Universiteit Utrecht

Introduction to *Introduction to Complex Systems*

Jason Frank
Mathematical Institute

Agenda

- Introductions
- Complexity and Complex Systems Science: Hype and Hope
- The Course
- What is a Complex System?
- An Example
- Chaos and Lyapunov exponents

Introductions

- **Me:**
  - Jason Frank
  - Mathematical Institute, Numerical Analysis
  - NWO Complexity (program board), UU Focus Area (steering committee)
  - Hans Freudenthal Building (HFG 612), j.e.frank@uu.nl, Tues-Thurs. http://www.staff.science.uu.nl/~frank011/Classes/complexity_intro/
- **You:**
  - "What's your major?", year?
  - Specialties?
  - Why complex systems?

Nomenclature

- *“Complexity”, “Complex Systems Science”*
  - Mathematics: combinatorial complexity, P=NP? TSP
  - Mathematics: computational complexity, operations counts
  - Computer science: (information theoretical) Kolmogorov complexity, information content.

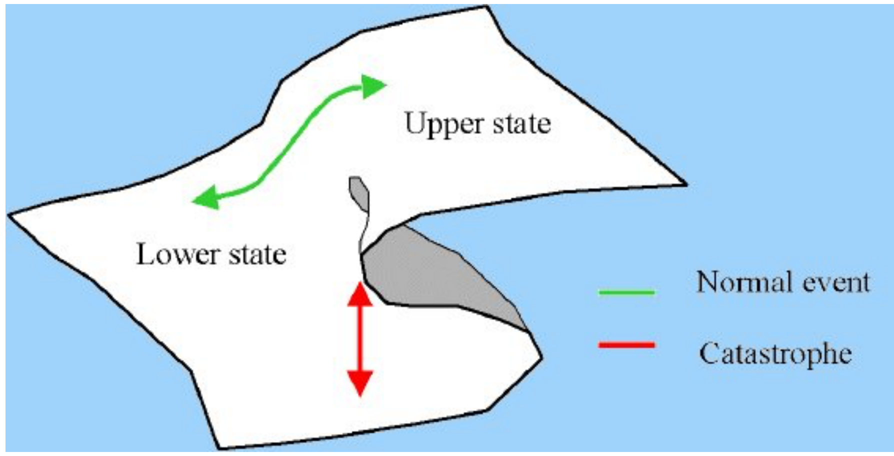
$$\begin{aligned}
&\text{Vector sum} = \mathcal{O}(N),\\
&\text{Matrix-vector product} = \mathcal{O}(N^2),\\
&\text{Matrix-inversion} = \mathcal{O}(N^3),\\
&\text{FFT} = \mathcal{O}(N \ln N)
\end{aligned}$$

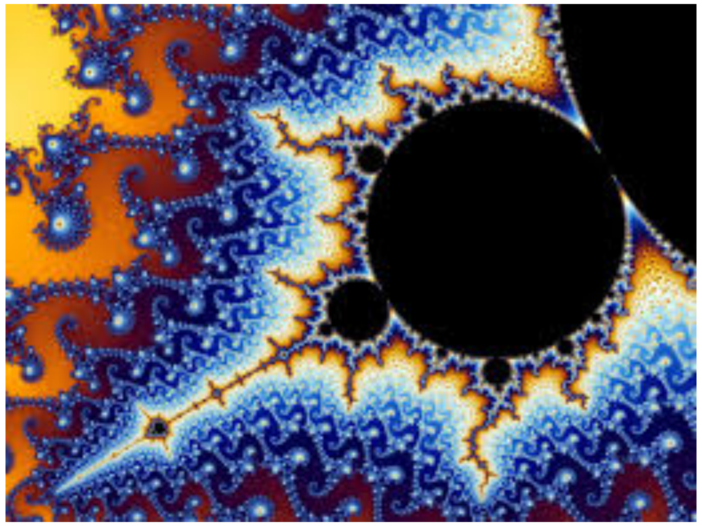
`abababababababababababababababab`

`4c1j5b2p0cv4w1x8rx2y39umgw5q85s7`

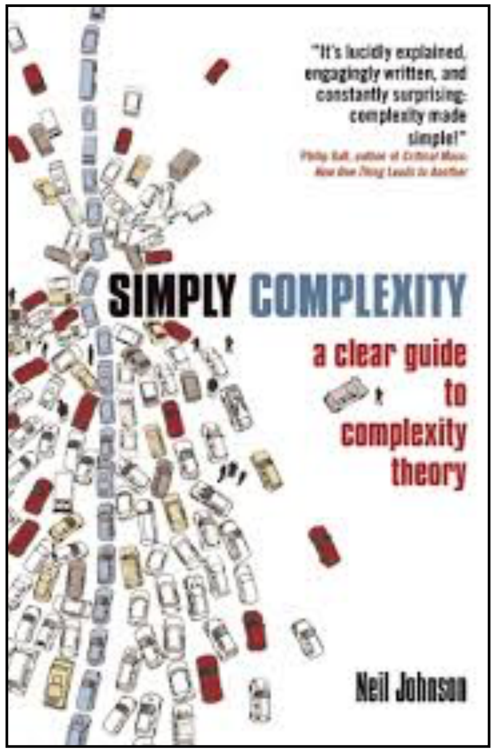
History

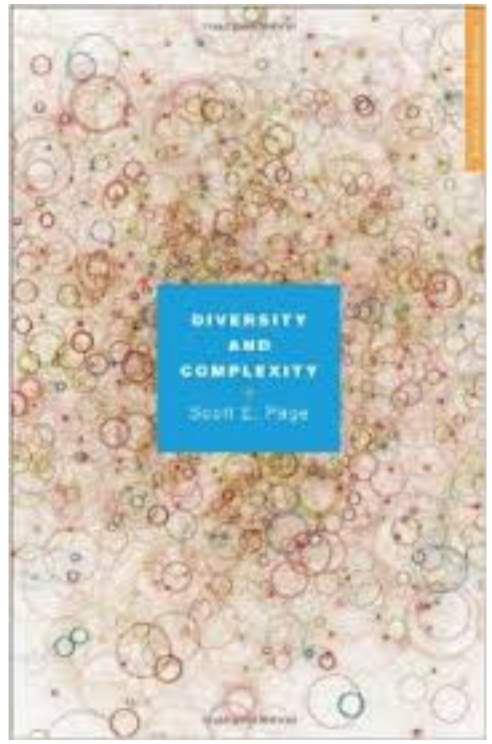
- Complex systems science has roots in:
  - Catastrophe theory (1960s)
  - Chaos theory (1980s)
  - Computational science (the third pillar)
  - many others

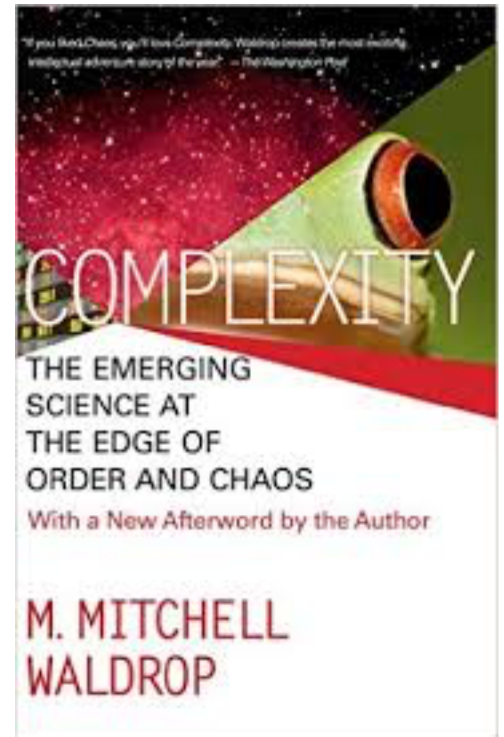


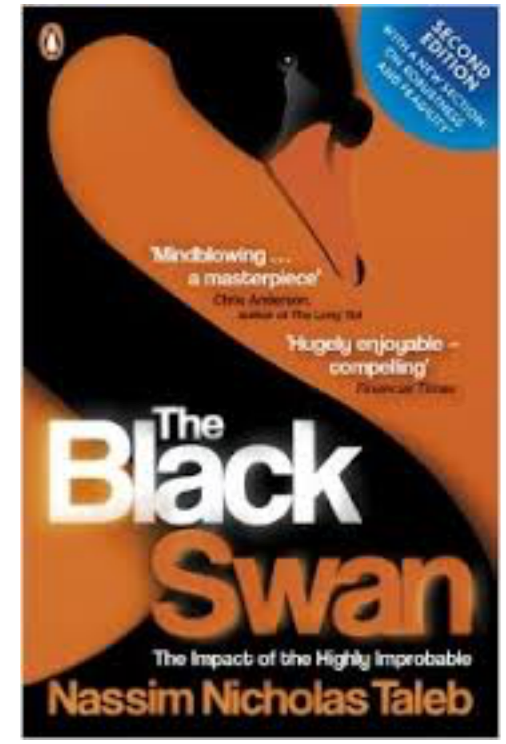


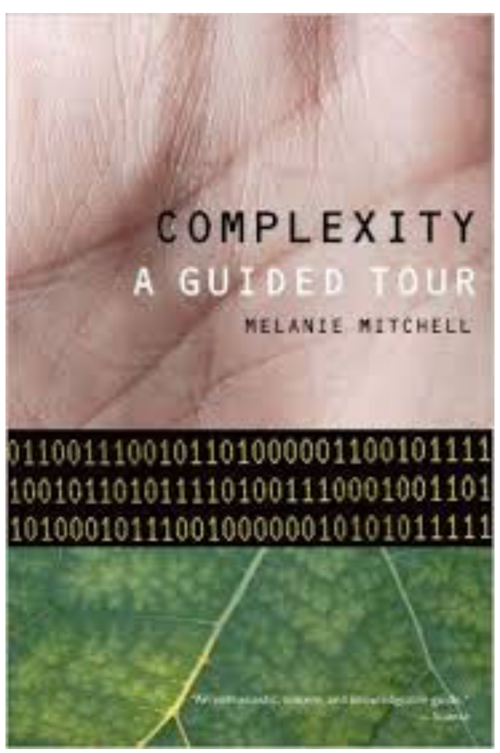
Introductory Books

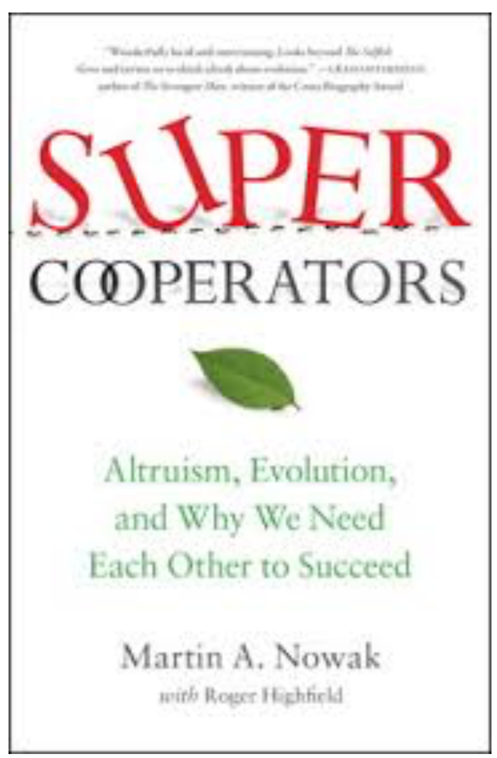


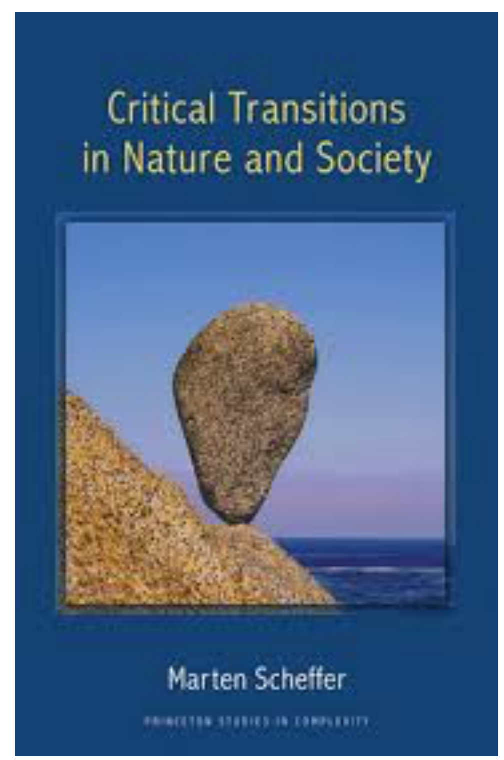


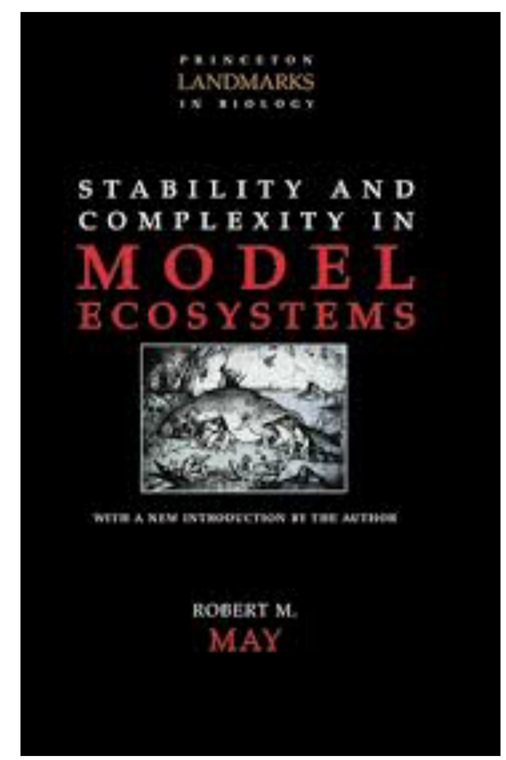








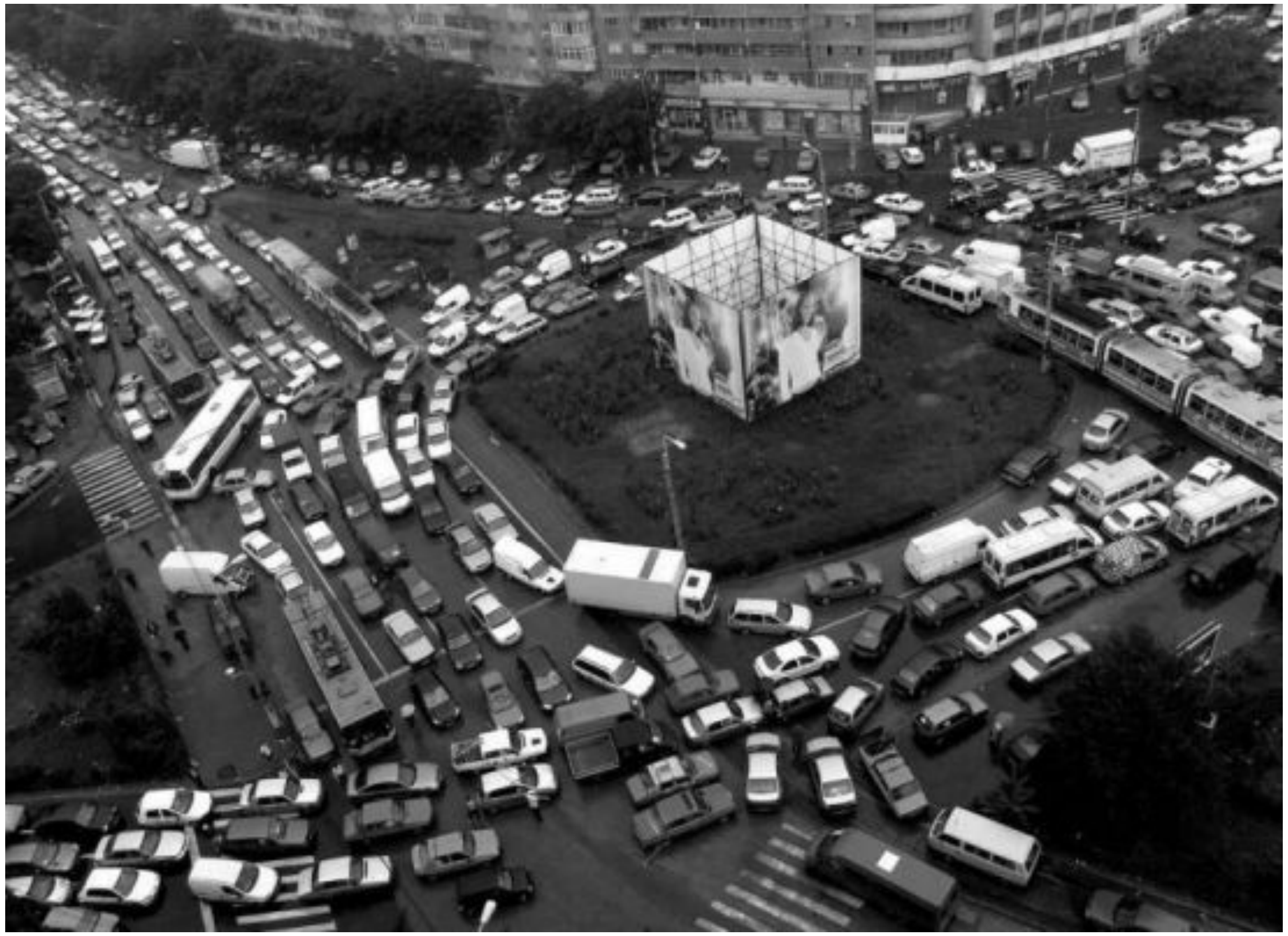




Examples of complex systems

We study complex systems to understand them, but ultimately we often need to be able to predict and control them…

Traffic
Transport
Networks



Examples of complex systems

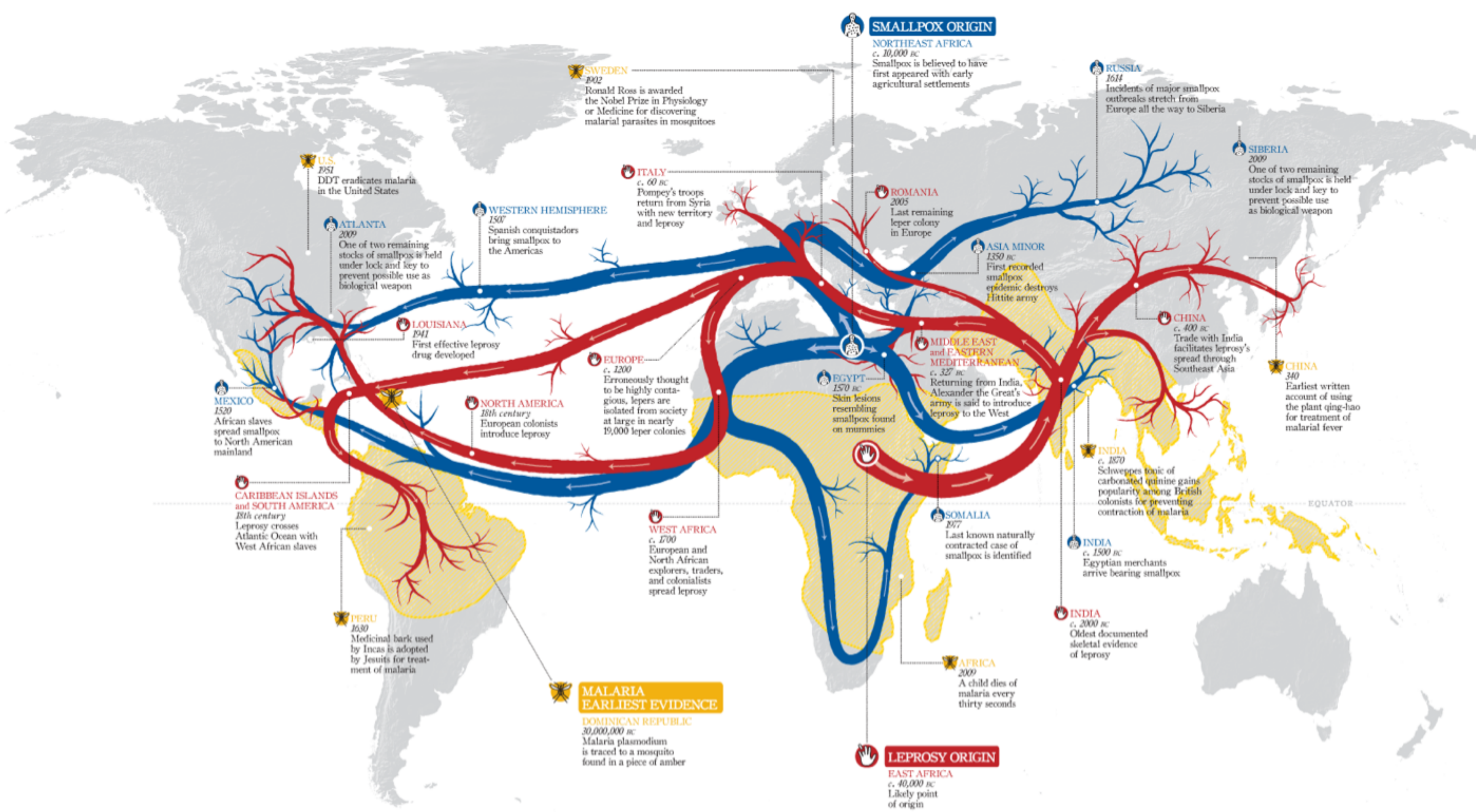
We study complex systems to **understand** them, but ultimately we often need to be able to **predict** and **control** them…

Trading and finance

Examples of complex systems

We study complex systems to understand them, but ultimately we often need to be able to predict and control them…

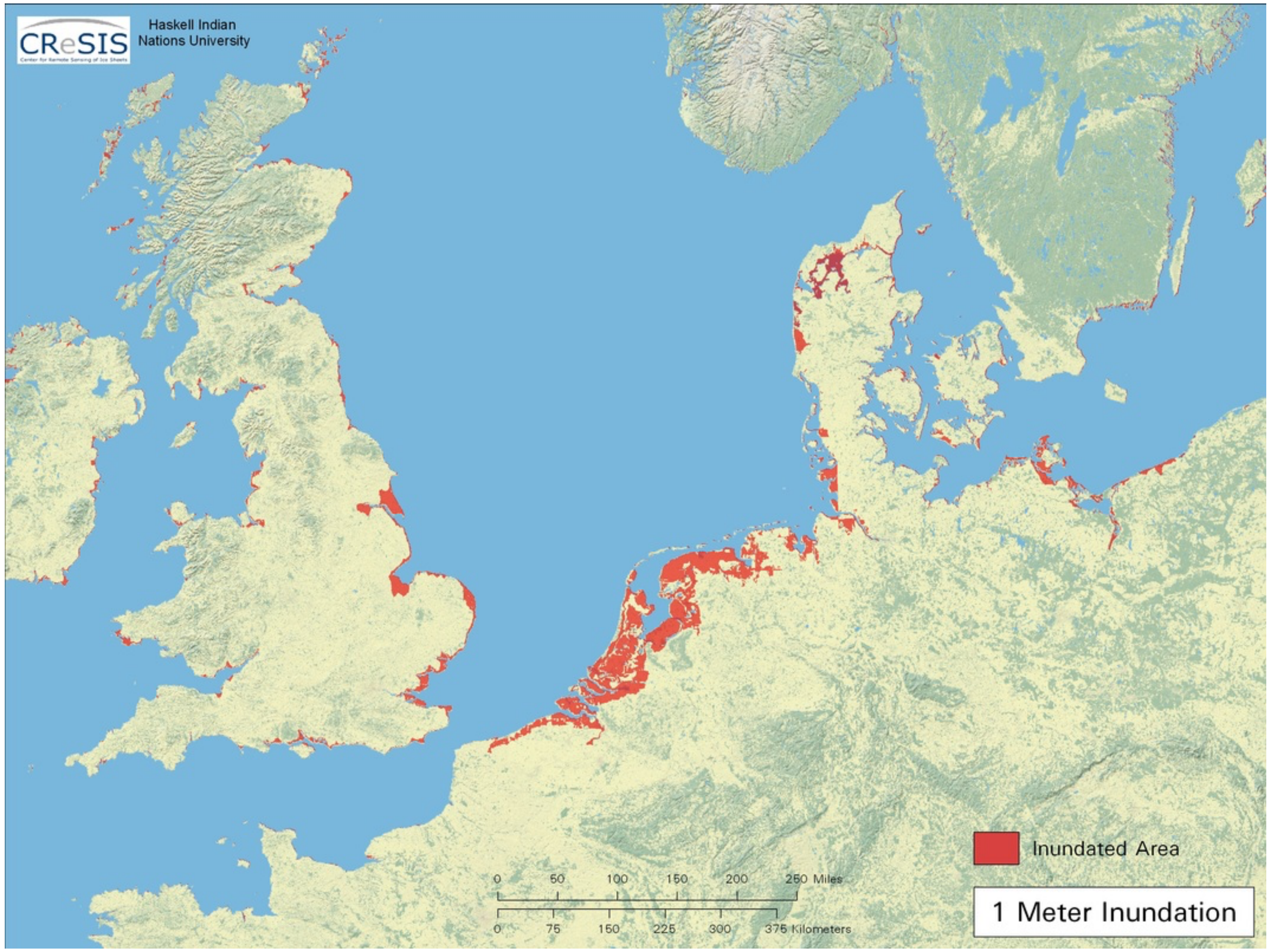
Epidemics



Examples of complex systems

We study complex systems to **understand** them, but ultimately we often need to be able to **predict** and **control** them…

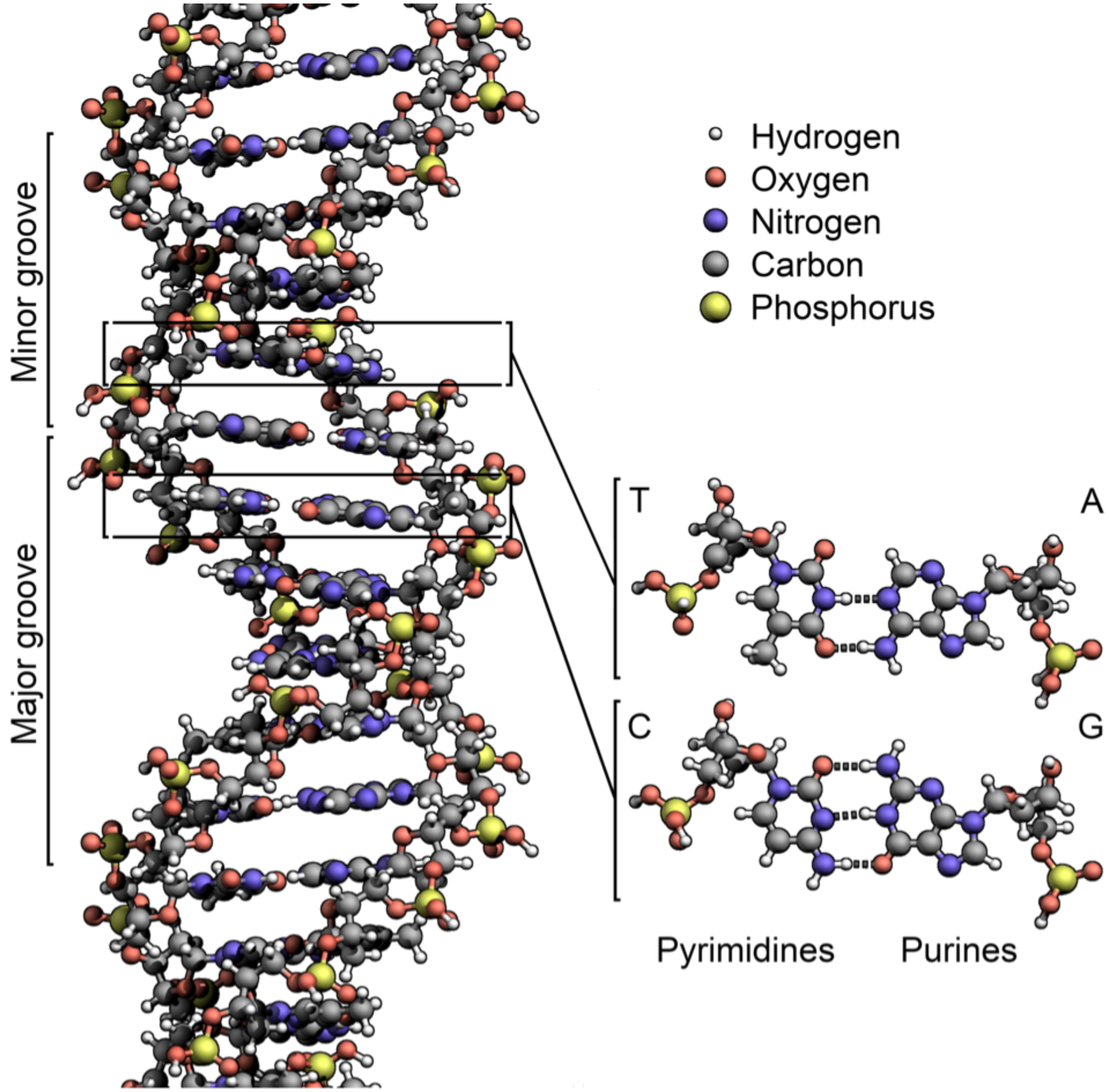
Climate and weather



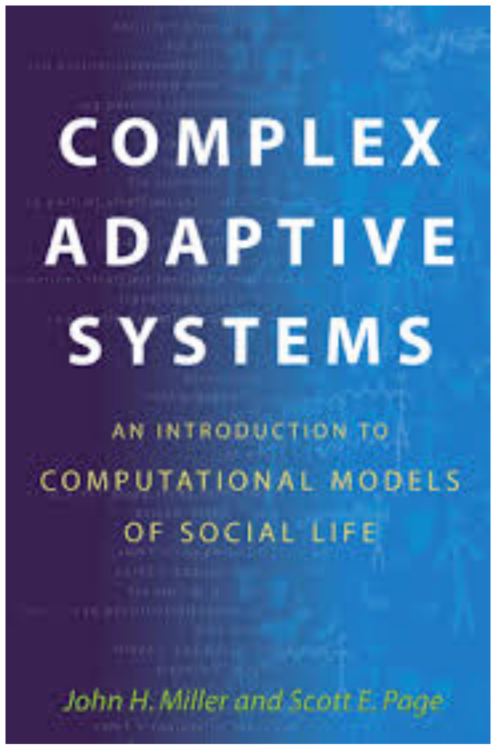
Examples of complex systems

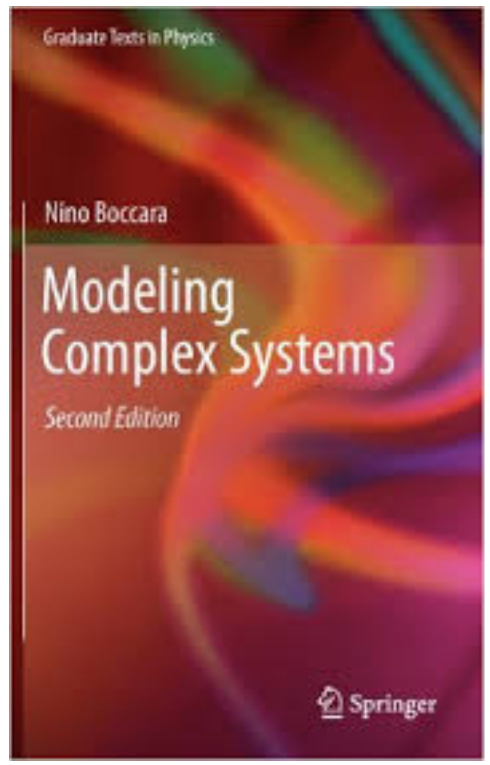
We study complex systems to understand them, but ultimately we often need to be able to predict and control them…

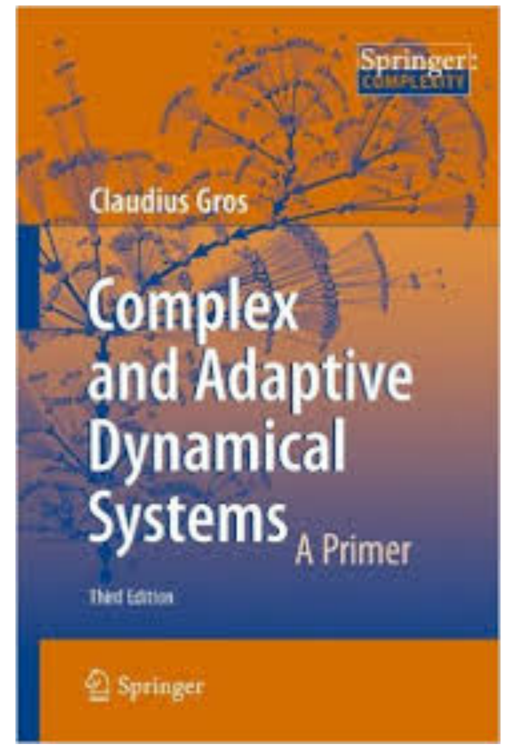
Biology
Ecology

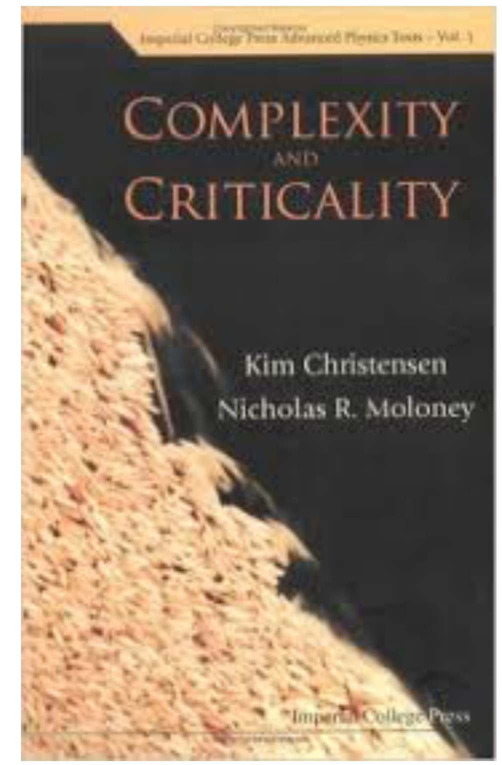


Course references:









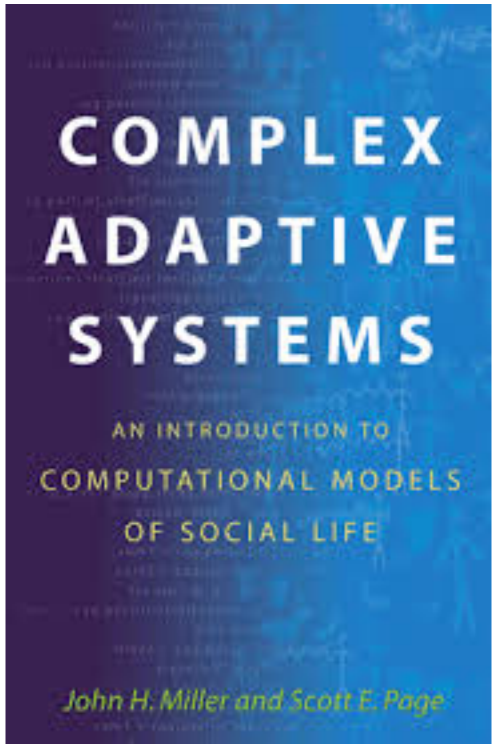
Course setup

- Weekly meeting: Tuesday at 13.15 - 15.00. The location changes - monitor Osiris.
- I will give lectures, largely focused on models, methods and techniques, some dynamics and some analysis.
- We will also have guest lectures from many different disciplines: *(if anything, Complexity science is multidisciplinary, we draw analogies and learn from each other)*
  - 27 October. **Guest lecture: Henk Stoof** (Physics)
  - 10 November. **Guest lecture: Vincent Buskens** (Sociology)
  - 17 November. **Guest lecture: Rob de Boer** (Theoretical Biology)
  - 24 November. **Guest lecture: Gábor Péli** (Economics)
  - 15 December. **Guest lecture: Henk Dijkstra** (Oceanography and climate)
  - 5 January. **Guest lecture: Koen Frenken** (Innovation studies)
- Scoring: class involvement (20%), two small projects (40%), one big project with oral exam (40%). You may work in groups of (up to) 3 persons.
- Software: Matlab, NetLogo, ??? *Code repository on the course homepage.*

What is a Complex System?

- Anything that can’t be understood completely?
- Anything nonlinear? unpredictable?
- There is no concensus.
- Any ideas?

What is a Complex System?

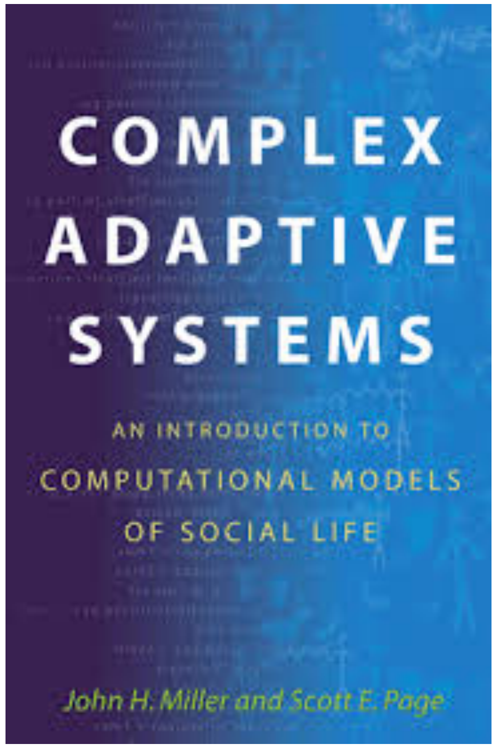


We are surrounded by complicated social worlds. These worlds are composed of multitudes of incommensurate elements, which often make them hard to navigate and, ultimately, difficult to understand. We would, however, like to make a distinction between complicated worlds and complex ones. In a complicated world, the various elements that make up the system maintain a degree of independence from one another. Thus, removing one such element (which reduces the level of complication) does not fundamentally alter the system's behavior apart from that which directly resulted from the piece that was removed. Complexity arises when the dependencies among the elements become important. In such a system, removing one such element destroys system behavior to an extent that goes well beyond what is embodied by the particular element that is removed.

Complexity is a deep property of a system, whereas complication is not. A complex system dies when an element is removed, but complicated ones continue to live on, albeit slightly compromised. Removing a seat from a car makes it less complicated; removing the timing belt makes it less complex (and useless). Complicated worlds are reducible, whereas complex ones are not.

While complex systems can be fragile, they can also exhibit an unusual degree of robustness to less radical changes in their component parts. The behavior of many complex systems emerges from the activities of lower-level components. Typically, this emergence is the result of a very powerful organizing force that can overcome a variety of changes to the lower-level components. In a garden, if we eliminate an insect the vacated niche will often be filled by another species and the ecosystem will continue to function; in a market, we can introduce new kinds of traders and remove old traders, yet the system typically maintains its ability to set sensible prices. Of course, if we are too extreme in such changes, say, by eliminating a keystone species in the garden or all but one seller in the market, then the system's behavior as we know it collapses.

What is a Complex System?

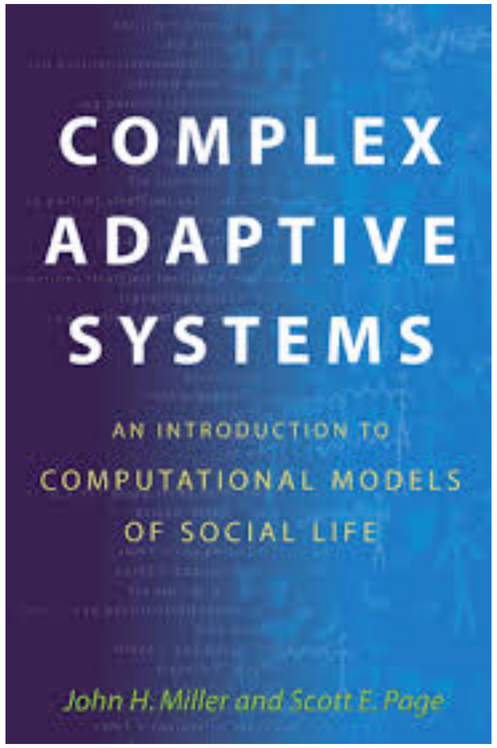


When a scientist faces a complicated world, traditional tools that rely on reducing the system to its atomic elements allow us to gain insight. Unfortunately, using these same tools to understand complex worlds fails, because it becomes impossible to reduce the system without killing it. The ability to collect and pin to a board all of the insects that live in the garden does little to lend insight into the ecosystem contained therein.

The innate features of many social systems tend to produce complexity. Social agents, whether they are bees or people or robots, find themselves enmeshed in a web of connections with one another and, through a variety of adaptive processes, they must successfully navigate through their world. Social agents interact with one another via connections. These connections can be relatively simple and stable, such as those that bind together a family, or complicated and ever changing, such as those that link traders in a marketplace. Social agents are also capable of change via thoughtful, but not necessarily brilliant, deliberations about the worlds they inhabit. Social agents must continually make choices, either by direct cognition or a reliance on stored (but not immutable) heuristics, about their actions. These themes of connections and change are ever present in all social worlds.

The remarkable thing about social worlds is how quickly such connections and change can lead to complexity. Social agents must predict and react to the actions and predictions of other agents. The various connections inherent in social systems exacerbate these actions as agents become closely coupled to one another. The result of such a system is that agent interactions become highly nonlinear, the system becomes difficult to decompose, and complexity ensues.

What is a Complex System?



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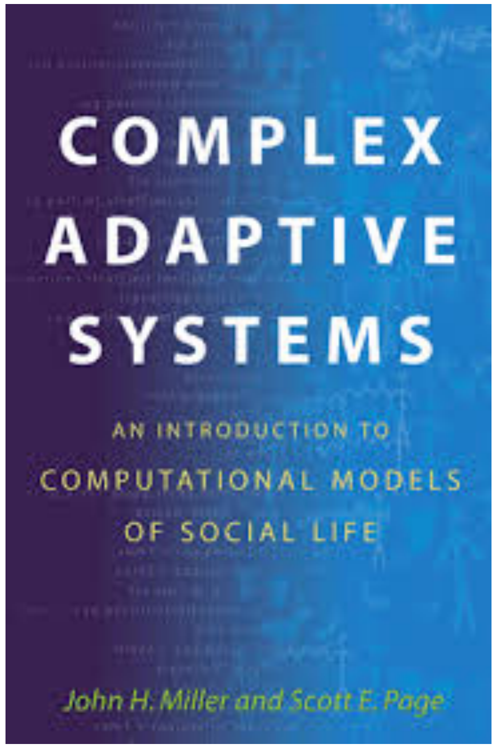

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What is a Complex System?

- Complexity is often juxtaposed with *reductionism*, the scientific approach of dissecting a problem and studying its parts.
- It is said that a complex system is one for which:
  - “the whole is greater than the sum of its parts”
  - a system whose behavior cannot be deduced from the behavior of its constituent elements - seems unlikely
  - more probably: we do not understand how the behavior of elements gives rise to the macroscopic behavior.
- We will roughly adopt the definition that a complex system is one composed of many parts, for which the behavior of interest is expressed at a system level, as opposed to an individual level.
- Clearly a multi-scale character, with emergent behavior (phase transitions)
- Adaptivity, optimization, criticality

Forest fire model (Miller & Page)

Lattice model (1D or 2D) of tree sites $i = 1, \ldots, N$.

Each site contains a tree ($X_i = 1$) or none ($X_i = 0$).

The production period consists of several phases:

1) Growth - a new tree grows at each site with probability g.
2) Fire - lightning strikes each tree with probability f. All trees at contiguous lattice sites burn.
3) Adaptation (optional) - each site adapts its growth rate, depending on its assessed risk.
4) Production - the surviving trees are counted as the forest production for the period.

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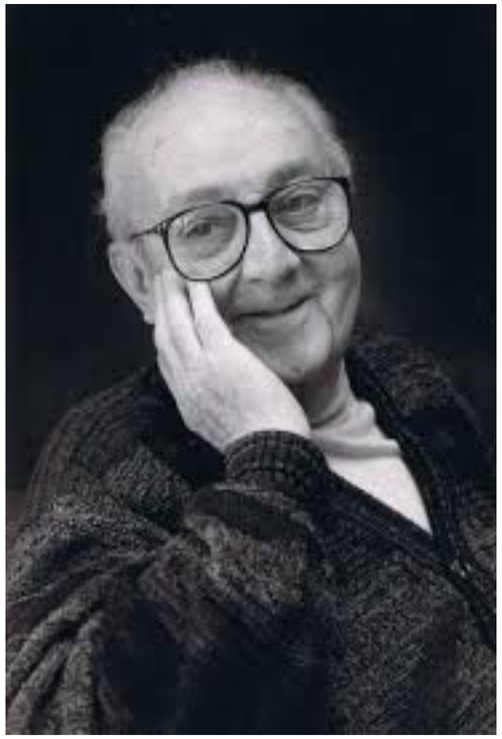
Conclusions from Forest Fire models:

- The models do not succumb easily to direct analysis (and this gets worse with increased complexity).
- *Do not confuse model with reality.*
- Models can be used to test assumptions and make statements like “a possible mechanism that could cause an observed phenomenon is…”
- Forest fire models have been used to model other phenomenon, like bank collapses.
- “All models are wrong but some are useful” - George Box



Dynamical systems

The basic components of most complex systems are dynamical systems.

Iterated maps:

$$\mathcal{D} \subset \mathbf{R}^N, \qquad F : \mathcal{D} \to \mathcal{D}$$

$$x_{n+1} = F(x_n), \qquad x_n \in \mathcal{D}$$

and *differential equations*:

$$\frac{dx}{dt} = f(x), \quad x(t) \in \mathcal{D}$$

Flow maps and *numerical methods*.

$$x(t+s) = \phi_t(x(s)) \qquad\qquad x_{n+1} = x_n + \Delta t f(x_n)$$

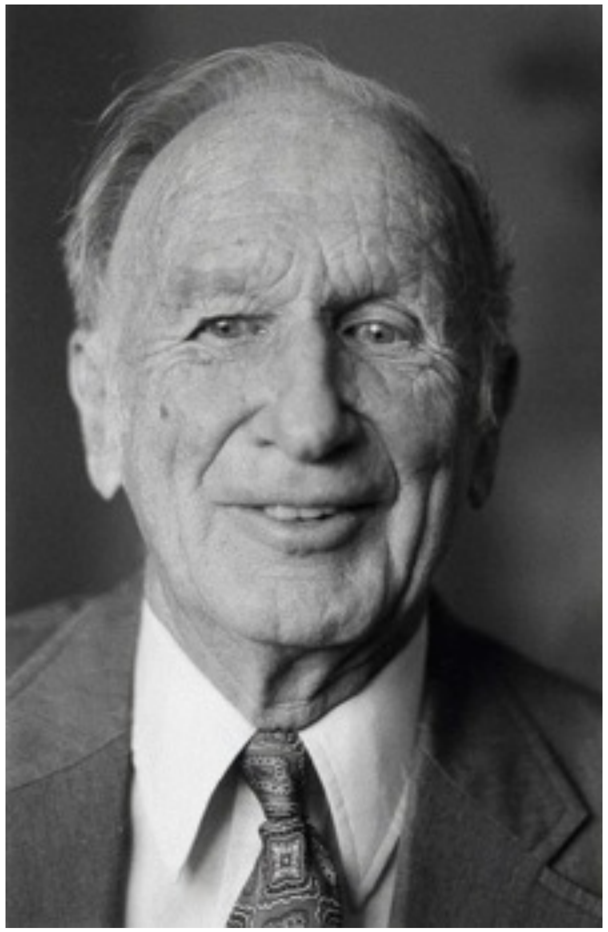
Dynamical systems

In general the dynamical systems may *depend explicitly on time* (nonautonomous), may *depend on random variables* (stochastic differential equations), and may be (highly) *nonsmooth or discontinuous*.

Example: see Forest fire models.

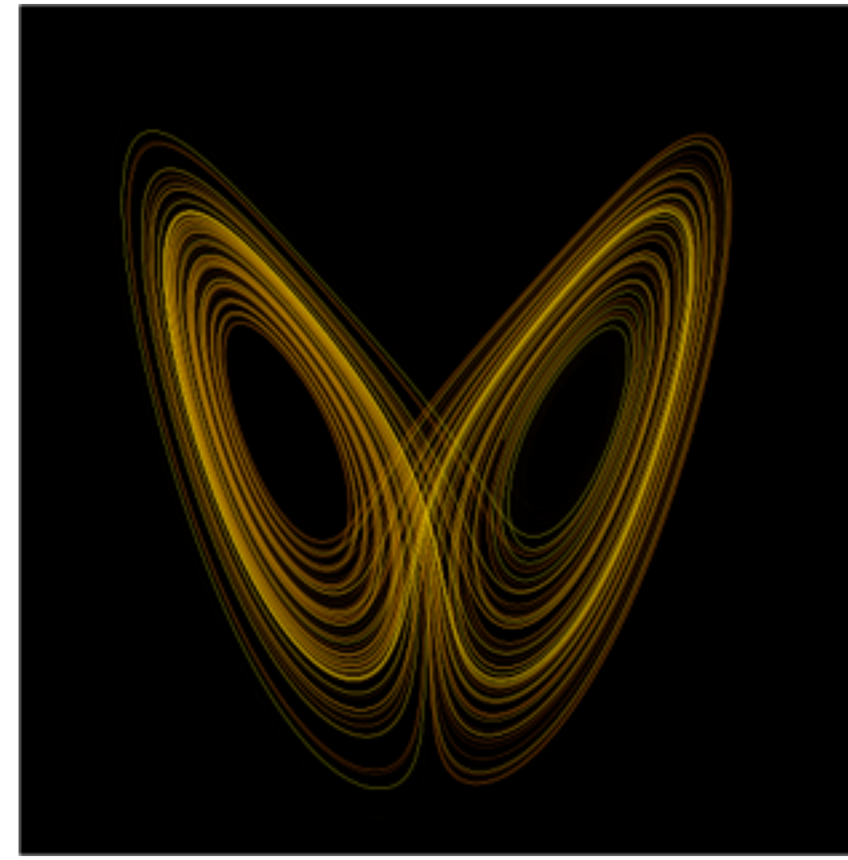
Typically they are nonlinear and chaotic.

Lorenz's 3-component model (1963)



E.N. Lorenz
1917 - 2008

Lorenz Attractor

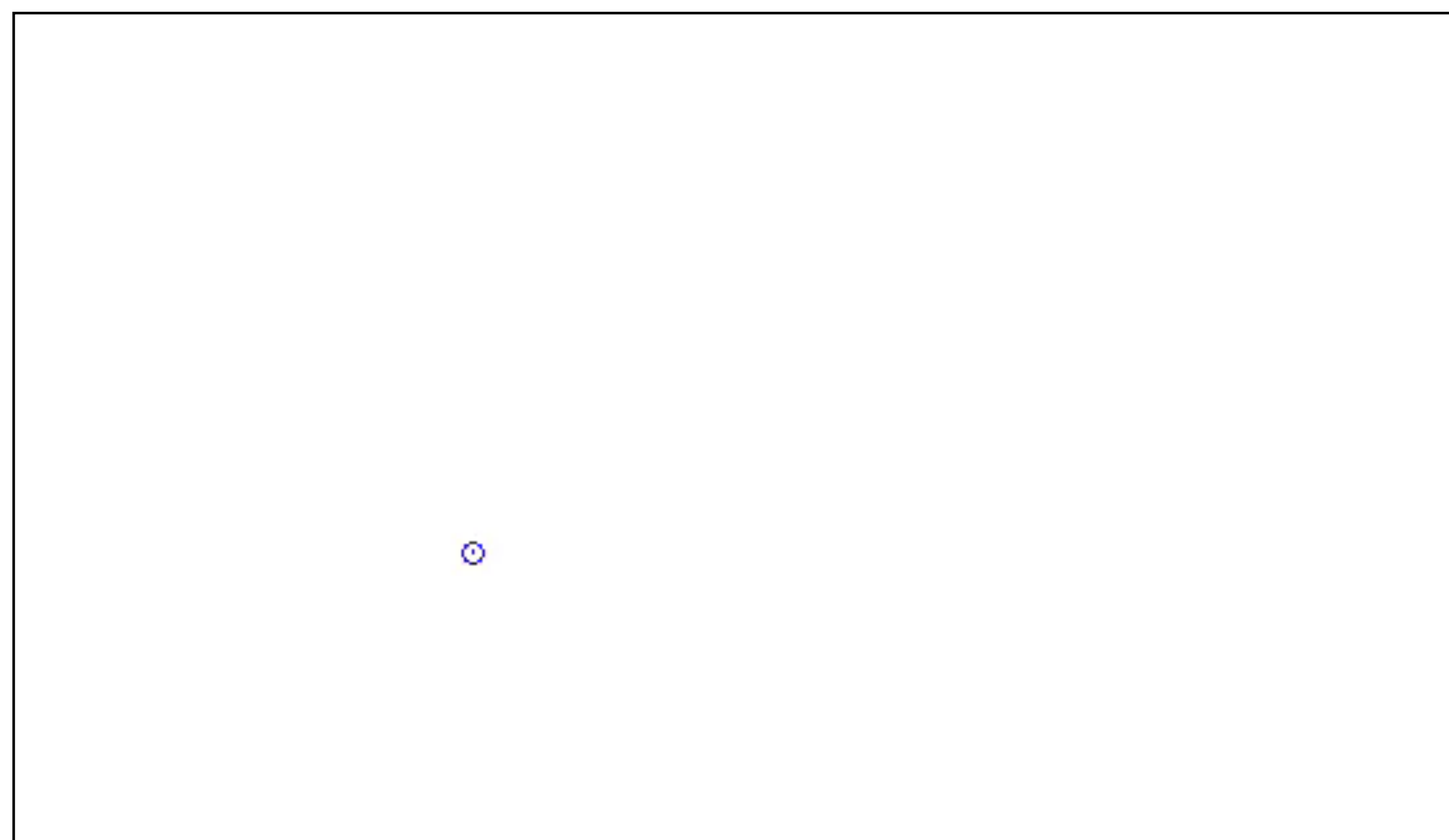


$$\frac{dy_1}{dt} = \sigma(y_2 - y_1)$$

$$\frac{dy_2}{dt} = y_1(\rho - y_3) - y_2$$

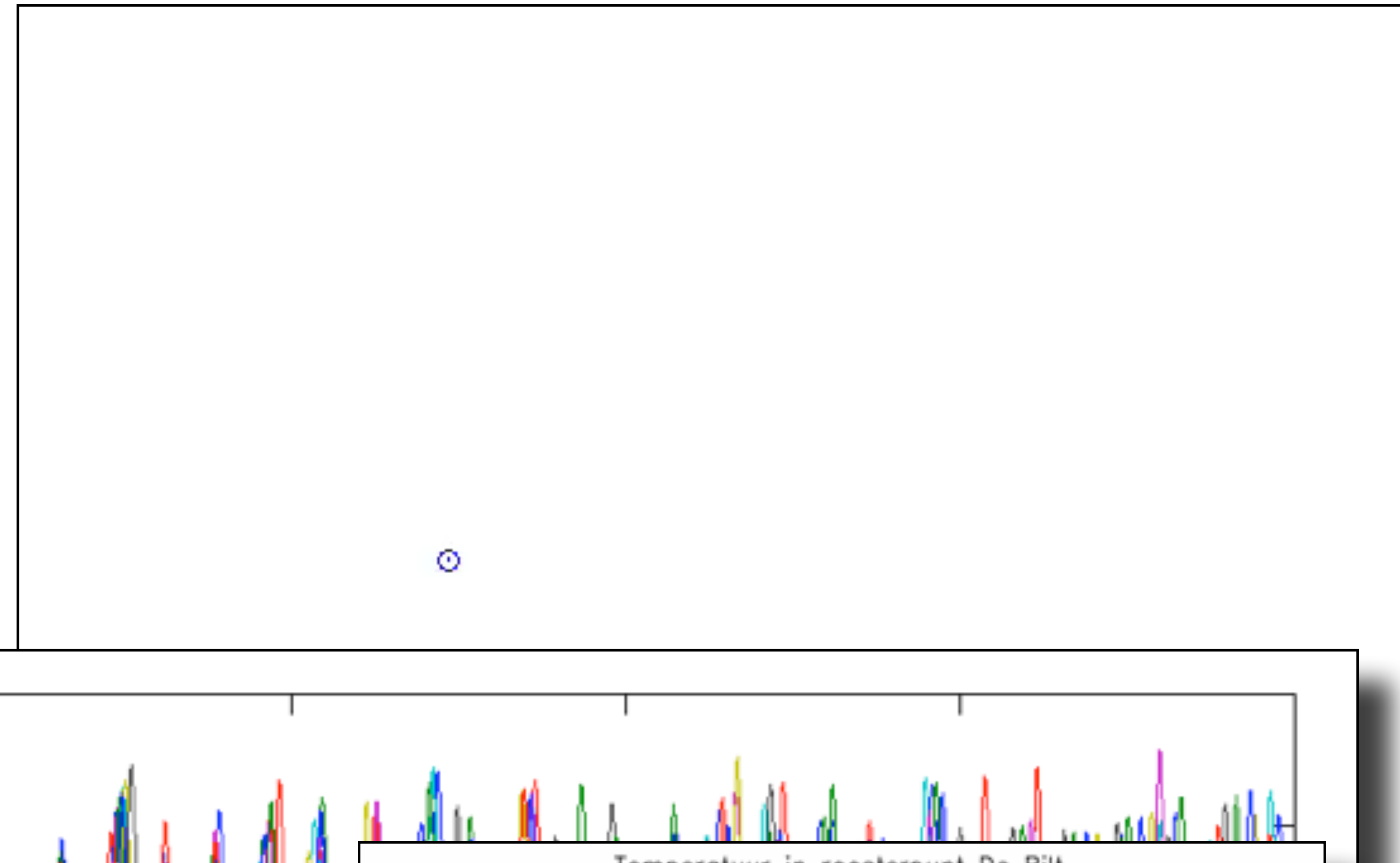
$$\frac{dy_3}{dt} = y_1 y_2 - \beta y_3$$

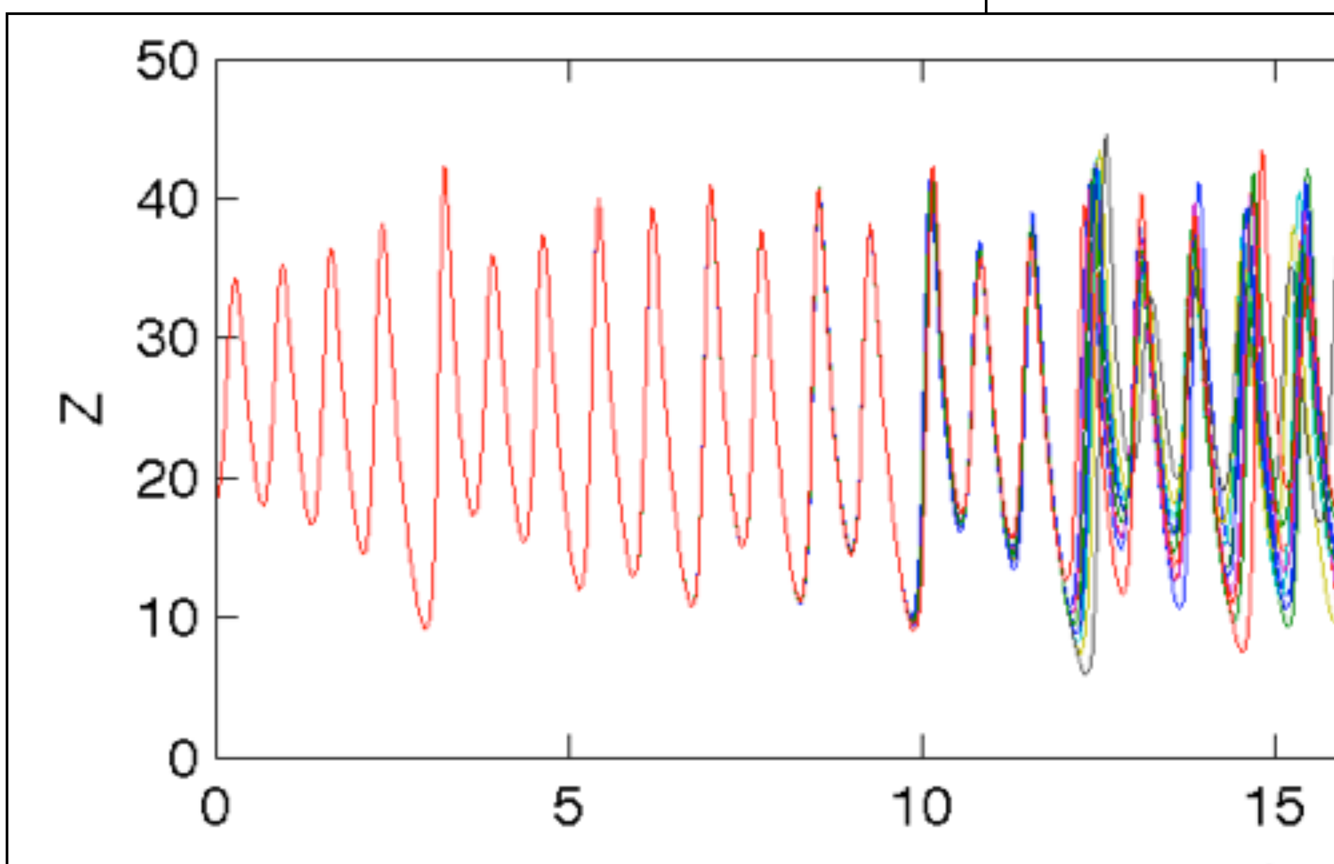
$$\sigma = 10, \quad \rho = 28, \quad \beta = 8/3$$



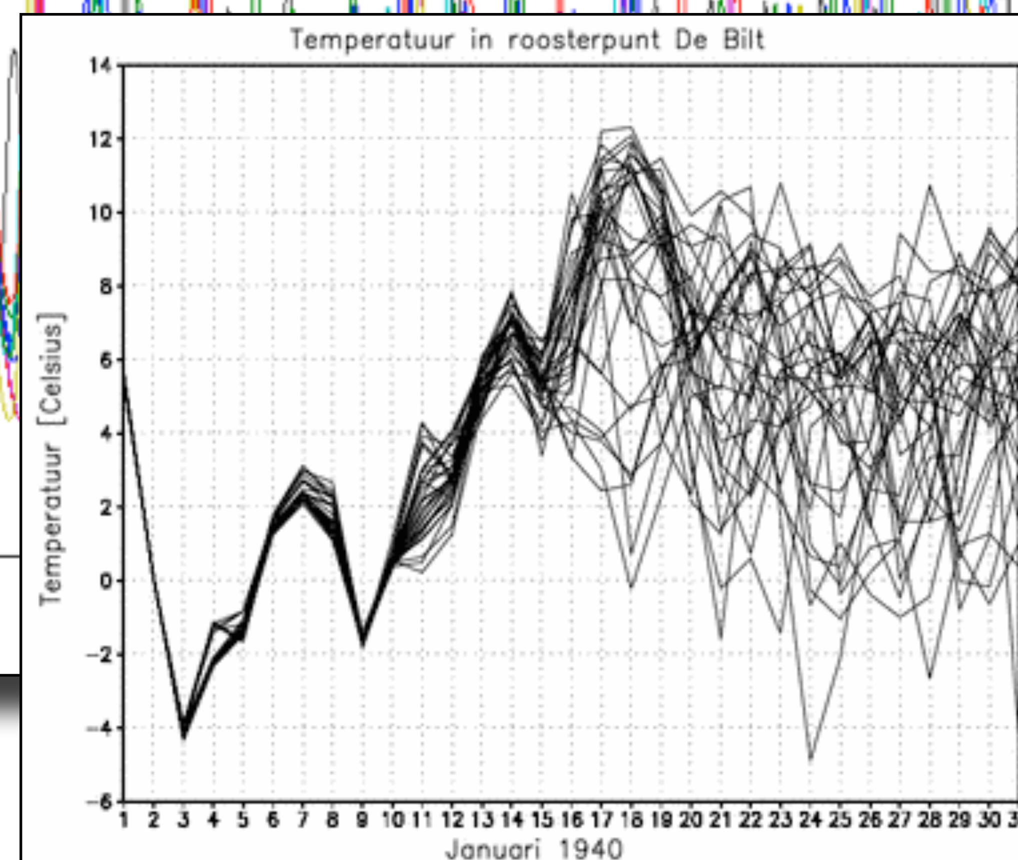
Chaotic behavior

Ten simulations of the Lorenz system with small perturbations to initial condition.





Just the variable z als functie van de tijd.



This effect is seen in climate simulations!

Chaos

A dynamical system is said to be "chaotic" if it possesses three properties:

- Sensitive dependence on initial conditions
- Topologically mixing (ergodic)
- Dense (unstable) periodic orbits

These properties are typically hard to prove for a given system, yet many practical nonlinear systems seem to exhibit them to some degree.

Sensitive dependence has the greatest consequences for predictability.

Lyapunov Exponents

Lyapunov exponents measure the exponential growth rate of a perturbation ε_0 to the initial condition.

Example.
$$x_{n+1} = A\,x_n$$
$$\tilde{x}_{n+1} = A\,\tilde{x}_n$$
$$\varepsilon_{n+1} = A\,\varepsilon_n, \quad \varepsilon_n = \tilde{x}_n - x_n$$

Suppose A has a basis of eigenvectors
$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, M$$
$$\varepsilon_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_M v_M$$
$$A\varepsilon_0 = \alpha_1 A v_1 + \cdots + \alpha_M A v_m = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_M \lambda_M v_m$$
$$\varepsilon_n = A^n \varepsilon_0 = \alpha_1 \lambda_1^n v_1 + \cdots + \alpha_M \lambda_M^n v_M$$

Lyapunov Exponents

Lyapunov exponents measure the exponential growth rate of a perturbation ε_0 to the initial condition. $\varepsilon(t) = e^{\lambda t}\varepsilon_0$

Example. $\dot{x} = f(x) \quad x(t) \in \mathbf{R}$

$$\dot{\tilde{x}} = f(\tilde{x})$$

$$\varepsilon(t) = \tilde{x}(t) - x(t)$$

$$\dot{\varepsilon} = f(x+\varepsilon) - f(x)$$

$$\dot{\varepsilon} \approx f'(x)\varepsilon$$

$$\dot{X} = f'(x(t))X, \quad X(0) = 1 \quad \Rightarrow \quad \varepsilon(t) = X(t)\varepsilon_0$$

$$e^{\lambda t} = X(t)$$

$$\lambda(t) = \frac{1}{t}\ln|X(t)|$$

$$\lambda(t) = \lim_{t\to\infty}\frac{1}{t}\ln|X(t)|$$

$\lambda > 0$ is expanding, $\lambda < 0$ is contracting

Lyapunov Exponents

Lyapunov exponents measure the exponential growth rate of a perturbation ε_0 to the initial condition. $\varepsilon(t) = e^{\lambda t}\varepsilon_0$

General: $\dot{x} = f(x) \qquad x(t) \in \mathbf{R}^M$

$$\dot{\tilde{x}} = f(\tilde{x})$$

$$\varepsilon(t) = \tilde{x}(t) - x(t)$$

$$\dot{\varepsilon} = f(x+\varepsilon) - f(x)$$

$$\dot{\varepsilon} \approx f'(x)\varepsilon$$

$$\dot{X} = f'(x(t))X, \quad X(0) = I \qquad \Rightarrow \qquad \varepsilon(t) = X(t)\varepsilon_0$$

There exists an orthonormal basis $\{v_i, i = 1, \ldots M\}$ such that

$$\lambda_i(t) = \lim_{t\to\infty} \frac{1}{t} \ln \|X(t)v_i\|, \quad i = 1, \ldots, M$$

(independent of initial condition for ergodic systems).

Source: Cross, Lecture Notes, CalTech

Computing Lyapunov exponents

Computation of Lyapunov exponents is tricky because of the great differences in scaling. Construct an orthogonal basis.

$$\dot{X} = A(t)X, \qquad X(t) = Q(t)R(t)$$

$$\dot{Q}R + Q\dot{R} = AQR$$

$$\dot{R} = Q^T AQR - Q^T \dot{Q}R = BR \qquad B = Q^T AQ - S$$

$$S = Q^T \dot{Q} = -S^T$$

$$\dot{Q} = (I - QQ^T)AQ - QS$$

$$\dot{R} = BR$$

These equations essentially project the original linear system onto a new basis in which the system becomes triangular. In this case the Lyapunov exponents can be determined from

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t B_{ii}(s)\, ds$$

Source: Dieci, Jolly, Van Vleck 2011

Computing Lyapunov exponents

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$$\dot{x} = f(x), \qquad A(t) = f'(x(t))$$

$$\dot{Q} = (I - QQ^T)AQ - QS \qquad B = Q^T AQ - S$$

$$\dot{R} = BR \qquad S = Q^T \dot{Q} = -S^T$$

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t B_{ii}(s)\, ds$$

We can choose: $Q(0)R(0) = X(0) \in \mathbf{R}^{M \times p}$
to compute the p largest Lyapunov exponents.

$$\begin{aligned} \lambda_i &> 0, \quad i = 1, \dots, p, \\ \lambda_{p+1} &= 0 \end{aligned} \qquad \Rightarrow \qquad Q = (q_1, \dots, q_p)$$

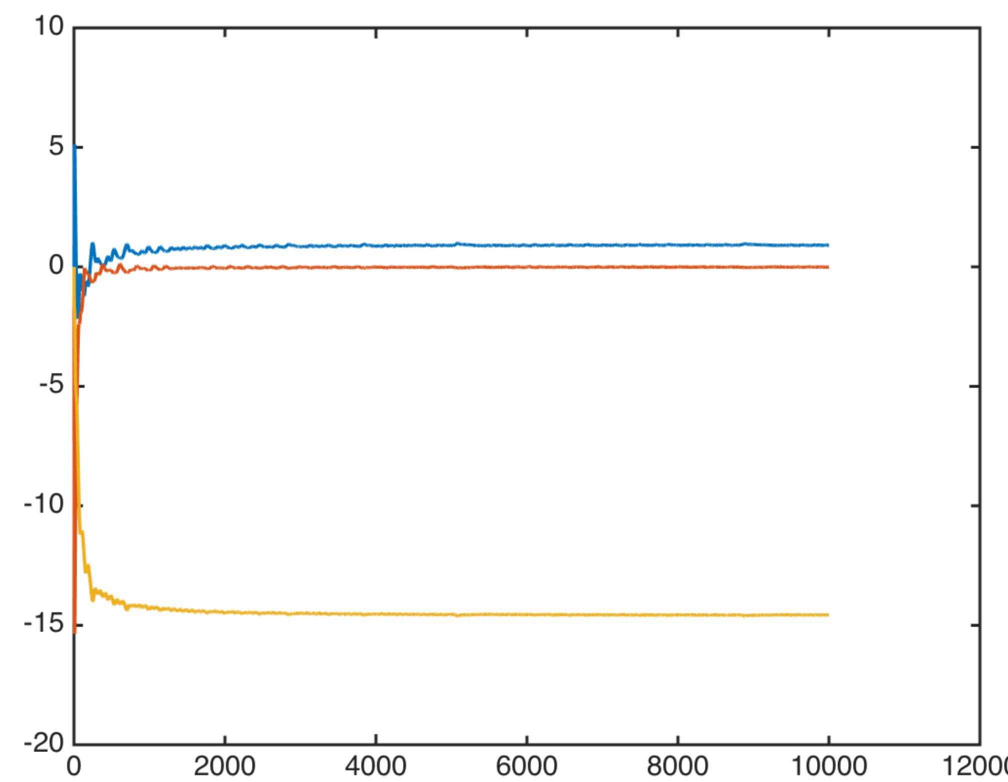
a basis for the unstable space.

Source: Dieci, Jolly, Van Vleck 2011

Computing Lyapunov exponents

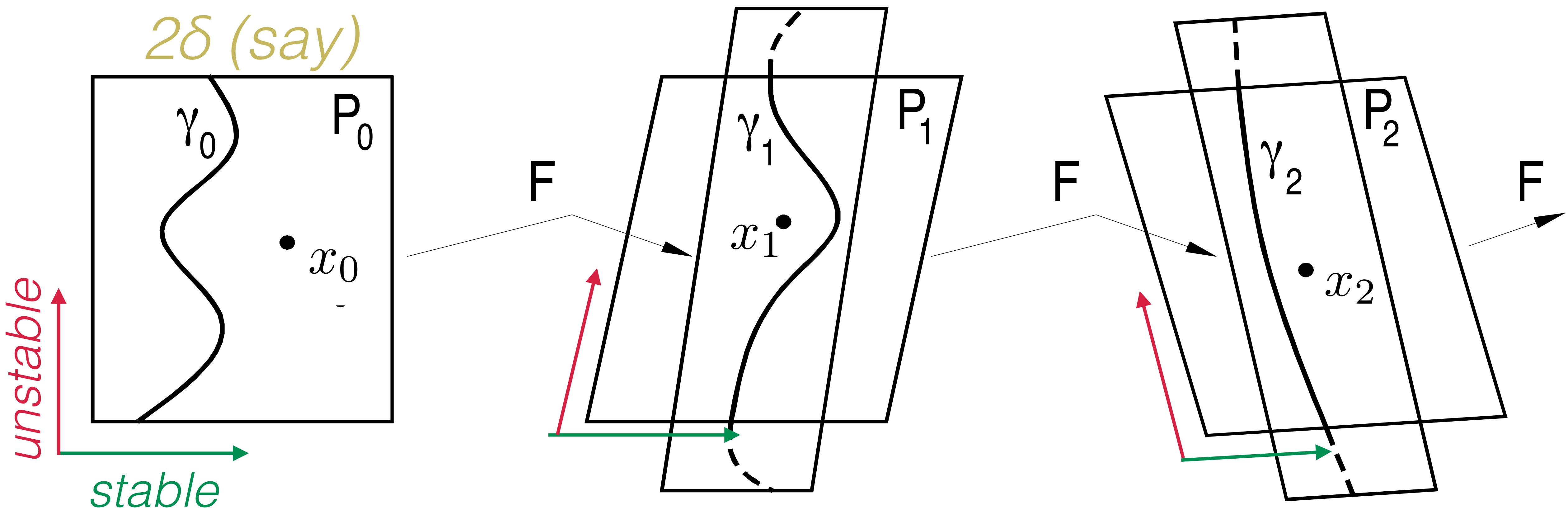
Some notes:

- The equations for Q and R need to be solved accurately (high order or adaptive step).
- It is important to preserve orthogonality in the columns of Q (projection).
- Orthogonalization via the modified Gram-Schmidt process.



Shadowing trajectories

Consider a hyperbolic system with a fixed dimension of stable and unstable manifolds. Suppose δ bounds the local error. And let F(x) be the exact time-Δt flow map.



Source: Cross, Lecture Notes, CalTech

Shadowing

Shadowing can also be used to construct an improved trajectory.

Local error $\pi_{n+1} = x_{n+1} - F(x_n)$

Refined solution $\tilde{x}_{n+1} = F(\tilde{x}_n)$

Determine Φ_n such that $\tilde{x}_n = x_n + \Phi_n$

$$\begin{aligned}\Phi_{n+1} &= \tilde{x}_{n+1} - x_{n+1} \\ &= F(\tilde{x}_n) - F(x_n) - \pi_{n+1} \\ &\approx F'(x_n)\Phi_n - \pi_{n+1}\end{aligned}$$

Let $Q_n = (U_n | S_n)$ be a basis of unstable/stable spaces

Project the above equation onto U_n and S_n.

Solve the stable (unstable) iteration forward (backward) in time.

$$\Phi_n = U_n \alpha_n + S_n \beta_n$$

$$\pi_n = U_n \eta_n + S_n \zeta_n$$

Source: Cross, Lecture Notes, CalTech

Shadowing

Shadowing results have been generalized to non-hyperbolic systems, etc.

By admitting uncertainty in the time step as well as in the discrete trajectory, Van Vleck is able to shadow a solution on the Lorenz attractor to time T=117.5 !

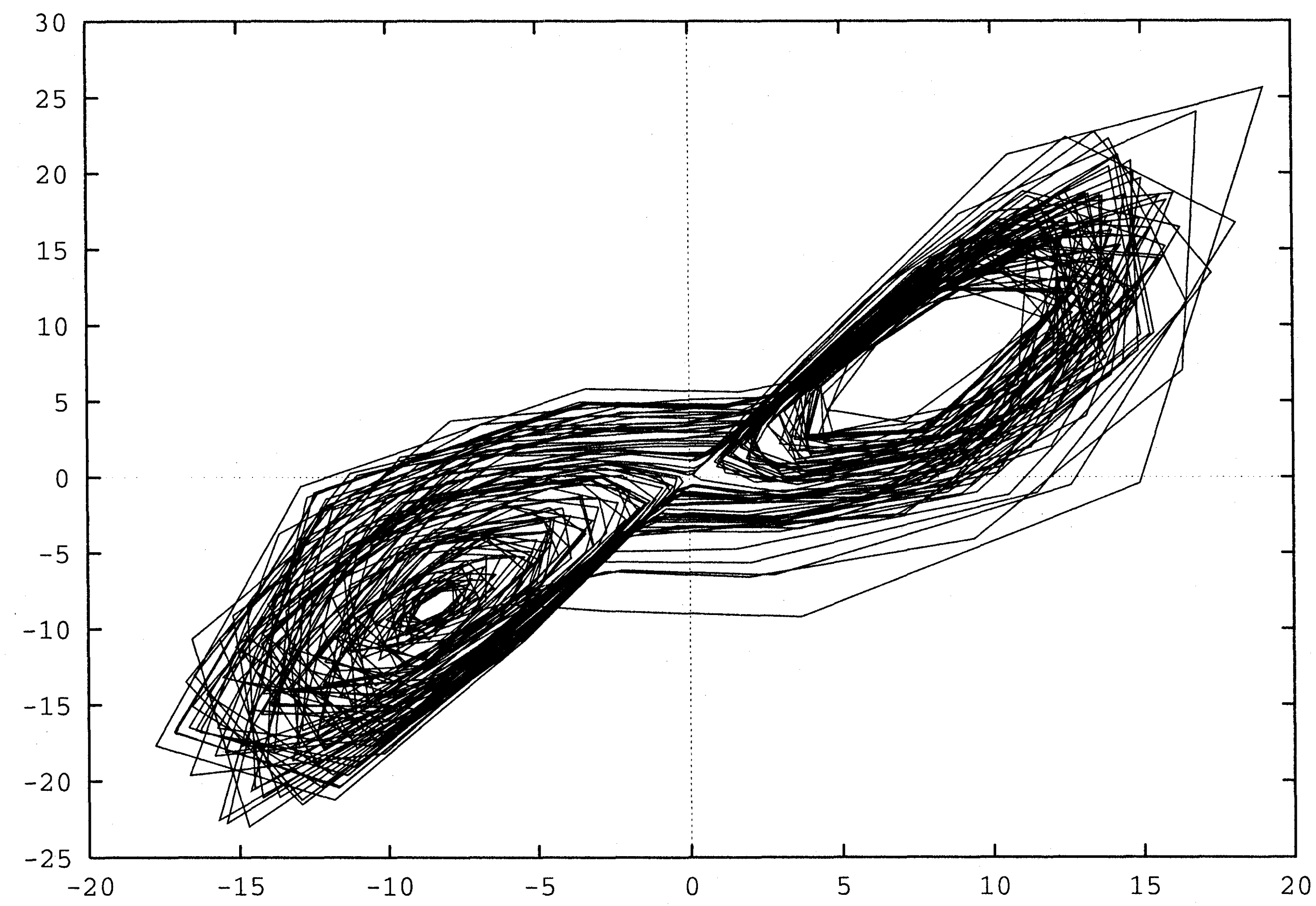


FIG. 2. *x − y plot of computed trajectory for* $T = 117.5$.

Source: Van Vleck, SIAM J. Sci. Comput., 1995

Synchronization: Pecora & Carroll 1990

The time series of the y_1 variable of one Lorenz trajectory was used as a driving function in a second. The trajectories of the second system were observed to converge to those of the first.

$$\frac{dy_1}{dt} = \sigma(y_2 - y_1)$$

$$\frac{dy_2}{dt} = y_1(\rho - y_3) - y_2 \qquad \frac{dY_2}{dt} = y_1(\rho - Y_3) - Y_2$$

$$\frac{dy_3}{dt} = y_1 y_2 - \beta y_3 \qquad \frac{dY_3}{dt} = y_1 Y_2 - \beta Y_3$$

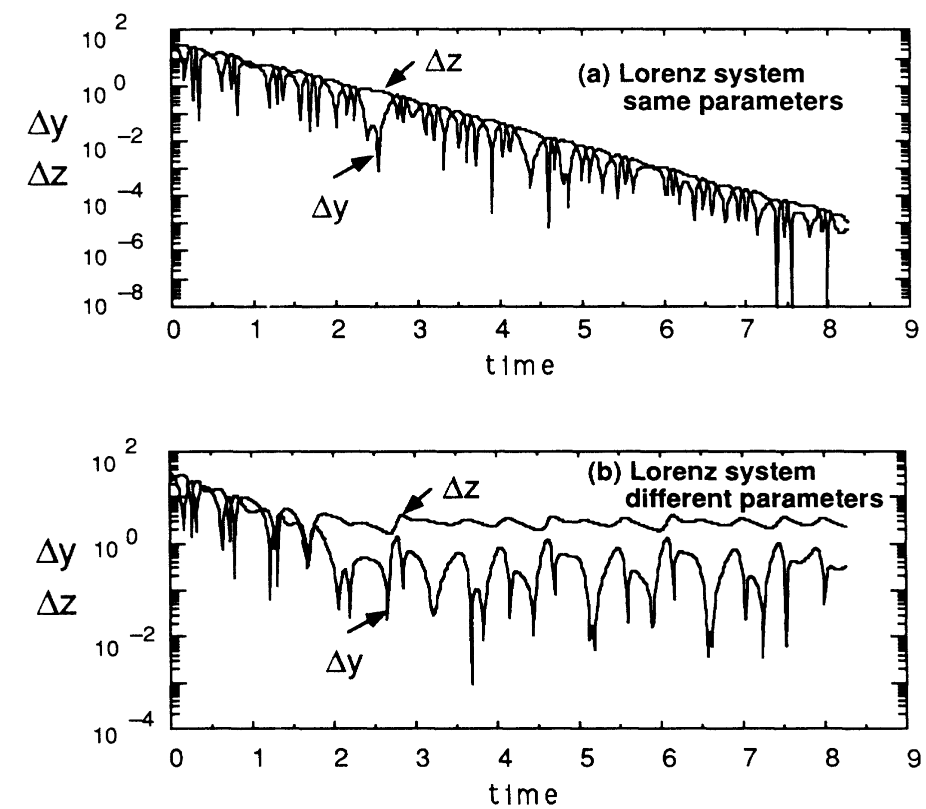


FIG. 2. The differences $y'-y$ and $z'-z$ between the response variables and their drive counterparts for the Lorenz system for (a) when parameters are the same for both systems and (b) when the parameters differ by 5%.

The same happened using y_2 as a driver.
With y_3 it didn't work.

Explanation: sub-Lyapunov exponents (negative for the first two components, positive for the last).

Source: Pecora & Carroll, Phys. Rev Lett, 1990

Synchronization

The observation of Pecora & Carroll has remarkable consequences for prediction and control:

Even though the Lorenz system is chaotic, knowing only the time series of one component is sufficient to predict the whole solution.

Similarly, even though the system is chaotic, it is only necessary to control one component to drive the whole system.

Sources

- M. Cross, *Introduction to Chaos*, Lecture Notes, Cal.Tech. (http://www.cmp.caltech.edu/~mcc/Chaos_Course/Outline.html)
- N. Balci, A.L. Mazzucato, J.M. Restrepo, G.R. Sell, Ensemble dynamics and bred vectors, *Monthly Weather Review*, 140 (2012) 2308—2334.
- L. Dieci, M.S. Jolly, E.S. Van Vleck, Numerical techniques for approximating Lyapunov exponents and their implementation, *J. Computational and Nonlinear Dynamics*, 2011.
- L.M. Pecora & T.L. Carroll, Synchronization in complex systems, *Physical Review Letters* 64 (1990) 821-824.
- E.S. Van Vleck, Numerical shadowing near hyperbolic trajectories, *SIAM J. Scientific Computing*, 16 (1995) 1177-1189.