Mathematical Modelling

Lecture 4

The devil's in the details (singular perturbations)

Review of regular perturbations

- Asymptotic expansions: $x \sim \varepsilon^{\alpha_0} x_0 + \varepsilon^{\alpha_1} x_1 + \cdots ,$
  - Assumed: $\alpha_0 < \alpha_1 < \cdots$
- Collecting terms: $\varepsilon^{\alpha_0} f_0(x_i) + \varepsilon^{\alpha_1} f_1(x_i) + \cdots = 0, \qquad 0 < \varepsilon \leq \varepsilon_0, \quad x \in D$
  - Consequently, $f_0(x_i) = 0, \;\; f_1(x_i) = 0, \;\; \cdots$
- $\alpha_0, \; \alpha_1, \; \cdots$ chosen to achieve balance, starting at the lowest order in ε
- Leads to a hierarchy of approximations, e.g.

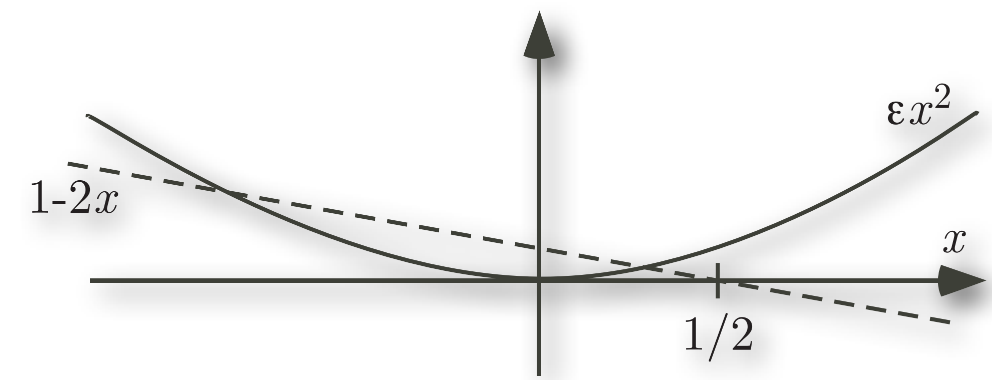
$$
\begin{aligned}
x &\approx x_0 \\
x &\approx x_0 + \varepsilon x_1 \\
x &\approx x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \\
&\vdots
\end{aligned}
$$

- Convergence in epsilon, divergent series
- Initial value problems (expansion of ODE *and* initial conditions)

Singular perturbations

Consider the solutions of the quadratic equation

$$\varepsilon x^2 + 2x - 1 = 0$$



where ε is a small parameter.

$$1 - 2x = \varepsilon x^2$$

One root converges to 1/2, one diverges in the limit $\varepsilon \to 0$

Using a regular perturbation expansion, we can approximate the root at x=1/2, but not the other root.

Instead we introduce a rescaling $\bar{x} = \varepsilon x$. The problem becomes

$$\bar{x}^2 + 2\bar{x} - \varepsilon = 0$$

for which both roots exist in the limit $\varepsilon \to 0$, and regular expansions can be applied.

Boundary layers

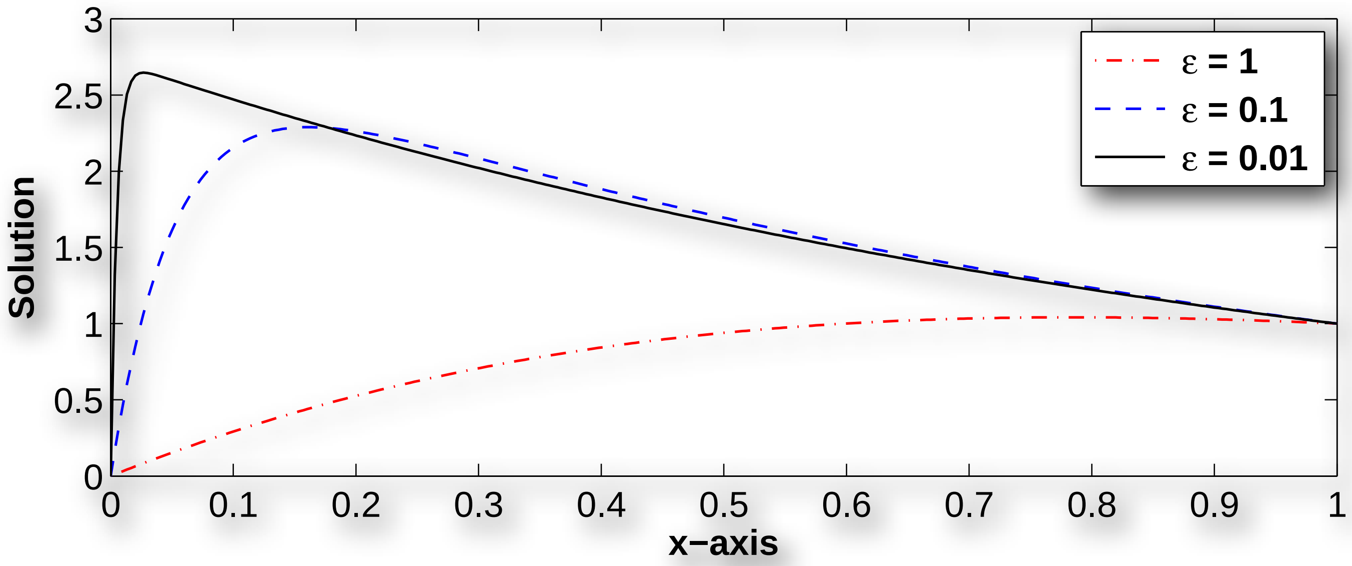
Singular perturbation problems arise also in differential equations. Typically when ε multiplies the highest derivative.

$$\varepsilon y''(x) + 2y'(x) + 2y(x) = 0, \quad 0 < x < 1$$
$$y(0) = 0, \quad y(1) = 1$$

Again the character of the problem changes for $\varepsilon = 0$: Both boundary conditions cannot be satisfied.

Exact solution $y(x) = \dfrac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}}, \quad r_\pm = (-1 \pm \sqrt{1 - 2\varepsilon})/\varepsilon$

Solution varies (rapidly) over a region of width ~ ε
"Boundary layer"



Solution has two parts: inner and outer layers

Boundary layers

Singular perturbation problems arise also in differential equations. Typically when ε multiplies the highest derivative.

$$\varepsilon y''(x) + 2y'(x) + 2y(x) = 0, \quad 0 < x < 1$$
$$y(0) = 0, \quad y(1) = 1$$

To obtain the outer solution we just apply the regular perturbation method: $y \sim y_0(x) + \varepsilon y_1(x) + \cdots$

$$\varepsilon(y_0''(x) + \varepsilon y_1''(x) + \cdots) + 2(y_0'(x) + \varepsilon y_1'(x) + \cdots) + 2(y_0(x) + \varepsilon y_1(x) + \cdots) = 0$$

$$y_0(0) + \varepsilon y_1(0) + \cdots = 0, \quad y_0(1) + \varepsilon y_1(1) + \cdots = 1$$

$\mathcal{O}(1):$ $\quad 2y_0' + 2y_0 = 0, \quad y_0(0) = 0, \; y_0(1) = 1 \quad \Rightarrow \quad y_0(x) = ae^{-x}$

One free parameter, can satisfy only one b.c. $\quad \Rightarrow \quad y_0(x) = e^{1-x}$

$\mathcal{O}(\varepsilon):$ $\quad y_0'' + 2y_1' + 2y_1 = 0, \quad y_1(1) = 1 \quad \Rightarrow \quad y_1(x) = (b - x/2)e^{1-x}$

$$\Rightarrow \quad y_1(x) = (1 - x)e^{1-x}/2$$

Boundary layers

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Typically when ε multiplies the highest derivative.

$$\varepsilon y''(x) + 2y'(x) + 2y(x) = 0, \quad 0 < x < 1$$
$$y(0) = 0, \quad y(1) = 1$$

Next we consider what happens in a neighborhood of the left boundary.
We rescale in x: $\bar{x} = \frac{x}{\varepsilon^\gamma}, \qquad \frac{d}{dx} = \frac{d\bar{x}}{dx}\frac{d}{d\bar{x}} = \frac{1}{\varepsilon^\gamma}\frac{d}{d\bar{x}}, \qquad \frac{d^2}{dx^2} = \frac{1}{\varepsilon^{2\gamma}}\frac{d^2}{d\bar{x}^2}$

$$Y(\bar{x}) = y(x)$$

$$\varepsilon^{1-2\gamma}Y'' + 2\varepsilon^{-\gamma}Y' + 2Y = 0$$

① ② ③

We want to choose γ such that the first term remains as $\varepsilon \to 0$.

We balance ① with either ② or ③ at the lowest order.

Boundary layers

$$\varepsilon^{1-2\gamma} Y'' + 2\varepsilon^{-\gamma} Y' + 2Y = 0$$

Balance ① ∼ ② : $1 - 2\gamma = -\gamma \;\Rightarrow\; \gamma = 1, \mathcal{O}(\varepsilon^{-1}), \quad ③ \sim \mathcal{O}(1)$

① ∼ ③ : $1 - 2\gamma = -0 \;\Rightarrow\; \gamma = 1/2, \mathcal{O}(1), \quad ② \sim \mathcal{O}(\varepsilon^{-1/2})$

$$Y'' + 2Y' + 2\varepsilon Y = 0$$

$$Y = Y_0 + \varepsilon Y_1 + \cdots$$

$$Y_0'' + \varepsilon Y_1'' + \cdots + 2(Y_0' + \varepsilon Y_1' + \cdots) + 2\varepsilon(Y_0 + \varepsilon Y_1 + \cdots) = 0$$

$$Y_0(0) + \varepsilon Y_1(0) = 0$$

$$\mathcal{O}(1): \quad Y_0'' + 2Y_0' = 0, \quad Y_0(0) = 0 \quad \Rightarrow \quad Y_0 = A + Be^{-2\bar{x}}$$

$$\Rightarrow \quad Y_0(\bar{x}) = A(1 - e^{-2\bar{x}})$$

Boundary layers

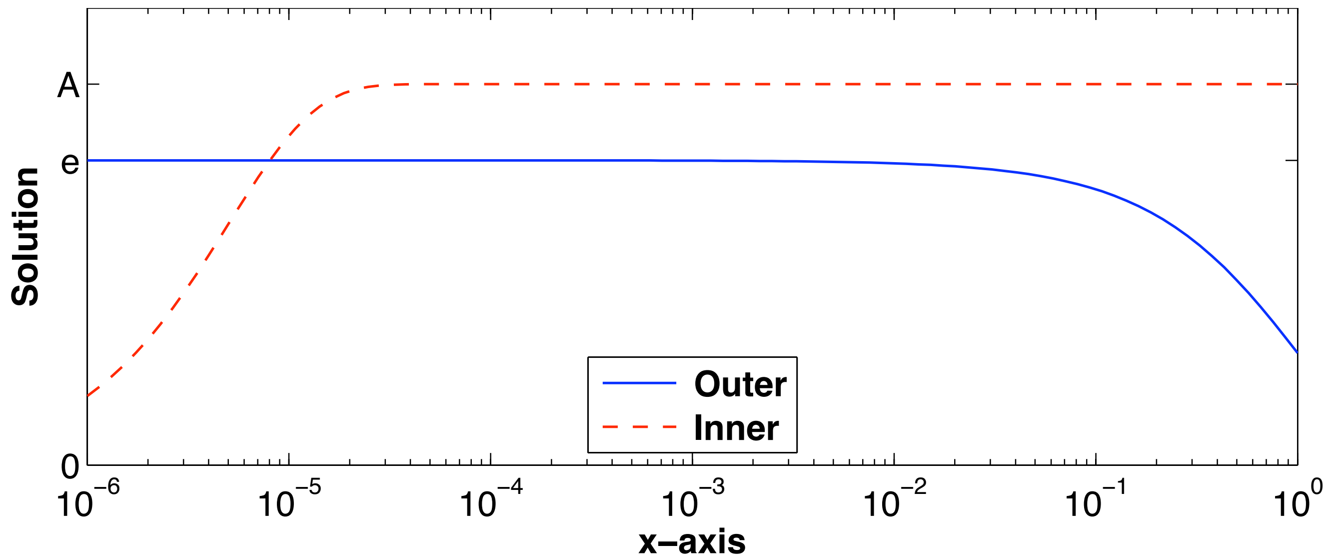
Next, we want to match the solutions in the "overlap region". We require the matching condition:

$$\lim_{\bar{x}\to\infty} Y_0 = \lim_{x\to 0} y_0$$

$$y_0(x) = e^{1-x} \to e$$

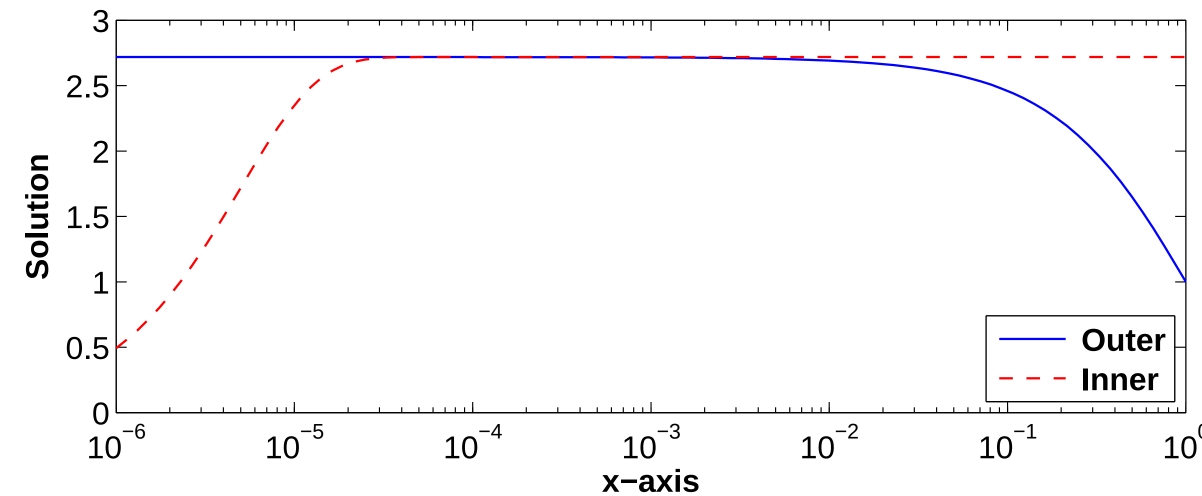
$$Y_0(\bar{x}) = A(1 - e^{-2\bar{x}}) \to A$$

$$\Rightarrow A = e$$



Since both solutions are constant outside of their respective regions, we can construct a composite solution:

$$y \sim y_0(x) + Y_0(\bar{x}) - y_0(0)$$
$$= e^{1-x} - e^{1-2x/\varepsilon}$$



Multiple boundary layers

Our second example illustrates multiple boundary layers and nonconstant coefficients:

$$\varepsilon^2 y'' + \varepsilon x y' - y(x) = -e^x, \quad 0 < x < 1$$
$$y(0) = 2, \quad y(1) = 1$$

For $\varepsilon = 0$ the solution is simply $y(x) = e^x$, which satisfies neither b.c.

Outer solution: $y(x) = e^x$

Boundary layer at x=0: $\bar{x} = \dfrac{x}{\varepsilon^\gamma}$

$$\underset{①}{\varepsilon^{2-2\gamma} Y''} + \underset{②}{\varepsilon \bar{x} Y'} - \underset{③}{Y} = \underset{④}{e^{\varepsilon^\gamma \bar{x}}}$$

Balance: $① \sim ② : \quad 2 - 2\gamma = 1 \ \Rightarrow \ \gamma = 1/2, \mathcal{O}(\varepsilon), \quad ③ \sim \mathcal{O}(1)$

$① \sim ③ : \quad 2 - 2\gamma = 0 \ \Rightarrow \ \gamma = 1, \mathcal{O}(1), \quad ② \sim \mathcal{O}(\varepsilon)$

$$Y'' + \varepsilon \bar{x} Y' - Y = -e^{\varepsilon \bar{x}} \qquad Y = Y_0 + \varepsilon Y_1 + \cdots$$

$\mathcal{O}(1) \quad Y_0'' - Y_0 = -1, \quad Y_0(0) = 2 \quad \Rightarrow \quad Y_0(\bar{x}) = 1 + Ae^{\bar{x}} + Be^{-\bar{x}} \quad \Rightarrow \quad B = (1 - A)$

Matching condition: $\lim_{\bar{x}\to\infty} Y_0 = \lim_{x\to 0} y_0 \qquad \lim_{\bar{x}\to\infty} 1 + Ae^{\bar{x}} = 1 \quad \Rightarrow \quad A = 0$

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For $\varepsilon = 0$ the solution is simply $y(x) = e^x$, which satisfies neither b.c.

Outer solution: $y(x) = e^x$

Boundary layer at x=1: $\tilde{x} = \dfrac{x-1}{\varepsilon^\gamma}$

$$\underset{①}{\varepsilon^{2-2\gamma}\tilde{Y}''} + \underset{②}{\varepsilon^{1-\gamma}(1+\varepsilon^\gamma \tilde{x})\tilde{Y}'} - \underset{③}{\tilde{Y}} = \underset{④}{-e^{1+\varepsilon^\gamma \tilde{x}}}$$

Balance: $① \sim ③: \quad 2 - 2\gamma = 0 \;\Rightarrow\; \gamma = 1, \mathcal{O}(1), \quad ② \sim \mathcal{O}(1)$

$$\tilde{Y}'' + (1+\varepsilon\tilde{x})\tilde{Y}' - \tilde{Y} = -e^{1+\varepsilon\tilde{x}} \qquad \tilde{Y} = \tilde{Y}_0 + \varepsilon\tilde{Y}_1 + \cdots$$

$$\tilde{Y}_0'' + \varepsilon\tilde{Y}_1'' + \cdots (1+\varepsilon\tilde{x})(\tilde{Y}_0' + \varepsilon\tilde{Y}_1' + \cdots) - (\tilde{Y}_0 + \varepsilon\tilde{Y}_1 + \cdots) = -e^{1+\varepsilon\tilde{x}}$$

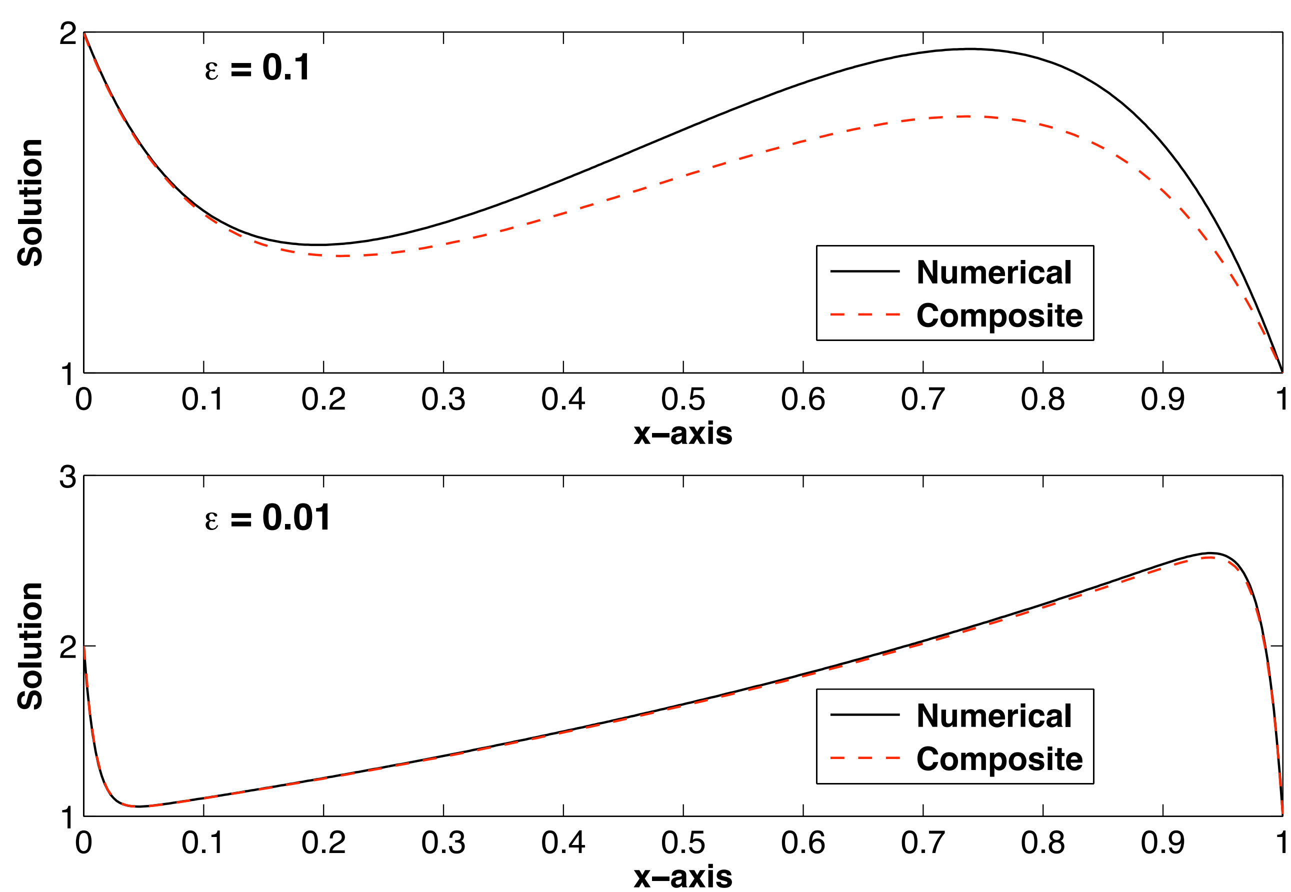
$$\mathcal{O}(1) \qquad \tilde{Y}_0'' + \tilde{Y}_0' - \tilde{Y}_0 = -e, \quad \tilde{Y}_0(0) = 1 \quad \Rightarrow \quad Y_0(\tilde{x}) = 1 + Ae^{r_+\tilde{x}} + Be^{r_-\tilde{x}} \qquad r_\pm = \frac{-1 \pm \sqrt{5}}{2}$$

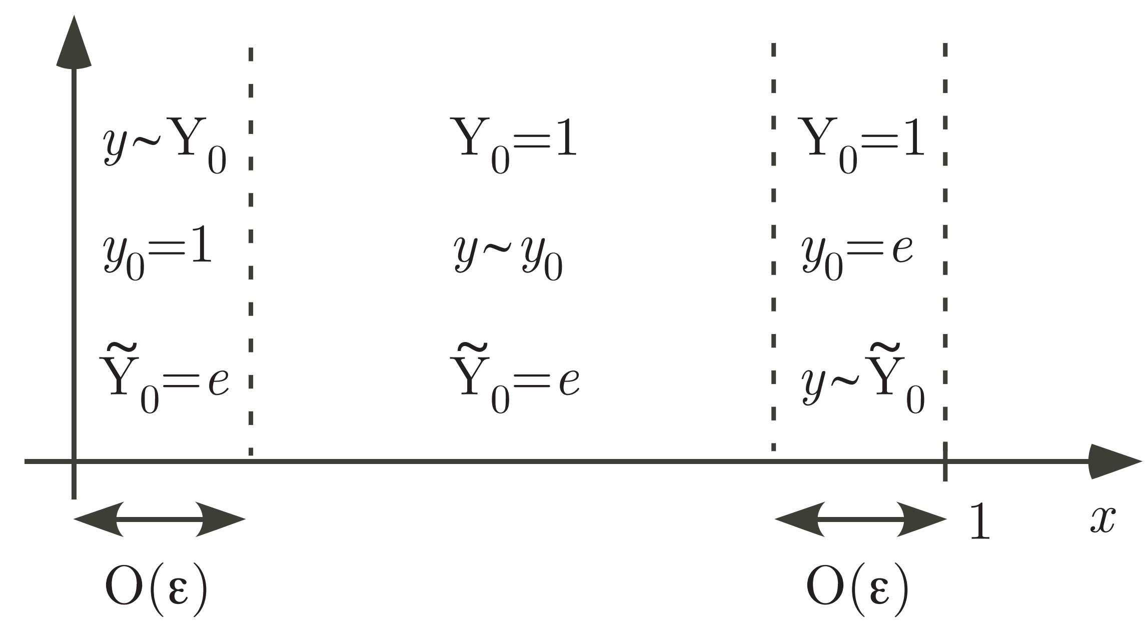
Matching condition: $\lim_{\tilde{x}\to-\infty} \tilde{Y}_0 = \lim_{x\to 1} y_0 \quad \Rightarrow \quad \tilde{Y}_0 = e + (1-e)e^{r_+\tilde{x}}$

Multiple boundary layers

Composite solution:

$$\tilde{y} \sim y_0(x) + Y_0(\bar{x}) + \tilde{Y}_0(\tilde{x}) - y_0(0) - y_0(1)$$
$$= e^x + e^{-x/\varepsilon} + (1-e)e^{r_+(x-1)/\varepsilon}$$





Two time-scales

Pendulum: $\theta'' + \sin(\theta) = 0, \quad \theta(0) = \varepsilon, \quad \theta'(0) = 0$

Regular perturbation analysis: $\theta \sim \varepsilon(\theta_0 + \varepsilon^\alpha \theta_1 + \cdots)$

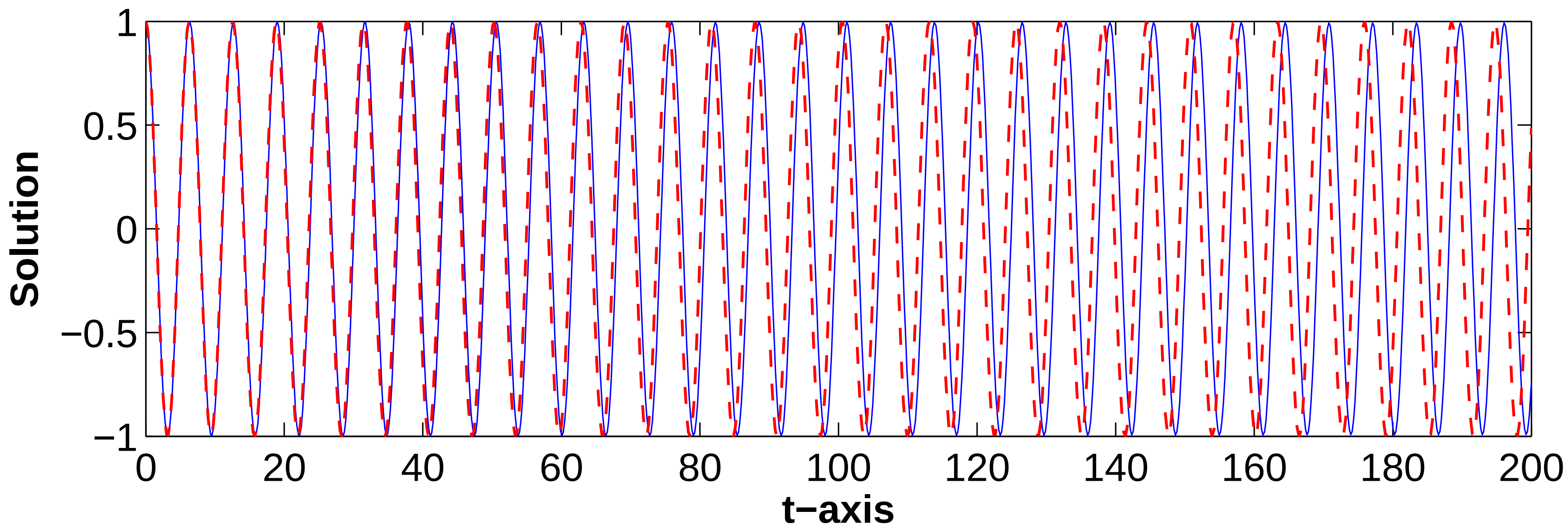
$$\sin(\theta) \sim \sin(\varepsilon(\theta_0 + \varepsilon^\alpha \theta_1 + \cdots)) \sim \varepsilon\theta_0 + \varepsilon^{\alpha+1}\theta_1 - \frac{1}{6}\varepsilon^3\theta_0^3 + \cdots$$

$$\varepsilon\theta_0'' + \varepsilon^{\alpha+1}\theta_1'' + \cdots + \varepsilon\theta_0 + \varepsilon^{\alpha+1}\theta_1 - \frac{1}{6}\varepsilon^3\theta_0^3 + \cdots = 0$$

$$\varepsilon\theta_0(0) + \varepsilon^{\alpha+1}\theta_1(0) + \cdots = \varepsilon$$

$$\varepsilon\theta_0'(0) + \varepsilon^{\alpha+1}\theta_1'(0) + \cdots = 0$$

$\mathcal{O}(\varepsilon) \qquad \theta_0'' + \theta_0 = 0, \quad \theta_0(0) = 1, \theta_0'(0) = 0 \qquad \theta_0(t) = A\cos(t + B) \quad \Rightarrow \quad \theta_0 = \cos t$



Two time-scales

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Regular perturbation analysis: $\theta \sim \varepsilon(\theta_0 + \varepsilon^\alpha \theta_1 + \cdots)$

$$\varepsilon\theta_0'' + \varepsilon^{\alpha+1}\theta_1'' + \cdots + \varepsilon\theta_0 + \varepsilon^{\alpha+1}\theta_1 - \frac{1}{6}\varepsilon^3\theta_0^3 + \cdots = 0$$

$$\varepsilon\theta_0(0) + \varepsilon^{\alpha+1}\theta_1(0) + \cdots = \varepsilon$$

$$\varepsilon\theta_0'(0) + \varepsilon^{\alpha+1}\theta_1'(0) + \cdots = 0$$

$\mathcal{O}(\varepsilon^3) \qquad \theta_1'' + \theta_1 = \frac{1}{6}\theta_0^3, \quad \theta_0(0) = 0, \theta_0'(0) = 0$

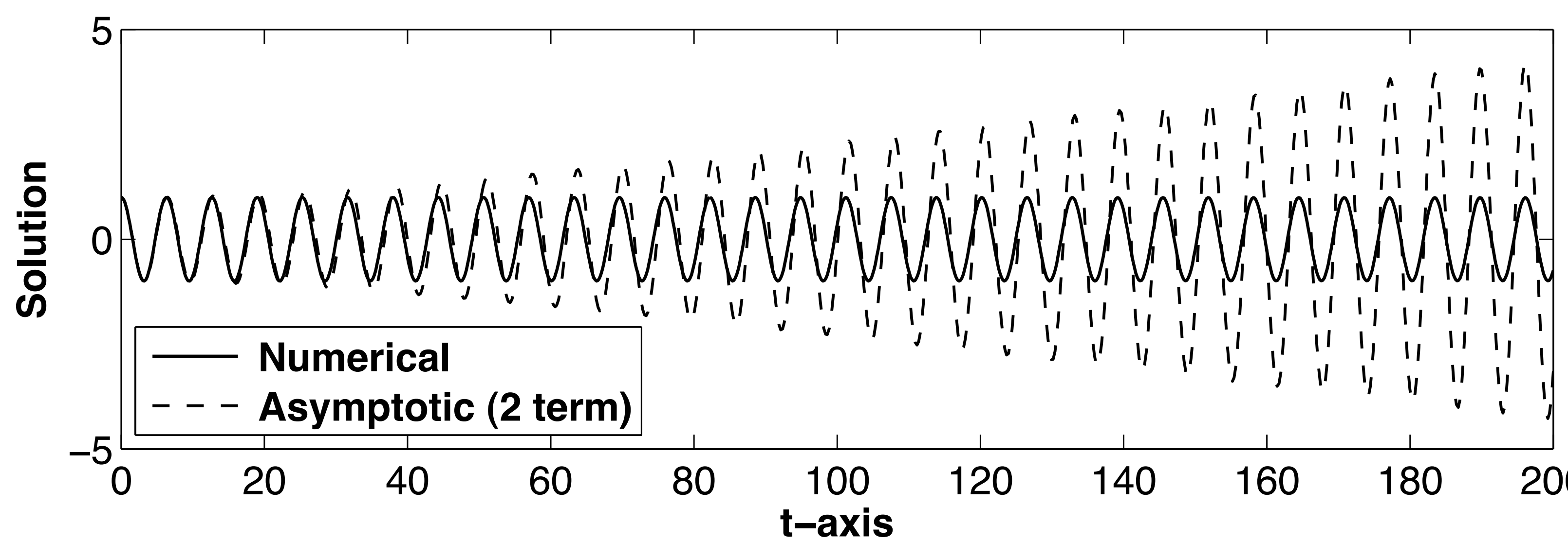
$$\theta_1'' + \theta_1 = \frac{1}{24}(3\cos t + 3\cos 3t)$$

$$\theta_1 = a\cos t + b\sin t - \frac{1}{16}t\sin t$$

$$\theta_1 = -\frac{1}{16}t\sin t$$

Secular term

$$\theta \sim \varepsilon\cos t - \frac{\varepsilon^3}{6}t\sin t + \cdots$$



Two time-scales

Pendulum: $\theta'' + \sin(\theta) = 0, \quad \theta(0) = \varepsilon, \quad \theta'(0) = 0$

The problem with the phase is that for the nonlinear problem, the phase is not constant, but changes slowly. We have two time scales: (1) the time scale on which the oscillations occur, (2) the time scale upon which the phase slowly changes.

We construct an approximation that explicitly uses these scales:

$$t_1 = t, \quad t_2 = \varepsilon^\gamma t$$

$$\frac{d}{dt} = \frac{dt_1}{dt}\frac{\partial}{\partial t_1} + \frac{dt_2}{dt}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon^\gamma \frac{\partial}{\partial t_2}$$

$$\frac{d^2}{dt^2} = (\frac{\partial}{\partial t_1} + \varepsilon^\gamma \frac{\partial}{\partial t_2})^2 = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon^\gamma \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} + \varepsilon^{2\gamma}\frac{\partial^2}{\partial t_2^2}$$

$$\theta \sim \varepsilon(\theta_0(t_1, t_2) + \varepsilon^\alpha \theta_1(t_1, t_2) + \cdots)$$

$$\varepsilon \frac{\partial^2}{\partial t_1^2}\theta_0 + \varepsilon^{\alpha+1}\frac{\partial^2}{\partial t_1^2}\theta_1 + 2\varepsilon^{\gamma+1}\frac{\partial^2}{\partial t_1 \partial t_2}\theta_0 + \cdots + \varepsilon\theta_0 + \varepsilon^{\alpha+1}\theta_1 - \frac{1}{6}\varepsilon^3\theta_0^3 + \cdots = 0$$

$$\varepsilon\theta_0(0,0) + \varepsilon^{\alpha+1}\theta_1(0,0) + \cdots = 0$$

$$\varepsilon\frac{\partial}{\partial t_1}\theta_0(0,0) + \varepsilon^{\alpha+1}\frac{\partial}{\partial t_1}\theta_1(0,0) + \varepsilon^{\gamma+1}\frac{\partial}{\partial t_2}\theta_0(0,0) + \cdots = 0$$

Two time-scales

Pendulum: $\theta'' + \sin(\theta) = 0, \quad \theta(0) = \varepsilon, \quad \theta'(0) = 0$

$$\theta \sim \varepsilon(\theta_0(t_1, t_2) + \varepsilon^\alpha \theta_1(t_1, t_2) + \cdots)$$

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$$\varepsilon\theta_0(0,0) + \varepsilon^{\alpha+1}\theta_1(0,0) + \cdots = 0$$

$$\varepsilon\frac{\partial}{\partial t_1}\theta_0(0,0) + \varepsilon^{\alpha+1}\frac{\partial}{\partial t_1}\theta_1(0,0) + \varepsilon^{\gamma+1}\frac{\partial}{\partial t_2}\theta_0(0,0) + \cdots = 0$$

$\mathcal{O}(\varepsilon)$ $\quad \frac{\partial^2}{\partial t_1^2}\theta_0 + \theta_0 = 0, \quad \theta_0(0,0) = 1, \quad \frac{\partial}{\partial t_1}\theta_0(0,0) = 0$

$$\Rightarrow \quad \theta_0 = A(t_2)\cos(t_1 + B(t_2)) \qquad A(0) = 1, \quad B(0) = 0$$

Two time-scales

Pendulum: $\theta'' + \sin(\theta) = 0, \quad \theta(0) = \varepsilon, \quad \theta'(0) = 0$

Next order in ε is ε^3. Applying balance as for singular perturbations:

$$\alpha + 1 = \gamma + 1 = 3$$

The next order term gives:

$$\mathcal{O}(\varepsilon^3) \qquad \frac{\partial^2}{\partial t_1^2}\theta_1 + \theta_1 + 2\frac{\partial^2}{\partial t_1 \partial t_2}\theta_0 = \frac{1}{6}\theta_0^3 = 0$$

$$\theta_1(0,0) = 0, \quad \frac{\partial}{\partial t_1}\theta_1(0,0) + \frac{\partial}{\partial t_2}\theta_0(0,0) = 0$$

or

$$\theta_1'' + \theta_1 = \frac{1}{24}[3\cos(t_1 + B) + 3\cos(3(t_1 + B))] + 2A'\sin(t_1 + B) + 2AB'\cos(t_1 + B)$$

To avoid a secular (growing in time) term, we may choose

$$A' = 0, \quad 2AB' = -\frac{1}{8} \quad \Rightarrow \quad A = 1, \; B = -t_2/16$$

$$\theta \sim \varepsilon\cos(t - \varepsilon^2 t/16) + \cdots$$