Mathematical Modelling

Lecture 3

The Big Picture (regular perturbations)

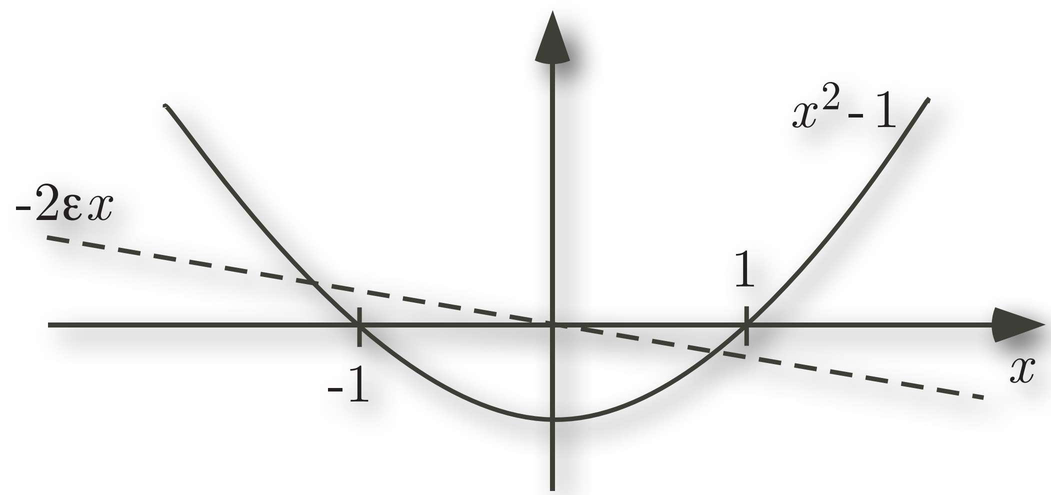
Regular expansions

Consider the solutions of the quadratic equation

$$x^2 + 2\varepsilon x - 1 = 0 \qquad\qquad (\varepsilon \ll 1)$$

where ε is a small parameter.

$$x^2 - 1 = -2\varepsilon x$$



Import are:

1. we are interested in the class of solutions as a function of ε.

2. we are interested in obtaining approximate solutions as ε tends to zero.

The number of solutions is 2, also in the limit $\varepsilon \to 0$ (*regular perturbation*). Consider instead the *singular perturbation* problem:

$$\varepsilon x^2 + 2x - 1 = 0$$

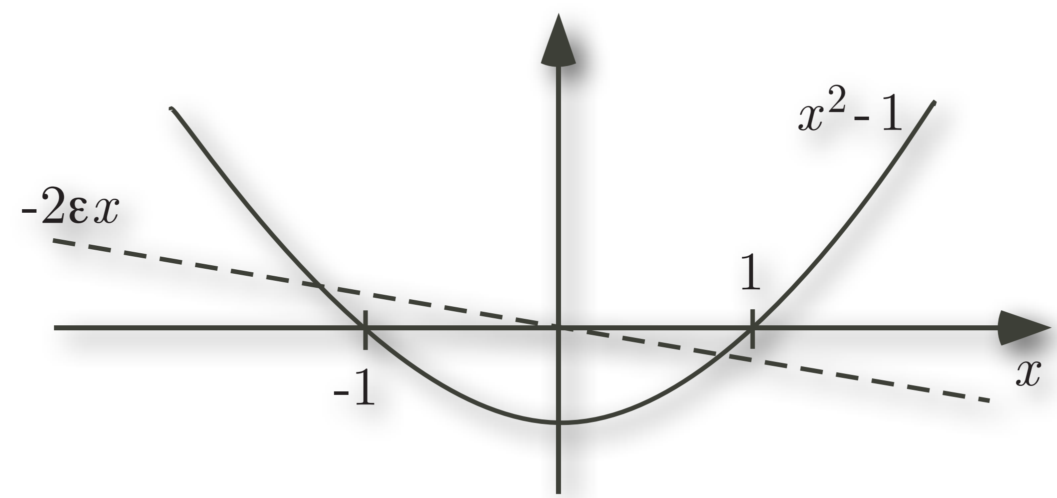
Regular expansions

Consider the solutions of the quadratic equation

$$x^2 + 2\varepsilon x - 1 = 0 \qquad (\varepsilon \ll 1)$$

where ε is a small parameter.

$$x^2 - 1 = -2\varepsilon x$$



Import are:

1. we are interested in the class of solutions as a function of ε.

$$x = -\varepsilon \pm \sqrt{1 + \varepsilon^2}$$

2. we are interested in obtaining approximate solutions as ε tends to zero.

Recall that $(1 + x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots, \quad (x < 1)$

(for common expansions, see Table 2.1). Hence,

$$x = -\varepsilon \pm \left(1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \cdots\right) = \pm 1 - \varepsilon \pm \frac{1}{2}\varepsilon^2 \mp \frac{1}{8}\varepsilon^4 + \cdots$$

Regular expansions

Consider the solutions of the quadratic equation

$$x^2 + 2\varepsilon x - 1 = 0 \qquad\qquad (\varepsilon \ll 1)$$

where ε is a small parameter.

Suppose we did not know the solution of the quadratic. We expect the solution to depend on ε. Generalizing Taylor's theorem, we try an expansion

$$x(\varepsilon) \sim x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots$$

Assumptions:

- Well-ordering $0 < \alpha < \beta < \cdots$

- Coefficients $x_0, x_1, x_2, \cdots$ do not depend on ε

Note that

$$\begin{aligned} x^2 &\sim (x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots)(x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots) \\ &\sim x_0^2 + 2\varepsilon^\alpha x_0 x_1 + 2\varepsilon^\beta x_0 x_2 + \varepsilon^{2\alpha} x_1^2 + \cdots \end{aligned}$$

Regular expansions

Consider the solutions of the quadratic equation

$$x^2 + 2\varepsilon x - 1 = 0 \qquad\qquad (\varepsilon \ll 1)$$

where ε is a small parameter.

Substituting the expansion into the quadratic equation gives

$$x(\varepsilon) \sim x_0 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots$$

$$x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \cdots + 2\varepsilon(x_0 + \varepsilon^\alpha x_1 + \cdots) - 1 = 0$$

Taking the limit $\varepsilon \to 0$ yields the relation

$$\mathcal{O}(1) \qquad x_0^2 - 1 = 0 \quad \Rightarrow \quad x_0 = \pm 1$$

Leaving $2\varepsilon^\alpha x_0 x_1 + \cdots + 2\varepsilon(x_0 + \varepsilon^\alpha x_1 + \cdots) = 0$

If the expansion is to hold as ε tends to zero, the terms of the same order in ε must equate independently. The term $2\varepsilon x_0$ must be balanced by $2\varepsilon^\alpha x_0 x_1$

$$\Rightarrow \alpha = 1$$

$$\mathcal{O}(\varepsilon) \qquad 2x_0 x_1 + 2x_0 = 0 \quad \Rightarrow \quad x_1 = -1$$

Regular expansions

Consider the solutions of the quadratic equation

$$x^2 + 2\varepsilon x - 1 = 0 \qquad\qquad (\varepsilon \ll 1)$$

where ε is a small parameter.

Continuing this way yields

$$\begin{array}{lll} \mathcal{O}(1) & x_0^2 - 1 = 0 \quad \Rightarrow \quad x_0 = \pm 1 \\ \mathcal{O}(\varepsilon) & 2x_0x_1 + 2x_0 = 0 \quad \Rightarrow \quad x_1 = -1 \\ \mathcal{O}(\varepsilon^2) & 2x_0x_2 + x_1^2 - 2x_1 = 0 \quad \Rightarrow \quad x_2 = \pm\frac{1}{2} \end{array}$$

Compare $x = -\varepsilon \pm \left(1 + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \cdots\right) = \pm 1 - \varepsilon \pm \frac{1}{2}\varepsilon^2 \mp \frac{1}{8}\varepsilon^4 + \cdots$

Formally, we require:

$$\lim_{\varepsilon\to 0} \frac{x - (1-\varepsilon)}{\varepsilon} = 0 \qquad\qquad \lim_{\varepsilon\to 0} \frac{x - (1 - \varepsilon + \frac{1}{2}\varepsilon^2)}{\varepsilon^2} = 0$$

i.e. the truncated expansions converge as $\varepsilon \to 0$.
But the (infinite) expansion typically does not converge for fixed ε.

Expansion of functions

Compound function:

$$f(\varepsilon) = \sin(e^{\varepsilon})$$

Start with the innermost function

$$e^{\varepsilon} = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots \qquad \sin(e^{\varepsilon}) = \sin\left(1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \cdots\right)$$

From Table 2.1 $\sin(1+y) \sim \sin(1) + y\cos(1) - \frac{1}{2}y^2\sin(1) + \cdots$

$$\sin(e^{\varepsilon}) \sim \sin(1) + \cos(1)\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \cdots\right) - \frac{1}{2}\sin(1)\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \cdots\right)^2 + \cdots$$

$$\sin(e^{\varepsilon}) \sim \sin(1) + \varepsilon\cos(1) + \frac{1}{2}\varepsilon^2[\cos(1) - \sin(1)] + \cdots$$

Expansion of functions

Compound function:

$$f(\varepsilon) = \frac{1}{[1-\cos(\varepsilon)]^3}$$

Start with the innermost function

$$\cos(\varepsilon) \sim 1 - \frac{1}{2}\varepsilon^2 + \frac{1}{24}\varepsilon^4 + \cdots$$

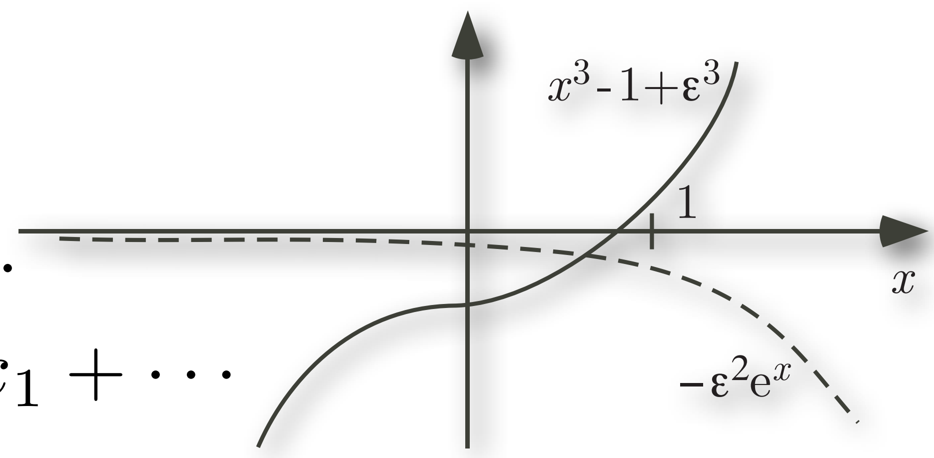
$$\frac{1}{1-\cos(\varepsilon)} \sim \frac{1}{\frac{1}{2}\varepsilon^2 - \frac{1}{24}\varepsilon^4 + \cdots} = \frac{2}{\varepsilon^2}\frac{1}{1-\frac{1}{12}\varepsilon^2+\cdots}$$

$$\frac{1}{[1-\cos(\varepsilon)]^3} \sim \frac{8}{\varepsilon^6}\left(1+\frac{1}{12}\varepsilon^2+\cdots\right)^3 = \frac{8}{\varepsilon^6}\left(1+\frac{1}{4}\varepsilon^2+\cdots\right)$$

Expansion of equations

Algebraic/transcendental equation:

$$x^3 + \varepsilon^2 e^x - 1 + \varepsilon^3 = 0$$



$$x \sim \varepsilon x_0 + \cdots$$

$$x \sim x_0 + \varepsilon^\alpha x_1 + \cdots$$

$$x \sim \frac{1}{\varepsilon} x_0 + \cdots$$

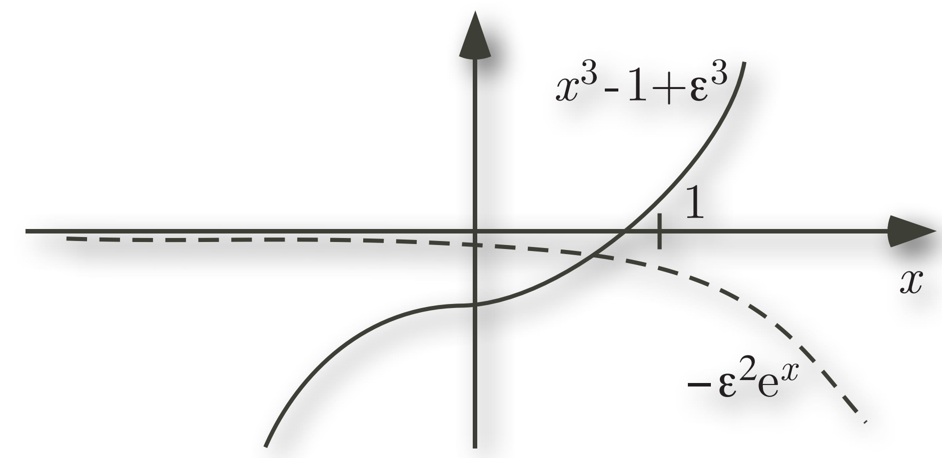
$$\begin{aligned} e^x &\sim e^{x_0 + \varepsilon^\alpha x_1 + \cdots} = e^{x_0} e^{\varepsilon^\alpha x_1 + \cdots} \\ &= e^{x_0}\left(1 + y + \frac{1}{2} y^2 + \cdots\right) \\ &= e^{x_0}\left(1 + (\varepsilon^\alpha x_1 + \cdots) + \frac{1}{2}(\varepsilon^\alpha x_1 + \cdots)^2 + \cdots\right) \\ &= e^{x_0}(1 + \varepsilon^\alpha x_1 + \cdots) \end{aligned}$$

$$x^3 \sim x_0^3 + 3\varepsilon^\alpha x_0^2 x_1 + \cdots$$

$$\Rightarrow \quad x_0^3 + 3\varepsilon^\alpha x_0^2 x_1 + \cdots + \varepsilon^2 e^{x_0}(1 + \varepsilon^\alpha x_1 + \cdots) - 1 + \varepsilon^3 = 0$$

Expansion of equations

Algebraic/transcendental equation:



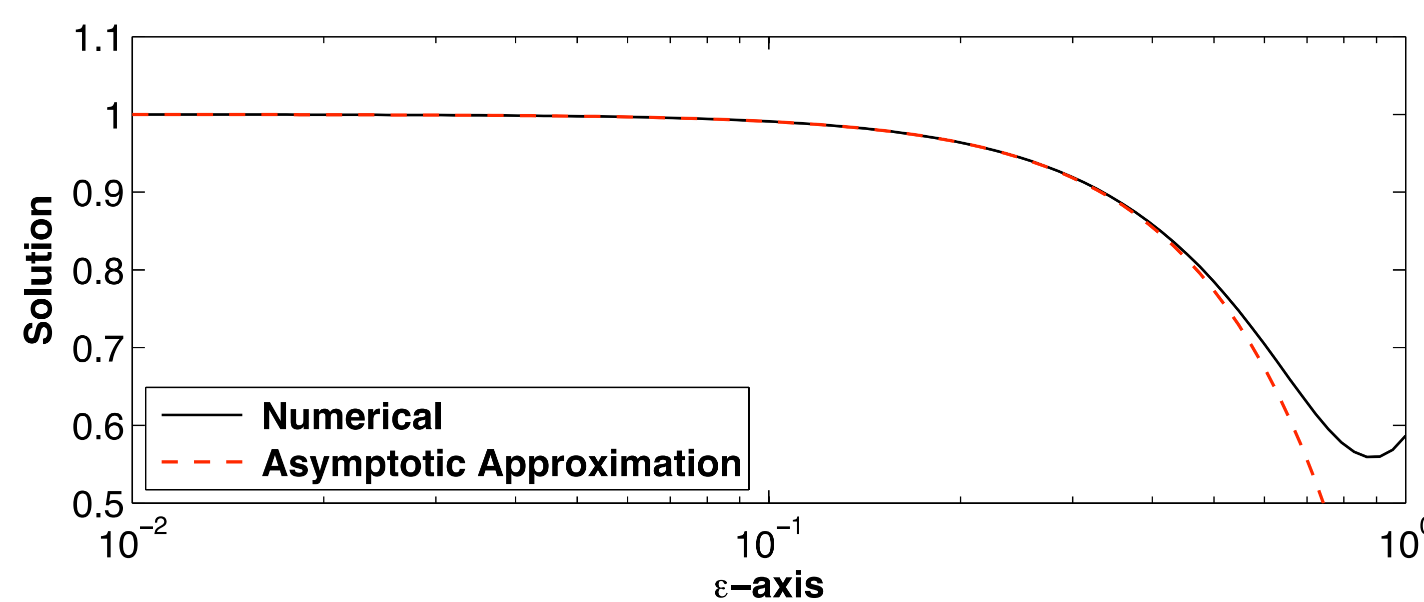
$$\Rightarrow \quad x_0^3 + 3\varepsilon^\alpha x_0^2 x_1 + \cdots + \varepsilon^2 e^{x_0}(1 + \varepsilon^\alpha x_1 + \cdots) - 1 + \varepsilon^3 = 0$$

$$\mathcal{O}(1) \qquad x_0^3 - 1 = 0 \quad \Rightarrow \quad x_0 = 1$$

$$\mathcal{O}(\varepsilon^2) \qquad 3x_0^2 x_1 + e^{x_0} = 0 \quad \Rightarrow \quad x_1 = -\frac{1}{3}e$$

Two-term approximation:

$$x \sim 1 - \frac{1}{3}\varepsilon^2 e + \cdots$$



Expansion of initial value problems

Rescaled projectile problem from the first lecture

$$\frac{d^2x}{dt^2} = -\frac{1}{(1+\varepsilon x)^2}, \qquad x(0)=0, \quad \frac{dx}{dt}(0)=1 \qquad \left(\varepsilon = \frac{v_0^2}{gR}\right)$$

In this case, expansion coefficients are functions of t:

$$x(t;\varepsilon) \sim x_0(t) + \varepsilon^\alpha x_1(t) + \varepsilon^\beta x_2(t) + \cdots$$

Note: $$\frac{1}{(1+\varepsilon x)^2} = 1 - 2\varepsilon x + 3\varepsilon^2 x^2 + \cdots$$
$$\sim 1 - 2\varepsilon(x_0 + \varepsilon^\alpha x_1 + \cdots) + 3\varepsilon^2(x_0 + \cdots)^2 + \cdots$$
$$= 1 - 2\varepsilon x_0 + \cdots$$

The expansion of the differential equation is

$$x_0'' + \varepsilon^\alpha x_1'' + \cdots = -1 + 2\varepsilon x_0 + \cdots$$

The expansions of the initial conditions are

$$x_0(0) + \varepsilon^\alpha x_1(0) + \cdots = 0, \quad x_0'(0) + \varepsilon^\alpha x_1'(0) + \cdots = 1$$

Expansion of initial value problems

The expansion of the differential equation is

$$x_0'' + \varepsilon^\alpha x_1'' + \cdots = -1 + 2\varepsilon x_0 + \cdots$$

The expansions of the initial conditions are

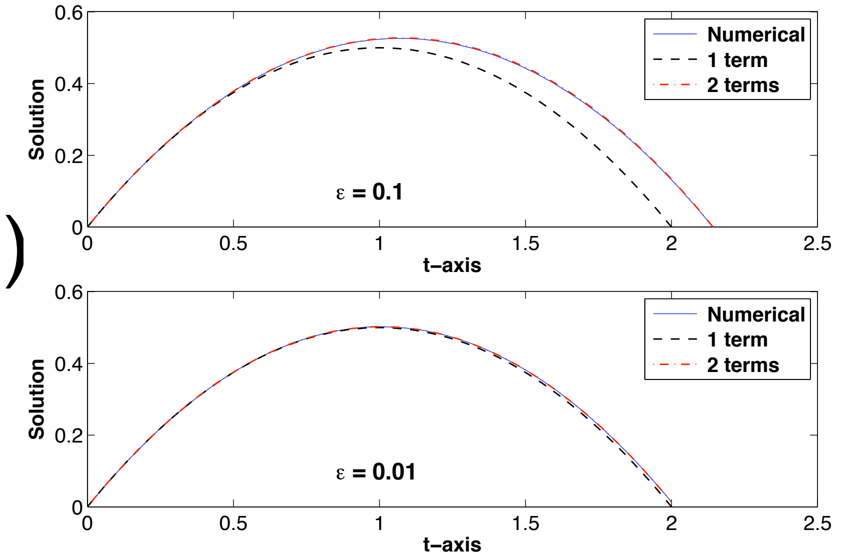
$$x_0(0) + \varepsilon^\alpha x_1(0) + \cdots = 0, \quad x_0'(0) + \varepsilon^\alpha x_1'(0) + \cdots = 1$$

Consistency conditions:

$$\mathcal{O}(1) \qquad x_0'' = -1 \quad x_0(0) = 0, \quad x_0'(0) = 1$$

$$\Rightarrow x_0(t) = t(1 - \frac{1}{2}t)$$

(term $2\varepsilon x_0$ must balance with $\varepsilon^\alpha x_1''$, $\Rightarrow \quad \alpha = 1$)



$$\mathcal{O}(\varepsilon) \qquad x_1'' = 2x_0 \quad x_1(0) = 0, \quad x_1'(0) = 0$$

$$\Rightarrow x_1(t) = \frac{1}{12}t^3(4 - t)$$

Two-term solution: $x(t) \sim t(1 - \frac{1}{2}t) + \frac{1}{12}\varepsilon t^3(4 - t) + \cdots$

Expansion of initial value problems

We can apply the same approach to systems

$$\frac{du}{dt} = 1 - ue^{\varepsilon(q-1)}, \quad u(0) = 0$$

$$\frac{dq}{dt} = ue^{\varepsilon(q-1)} - q, \quad q(0) = 0$$

$$u \sim u_0(t) + \varepsilon u_1(t) + \cdots,$$

$$q \sim q_0(t) + \varepsilon q_1(t) + \cdots$$

$$\begin{aligned} e^{\varepsilon(q-1)} &\sim 1 + \varepsilon(q-1) + \frac{1}{2}\varepsilon^2(q-1)^2 + \cdots \\ &\sim 1 + \varepsilon(q_0 + \varepsilon q_1 + \cdots - 1) + \frac{1}{2}\varepsilon^2(q_0 + \varepsilon q_1 + \cdots - 1)^2 \\ &\sim 1 + \varepsilon(q_0 - 1) + \cdots \\ ue^{\varepsilon(q-1)} &\sim (u_0 + \varepsilon u_1 + \cdots)\left[1 + \varepsilon(q_0 - 1) + \cdots\right] \\ &\sim u_0 + \varepsilon\left[u_0(q_0 - 1) + u_1\right] + \cdots \end{aligned}$$

$$u_0' + \varepsilon u_1' + \cdots = 1 - u_0 - \varepsilon(u_0(q_0 - 1) + u_1) + \cdots$$

$$q_0' + \varepsilon q_1' + \cdots = u_0 - q_0 + \varepsilon(u_0(q_0 - 1) + u_1 - q_1) + \cdots$$

Expansion of initial value problems

$$u_0' + \varepsilon u_1' + \cdots = 1 - u_0 - \varepsilon(u_0(q_0 - 1) + u_1) + \cdots$$
$$q_0' + \varepsilon q_1' + \cdots = u_0 - q_0 + \varepsilon(u_0(q_0 - 1) + u_1 - q_1) + \cdots$$

Consistency conditions

$$\mathcal{O}(1) \qquad u_0' = 1 - u_0 \quad q_0' = u_0 - q_0, \quad u_0(0) = q_0(0) = 0$$
$$\Rightarrow u_0(t) = 1 - e^{-t}, \quad q_0(t) = 1 - (1+t)e^{-t}$$

$$\mathcal{O}(\varepsilon) \qquad u_1' = -u_1 - u_0(q_0 - 1) \quad q_1' = -q_1 + u_1 + u_0(q_0 - 1),$$
$$u_1(0) = q_1(0) = 0$$
$$\Rightarrow \quad u_1(t) = \frac{1}{2}(t^2 + 2t - 4)e^{-t} + (2+t)e^{-2t},$$
$$q_1(t) = \frac{1}{6}(t^3 - 18t + 30)e^{-t} - (2t+5)e^{-2t}$$

$$u(t) \sim 1 - e^{-t} + \varepsilon\left(\frac{1}{2}(t^2 + 2t - 4)e^{-t} + (2+t)e^{-2t}\right) + \cdots$$
$$q(t) \sim 1 - (1+t)e^{-t} + \varepsilon\left(\frac{1}{6}(t^3 - 18t + 30)e^{-t} - (2t+5)e^{-2t}\right) + \cdots$$