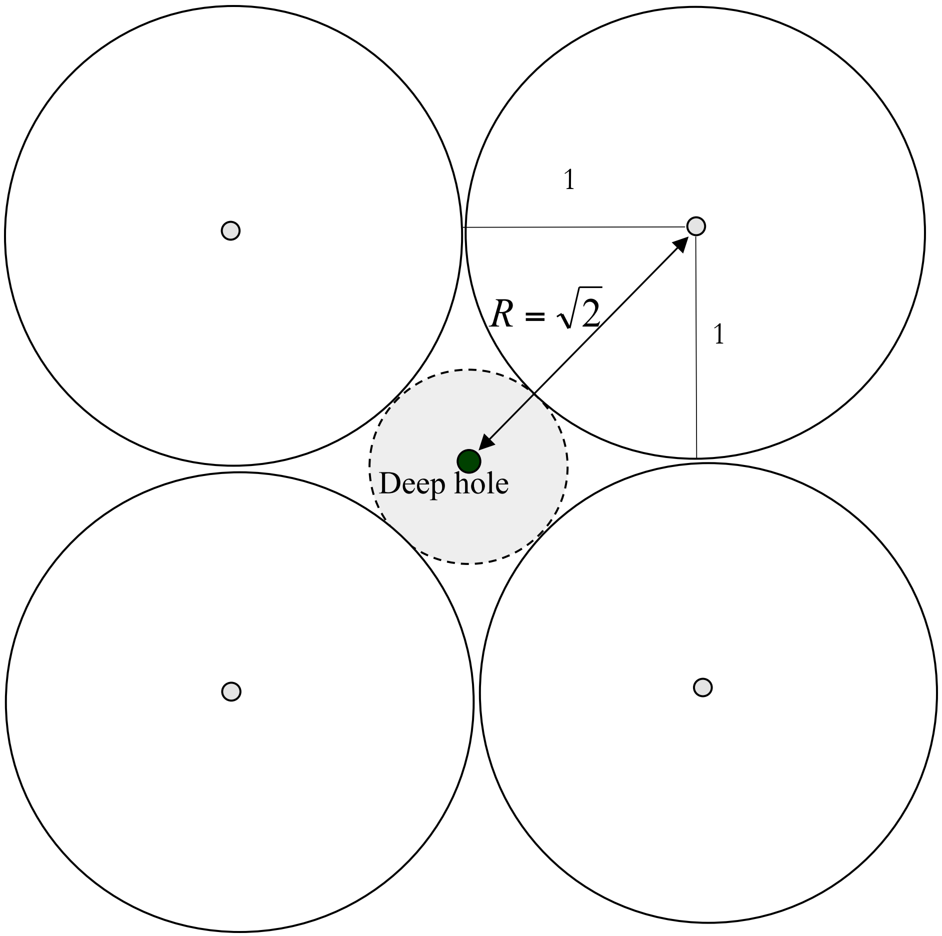
Really deep holes invite entry of potential competitors.

Deep holes (increasing in depth with dimension *n*) may serve as competition free 'safe havens' for small niche specialist organizations once the big niche firms have distributed their markets.

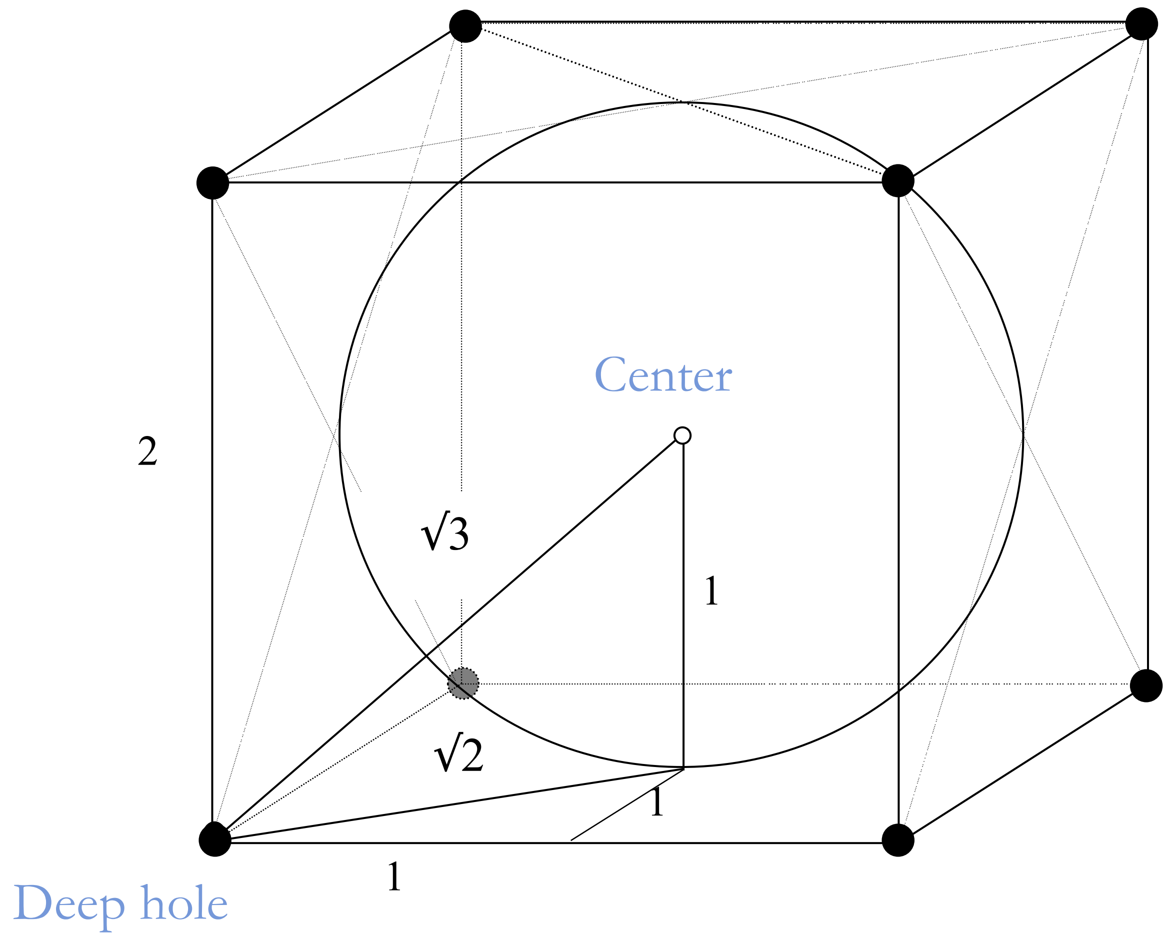
Péli, G. and B. Nooteboom. 1999. "Market Partitioning and the Geometry of the Resource Space." *American Journal of Sociology*, 104: 1132-1153.

Really deep holes invite entry of potential competitors.

Square-lattice packing – simple, but 'loose'.


1
$R = \sqrt{2}$
1
Deep hole

Deep holes in Z^N:



Deep hole distance in square lattice $\mathbf{Z}^n$:

$$R_{\mathbf{c}}^{(n)} = r\sqrt{n}$$

goes to infinity with *n!*

In four dimensions:

$r\sqrt{4} = 2r$

That is, the same large spherical niche can be inserted into a complete niche packing without overlap!

Packing density doubles.

If $n > 4$, the new niches can be even larger.

The square lattice based niche arrangement builds up easily, but its looseness makes it instable in higher dimensional spaces.