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Lin & Segel's Standing Gradient Problem Revisited: A Lesson in Mathematical Modeling and Asymptotics*

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Abstract. We revisit a physiological standing gradient problem of Lin and Segel from their landmark text on mathematical modeling [C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM, Philadelphia, 1988] with a view to giving it an up-to-date perspective. In particular, via an alternative nondimensionalization, we show that the problem can be analyzed using the tools of singular perturbation theory and matched asymptotic expansions. In the spirit of the aforementioned authors, the development is didactic in style. Solving the problem requires many of the necessary skills of continuous modern mathematical modeling: formulation from a physical description of the process, scaling and asymptotic simplification, and solution using advanced perturbation (boundary layer) techniques.

Key words. scaling, asymptotics, mathematical modeling, formulation, boundary layers, standing gradient, osmosis, active transport

AMS subject classifications. 97M, 97M60, 34-01, 76-01

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1. Introduction.

The art and science of mathematical modeling (applied mathematics/industrial mathematics are near synonyms) are perhaps more popular now than ever. Certainly public awareness of the discipline has increased. Mathematics is the invisible glue which binds our technology together. This is obvious to experts, but less so to the man on the street, who is commonly almost proud of the fact that "I was never any good at math," so the argument will bear indefinite repeating. Put simply, it is hard to think of any technology in which mathematics is not in some way inextricably involved. Weather prediction, climate change, flood prevention; electricity, water supply, sewage treatment; roads, buildings; airline scheduling, supermarket restocking; they all involve mathematics at some fundamental level, but not one which is necessarily visible to the consumer (or indeed the provider). Almost all science is built around the concept of a mathematical model. If one wants to find out how climatic temperatures fluctuated during the last hundred thousand years (a lot is the answer, with some alarmingly sudden switches), one might drill an ice core in Greenland and painstakingly measure the variation of oxygen isotopes in the ice. This variation is

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sensitive to the temperature at the time the water which forms the ice was evaporated from the ocean. But to find that temperature, one needs a mathematical model that tells how the two are related. Without the model, the huge expenditure to drill the ice is pointless.

The classical models of fluid flow, heat transfer, solid mechanics, and electromagnetism were developed in the 19th century. Early in the 20th century, mathematics was spurred on by advances in physics. Applied mathematics became synonymous with continuum mechanics around the time of the second world war. Fluid and solid mechanics formed the basis of the Cambridge school, developed first by G. I. Taylor and later George Batchelor, which flourishes to this day. More recently, fluid mechanics has been heavily supported by numerical analysis as computing power has increased.

I became aware of the modeling book of Lin and Segel [12] in the late 1980s when it was reissued as a classic in applied mathematics by SIAM. But it is only recently that a colleague, Andrew Fowler, pointed out how new and innovative their approach seemed at the time. This was corroborated by the recent review of Mark Holmes' new modeling book [7] in *SIAM Review* [14] by J. David Logan. Lin and Segel were using what were then regarded as exotic tools such as nondimensionalization and scaling to simplify real continuous problems into tractable systems of differential equations. In so doing they espoused the philosophy that problems in all areas where quantitative ideas occur can be dealt with mathematically. More or less simultaneously, Alan Tayler [22] and Leslie Fox were following a similar avenue and, in comparison to Cambridge, a different school was emerging at Oxford which focused on the applications of mathematics in industry (or industrial mathematics). The famous Oxford Study Groups have now been successfully exported worldwide, and the Oxford school is a world leader in the wider applications of mathematics.

Mathematical modeling is a subject which thus developed intensively over the last century, and since the 1960s it has widened its influence from such prewar endeavors as aeronautics to encompass applications in increasingly diverse fields: materials science, semiconductors, groundwater quality, ore smelting, drug design, finance, and cooking, to name but a few. In the 1970s, mathematical biology emerged as a new and exciting application area providing new paradigms in the study of reaction-diffusion equations. The other most obvious recent growth area has been in financial mathematics, which is following a similar pattern to mathematical biology. There are many other fields of application which are thriving, and some of these are ripe for similar sorts of explosive growth, with the right seeding. These include environmental mathematics, materials science, bioengineering, and general methods of risk, uncertainty, and data analysis.

Much of this development has been aided by the formation of the European Study Groups with Industry (ESGI) in Oxford in 1968 and their subsequent export round the world, to Canada, the U.S., China, India, South Africa, Mexico, and so on, and later the formation of the European Consortium for Mathematics in Industry (ECMI). At an international level the establishment of SIAM consolidated the early activities and it remains the keystone of such mathematical activity. This high-level modeling activity provides potential benefits to industry which go far beyond the more mundane levels alluded to earlier: a kind of super-glue. Gradually, the concept of "industrial mathematics" has become imbedded in the mathematical community. In this context, "industry" is interpreted in a very broad sense: the remit of these groups includes more than collaboration with industry (for example, problems may come from biology, finance, the environment, or energy). I do not wish to get involved in a discussion of whether there is any difference between industrial mathematics,

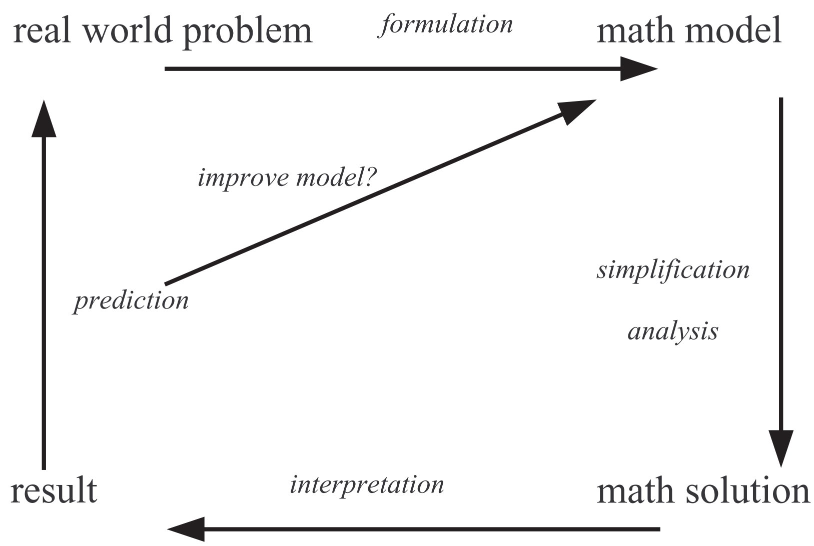

Fig. 1 *The process of modeling real problems mathematically.*

mathematical modeling, and indeed applied mathematics. Suffice it to say that the area thrives when it does what it claims to do: use mathematics to solve problems arising outside of mathematics.

The essential program of the applied mathematician is this: identify the problem of concern, build a mathematical model, analyze and solve it, then apply the results. The emphasis is on the science of the application, i.e., in developing an understanding of the underlying mechanisms involved. It is an iterative procedure, since if the results do not explain or fit the observations, one adapts the model and repeats the cycle, although this process is not often visible in published research. This procedure of necessity involves an active collaboration with the source of the application. The art of modeling is a subtle one and not at all trivial, involving a thorough understanding of many different scientific disciplines and a practical knowledge of how equations "work." Even the art of *formulating* a mathematical model is a complex interdisciplinary exercise. The task of analysis of the resulting complex, nonlinear systems often involves an array of asymptotic methods. For any particular problem, the basic phenomena (e.g., physical, economic, etc.) are described by corresponding mathematical expressions; nondimensionalization and scaling techniques are used leading to mathematical simplification. The reduced model is then analyzed using approximate or asymptotic methods where appropriate. Numerical solutions may also be obtained to complement and extend the approximate solutions. Finally, the mathematical solutions must be interpreted in the context of the original problem. Gaining insight into, and understanding of, the original problem is at least as important as the mathematics itself. The process is summarized in Figure 1.

Citing historical precedent, one can predict with confidence that novelty will follow from the application of mathematics to real world problems [22], [19]. An obvious example of this is Newton and the development of calculus. This arose from his practical investigations in mechanics and geometry and the need to measure areas under curves. Similarly, differential geometry was invented in the context of making maps and surveying a region of the earth. Fourier's desire to understand heat conduction problems led to the development of modern Fourier theory. Fluid mechanics has fueled perturbation techniques (Prandtl boundary layers, singular perturbation theory, multiple scales, exponential asymptotics) which now form a deep platform for the study of differential equations and have led to significant scientific advances in many areas outside fluid mechanics. Indeed, our current understanding of chaos and turbulence arose from Lorentz's investigation of a simplified model for convection in the atmosphere. As is typical in applied mathematics, the scientific understanding which arose as a result has turned out to be a common (universal) feature of many nonlinear systems.

It is in the context of the above discussion that the significance of the milestone text of Lin and Segel [12] becomes clear. The book includes a discussion on the nature of applied mathematics and introductory material on ODEs, PDEs, random processes, Fourier analysis, and continuum mechanics. But the heart of the book and its most novel aspect (at the time of publication) is a description of the process of nondimensionalization, scaling, and simplification of systems of differential equations and their subsequent solution using asymptotic methods. In particular, they demonstrate how to deal with a real problem from a qualitative description through the mathematical formulation, simplification, approximate solution, and, importantly, the interpretation of the mathematical results in terms of the original process. Especially noteworthy is the attention paid to the formulation of the problem and interpretation of the solutions, a key element of real mathematical modeling (see Figure 1).

In the European context, in an attempt to unify mathematics syllabi across Europe, the Mathematics Subject Area Group was unanimous[1] in identifying three skills which it believes *every* mathematics graduate should acquire: the abilities to conceive a proof, to solve problems using mathematical tools, and *to model a situation.* To this day, this last key skill is often neglected in third-level mathematics courses. There is still a tendency in many universities to relabel a differential equations course as a course in mathematical modeling, paying lip service to the importance of teaching real modeling skills. This is emphasized in [1], an extreme (but thought-provoking) view on the mathematical needs of industry from the point of view of a pure mathematician who moved from academia to industrial consultancy. In particular, from the point of view of the industrialist who needs mathematical support, Beauzamy claims that "it (mathematics) brings solutions which nobody understands to questions nobody asked." Of course, there are now many textbooks which have helped to popularize and extend the skills of the mathematical modeler, for example, [4], [5], [9], [16], [19], (all from the Oxford school), [3], [13], [10], [15], and, most recently, [7]. This list is, of course, far from exhaustive.

One can surmize that Lin and Segel's book has been used extensively as a course textbook around the world. One particular problem which they study is associated with standing gradient flow in physiology, and on closer examination it became clear to me that this problem deserves a reappraisal from the point of view of basic modeling techniques. The problem is presented by Lin and Segel as a physiological phenomenon whose mechanism is not well understood. Within the body, it was known that diffusion on its own could not explain the way in which solute is secreted by various tissues into the fluid adjacent to them. Physiologists had proposed a qualitative hypothesis

[1] "Tuning educational structures in Europe" in *Towards a Common Framework for Mathematics Degrees in Europe*, Newsletter of the European Mathematical Society, September 2002, pp. 26–28, http://www.emis.de/newsletter/newsletter45.pdf.

of *active transport* by which the solute is chemically pumped into the ends of long thin channels, a structural feature of tissues that transport fluid. In the spirit of the discussion on climatic temperatures in the first paragraph of this introduction, what was lacking was a quantitative mathematical model whose predictions could be compared with experimental results. Lin and Segel formulate such a mathematical model (and ultimately successfully match its predictions with experiment). Using scaling arguments, they present the final dimensionless model as a regular perturbation problem. Though the problem involves advection and diffusion, there is no mention of a Péclet number. The driving force term is modeled as being small (so that leading order solutions are trivial). We will show via a more informative nondimensionalization that the problem can be treated as a singular perturbation problem. Indeed, our alternative scaling shows the power of scaling in obtaining a simple leading order solution (where the driving force term makes an $O(1)$ contribution) which captures the essential mechanism of the process. Matched asymptotic expansions are used to improve on this and flesh out the details. In fact Lin and Segel's somewhat arbitrary choice of a step function source term also gives rise to the issue of exponentially small terms, but these can be resolved by continuity arguments in this example.

In summary, this is an excellent didactic example which deserves a reappraisal, and one might regard this note as an appendix to the Lin and Segel chapter on this topic. The present paper is predominantly concerned with aspects of the technical solution of the problem. The interpretation of the results is not discussed for the simple reason that the original presentation of Lin and Segel is very thorough. Finally, we note that, as is the case with most problems in applied mathematics, the model has universal features which would allow it to be adapted to many other physical situations, e.g., modeling the absorption of nutrients into the roots of a plant [21].

In section 2, we present the basic model; in section 3, we scale it; in section 4, we obtain various numerical and asymptotic solutions. Finally, section 5 is a short summary.

2. The Model. We consider the situation in Figure 2. Solutes (i.e., salts) are secreted by various tissues into the fluid (solvent) that is adjacent to them, or "bathes" them. For example, several different salts are secreted by the tissues of the kidney into the fluid that flows in kidney tubules. In the cases of interest the solute concentration in the bathing fluid is usually greater than the average concentration of solute in the secreting tissue, so the principal mechanism for solute secretion cannot be diffusion. Another possible mechanism is active transport by the membrane that forms the boundary between the tissues and the bathing fluid. Here, chemical "pumps" can expend energy to move solute across a membrane in a given direction, more or less independently of the concentrations of solute in the fluids bathing the membrane. In particular, an actively transporting membrane can pump solute from a region of low concentration to a region of high concentration.

It has been shown experimentally that active transport cannot be directly involved in the phenomenon of interest, i.e., the solute is not pumped directly into the bathing fluid.

As explained in [12], one hypothesis proceeds from the observation that long narrow channels are *consistently* a structural feature of tissues that transport fluid, e.g., infoldings in kidney tubules. The conjecture is that solute is actively transported in the "far" end (the closed end) of the channels, deep in the secreting tissue (see Figure 2). The relatively high concentration in the far end is then diluted as the fluid passes toward the opening of the channel because water is drawn into the channel by

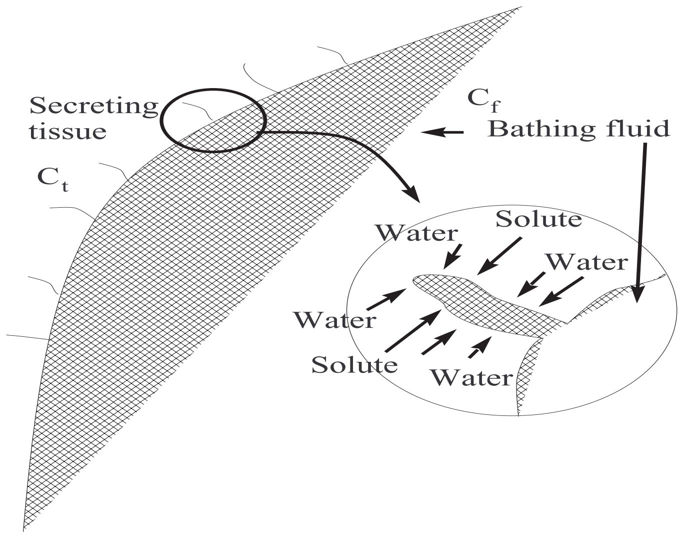

Fig. 2 *Long thin channels, as found in secreting tissues. Solute is actively pumped into each channel near its closed end; water is also drawn in by osmosis along the length of the channel so that the solution that emerges at the open end of the channel (where it meets the bathing fluid) is considerably diluted.*

osmosis due to this high concentration. A *standing gradient* of solute concentration is soon set up. The concentration is high at the closed "far" end of the channel (the left-hand end in the model problem of Figure 3) and decreases to a relatively low value at the other open "secreting" end.

Following [12], we model the problem as in Figure 3. We consider the transport of the solute and solution in a cylinder of radius r, length L with active transport of the solute across the boundary occurring over a short length at the tip of the cylinder ($x \in [0, \delta]$) and osmosis of water into the cylinder occurring along its length.

Solute Transport. Dimensional variables will be labeled with an asterisk. The solute (with concentration c^*) is transported into the tube via active transport and moves along the tube via advection and diffusion. The conservation law is

$$\frac{\partial c^*}{\partial t^*} + \nabla \cdot (\mathbf{u}^* c^*) = D \nabla^2 c^*, \tag{1}$$

where D is the diffusion constant.

Solution Transport. The solution fluid advects along the tube while solvent seeps into the tube along its length via osmosis. The conservation law is

$$\frac{\partial \rho}{\partial t^*} + \nabla \cdot (\rho \mathbf{u}^*) = 0, \tag{2}$$

where $\mathbf{u}^*$ is the fluid velocity and ρ is the density. If the density is constant, this simplifies to

$$\nabla \cdot \mathbf{u}^* = 0. \tag{3}$$

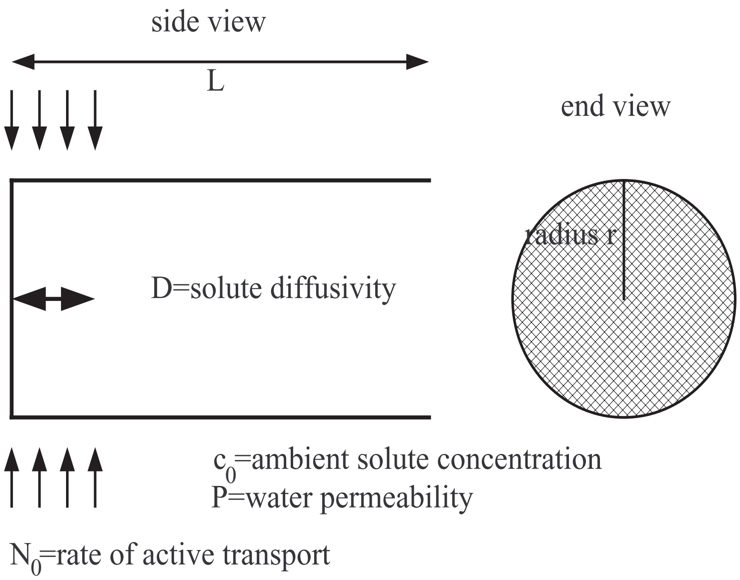

Fig. 3 *The simplified standing gradient flow system. The left-hand end is closed, the right-hand open.*

One would also have to include a model for the flow in the two-dimensional case. Active transport and osmosis through the side walls are included as flux boundary conditions for (1) and (3), but in the one-dimensional model that we propose below (where the side wall is no longer a boundary), this effect will be modeled as a source term.

One-Dimensional Model. In the case of steady one-dimensional flow, (1) and (3) reduce to

$$\frac{d[u^*(x^*)c^*(x^*)]}{dx^*} = D\frac{d^2c^*(x^*)}{dx^{*2}} + \frac{2}{r}N^*(x^*), \tag{4}$$

$$\frac{du^*(x^*)}{dx^*} = P\frac{2}{r}(c^*(x^*) - c_0), \tag{5}$$

where source terms have been added to the right-hand side of this one-dimensional model to account for the effects of active transport and osmosis through the side walls of the higher-dimensional model. N^* is the active transport rate per unit area of lateral boundary, c_0 is the ambient concentration, $\hat{P}$ is the rate at which solvent enters via osmosis per unit length of perimeter, and we define $P \equiv \hat{P}/\rho$. Note that the ratio of the cross-sectional area of the cylinder to its circumference is $2\pi r/\pi r^2 = 2/r$.

The appropriate boundary conditions for the one-dimensional model are

$$\begin{aligned} &u^*(x^* = 0) = 0, \\ &\frac{dc^*}{dx^*}(x^* = 0) = 0, \\ &c^*(x^* = L) = c_0, \end{aligned} \tag{6}$$

corresponding to no flow and zero solute flux at $x^* = 0$ and the solute concentration adjusting to the ambient concentration at $x^* = L$.

The mathematical problem thus reduces to solving this coupled system of ODEs for the solute concentration $c^*(x^*)$ and the solution velocity $u^*(x^*)$.

3. Scaling the Problem. In the original approach to the problem [12], Lin and Segel went to great lengths to justify the scales at which they finally arrived via a somewhat circuitous route. Perhaps too much attention is paid to the attempts to find suitable scales for each individual quantity and not enough to the fact that the ultimate aim of scaling is to bring out the balances in the equations, i.e., to make explicit which terms dominate the process. In this problem there are only three variables u^*, c^*, x^* and there are a limited number of possible balances. The least obvious is perhaps the velocity scale, as there is no single explicit velocity in the problem. The fluid is driven by the osmosis and this in turn is driven by the active transport of the solute so the velocity scale has to be inferred. We begin by writing

$$\begin{aligned} c^*(x^*) &= c_0 + c(x)C_s, \\ u^*(x^*) &= u(x)V_s, \\ x^* &= xL, \\ N^*(x^*) &= N_0 N(x), \end{aligned} \tag{7}$$

where N_0 is a typical value of the dimensional active transport rate; see Table 1. The velocity scale is then chosen by balancing the larger of the advective terms ($c^* \frac{du^*}{dx^*} \sim c_0 V_s/L$) and the last term in (4), i.e., balancing advection and active transport (a similar result would be obtained if we balanced diffusion and the active transport term). Once this has been chosen, the concentration scale is obtained by subtracting off the ambient concentration and then balancing both sides of (5). This is important because c_0 is not a good overall scale for c^*, which will typically show small fluctuations (as measured by $C_s \ll c_0$, to be verified a posteriori) about an "average value" c_0. It also means that the two advective terms are of differing sizes. Thus we choose

$$\begin{aligned} V_s &= \frac{2N_0 L}{rc_0}, \\ C_s &= \frac{V_s r}{2LP} = \frac{N_0}{c_0 P}, \end{aligned} \tag{8}$$

and the dimensionless system is

$$\frac{d(u(1+\varepsilon c))}{dx} = \frac{\varepsilon}{\mathrm{Pe}} \frac{d^2 c}{dx^2} + N(x), \tag{9}$$

$$\frac{du}{dx} = c, \tag{10}$$

where $\mathrm{Pe} \equiv \frac{V_s L}{D} \equiv \frac{2N_0 L^2}{Drc_0}$, $\varepsilon \equiv C_s/c_0 \ll 1$. Pe/ε is a modified Péclet number: it is a measure of the relative sizes of the larger advective term and diffusion (cf. the modified Reynolds number of thin film flow [17], [18]). The parameter values suggested in [12] are reproduced in Table 1. These will be used to inform the nondimensionalization process although we will allow ourselves a little latitude in the development of our approximate solutions. From the typical values, we verify that $\varepsilon = 1/18 \ll 1$, i.e.,

Table I *Parameter values as given in* [12].

Parameter	Units	Minimum value	Typical value	Maximum value
r	cm	10^{-6}	5×10^{-6}	10^{-4}
L	cm	4×10^{-4}	10^{-2}	2×10^{-2}
δ	cm	4×10^{-5}	10^{-3}	2×10^{-3}
D	cm^2 /sec	10^{-6}	10^{-5}	5×10^{-5}
N_0	$\mathrm{mOsm/cm}^2$ sec	10^{-10}	10^{-7}	10^{-5}
P	cm^4 /sec mOsm	10^{-6}	2×10^{-5}	2×10^{-4}
c_0	$\mathrm{mOsm/cm}^3$	-	3×10^{-1}	-

$C_s \ll c_0$ and $\mathrm{Pe} = 4/3 = O(1)$. The dimensionless boundary conditions are simply

$$\begin{aligned} &u(x=0) = 0, \\ &\frac{dc}{dx}(x=0) = 0, \\ &c(x=1) = 0, \end{aligned} \tag{11}$$

and the problem is driven by the inhomogeneous active transport term. This approach has ignored the details of the active transport term thus far. As $N^* = N_0\, N(x)$, there is another length scale implicit in $N^*(x^*)$ which may be different from L. Lin and Segel use the particular step function form

$$\begin{aligned} N^*(x^*) &= N_0, \quad 0 \le x^* \le \delta, \\ &= 0, \quad \delta < x^* \le L, \end{aligned} \tag{12}$$

where the interval of active transport is $[0, \delta]$. In the present formulation this becomes

$$\begin{aligned} N(x) &= 1, \quad 0 \le x \le \gamma, \\ &= 0, \quad \gamma < x \le 1. \end{aligned}$$

Though the parameter $\gamma \equiv \delta/L \approx 0.1 \ll 1$, we will first seek solutions on the assumption that $\gamma = O(1)$. This has ramifications which we discuss later.

We thus seek a perturbation solution of (9), (10), and (11) based on $\varepsilon \ll 1$.

4. Perturbation Solutions.

4.1. A Leading Order Approximation. One of the advantages of the approach here (in comparison with the original [12]) is that we can obtain a very simple leading order approximation purely on the basis of one small parameter $\varepsilon \ll 1$. The leading order approximation on $0 < x < 1$ is then

$$\frac{du}{dx} = N(x), \tag{13}$$

$$\frac{du}{dx} = c, \tag{14}$$

subject to boundary conditions (11), and this yields the simple solutions

$$\begin{aligned} u(x) = \int N(x)\, dx &= x, \quad 0 \le x \le \gamma, \\ &= \gamma, \quad \gamma \le x \le 1, \end{aligned} \tag{15}$$

$$c(x) = N(x). \tag{16}$$

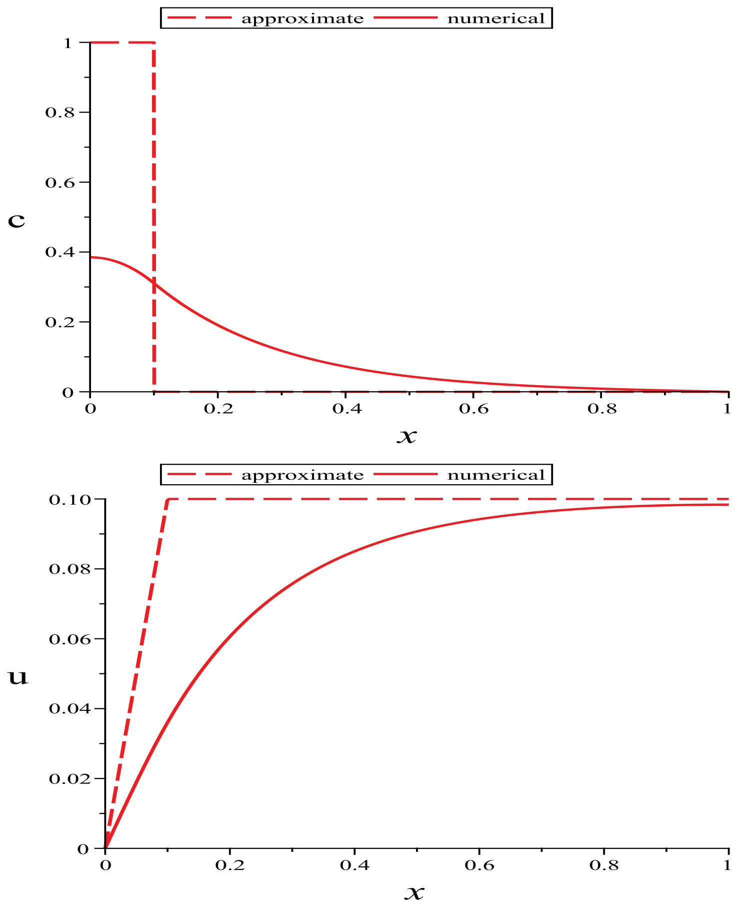

Fig. 4 *Comparison of numerical and leading order asymptotic solution for the case* $\varepsilon = 1/18$, $\mathrm{Pe} = 4/3$, $\gamma = 0.1$.

The simplicity of this basic result was perhaps not brought out previously, although for the parameter values of Table 1 this should be regarded as a first step in the development of the solutions. The whole process is driven by the active transport terms, and the basic result is that u and c mimic $N(x)$ and its integral. From the point of view of elucidating the fundamental mechanism of the process, some might regard the analysis as being finished at this point with the understanding that the diffusion terms omitted at leading order will have a smoothing effect on the solutions. But, of course, one can technically improve upon this leading order model. Before proceeding we provide a comparison of the numerical solution of the full problem and the approximation (15), (16) using the actual parameter values in Figure 4 and a much smaller value of ε in Figure 5. In Figure 5, the approximation is not bad, but the same cannot be said of Figure 4 particularly at the left-hand edge. In fact, as we discuss later, this is an artifact of the actual parameter values $\varepsilon \ll 1, \gamma \ll 1$ and in particular the fact that $\sqrt{\varepsilon} \sim \gamma$. In the interests of generality, we will first develop solutions on the basis that $\gamma = O(1)$, noting that the typical value of δ in Table 1 is open to discussion and may in fact be five times as large. We show later for the "typical" parameter values that there is a boundary layer of $O(\sqrt{\varepsilon})$ about $x = \gamma$ and

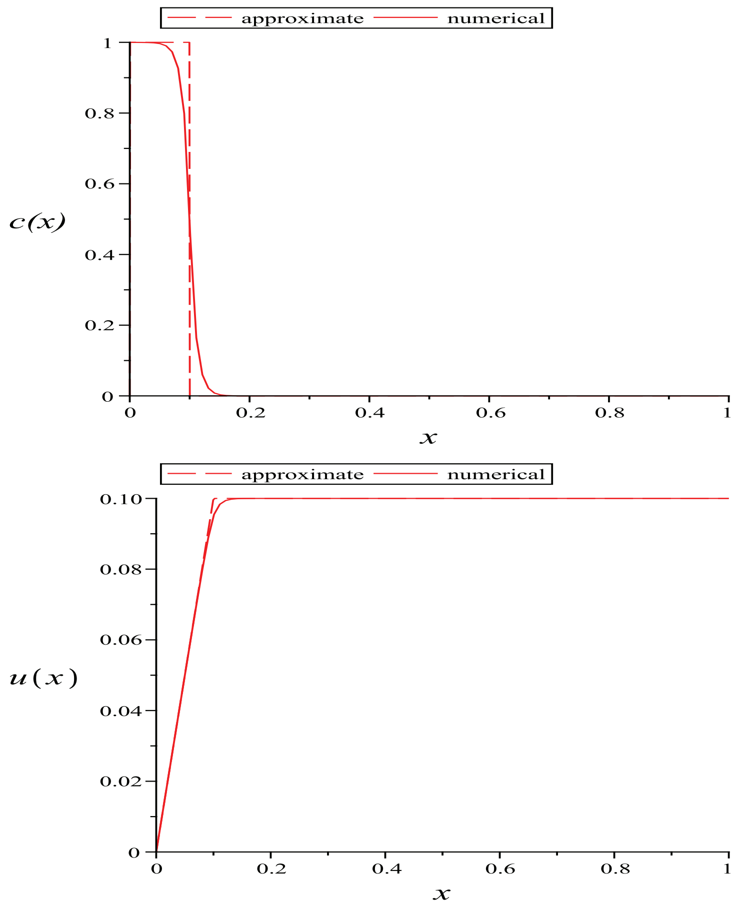

Fig. 5 *Comparison of numerical and leading order asymptotic solution for the case* $\varepsilon = 10^{-4}$, $\mathrm{Pe} = 1,\ \gamma = 0.1$.

so the neighborhood of $x = 0$ requires a boundary layer rescaling. To illustrate the basic soundness of the approach, we demonstrate that the approximation improves if $\gamma = O(1) = 0.5$, say, or if we take much smaller values of ε (moving the boundary layer away from $x = 0$). This is illustrated in Figure 5.

In the next section we show that the asymptotic error for the leading order solutions (15) and (16) is expected to be $O(\sqrt{\varepsilon})$. By way of illustration of the consistency of the asymptotic approach, the error in the asymptotic solution for $u(x)$ in (15) was computed for the parameter values illustrated in Figure 5 while varying the value of ε. The resulting error plot is shown in Figure 6. Taking the numerical solutions to be exact, we see that for $\varepsilon = 10^{-2}$, 10^{-3}, and 10^{-4} the maximum error (which occurs at $x = \gamma$) is, respectively, $0.43\sqrt{\varepsilon}$, $0.47\sqrt{\varepsilon}$, and $0.45, \sqrt{\varepsilon}$.

4.2. Boundary Layer Analysis. Some of the difficulty in this problem arises because of the choice of $N^*(x^*)$ or the dimensionless $N(x)$ as a step function in (12). In particular, when integrating the ODEs this guarantees that c, u are not analytic (there is a salient undergraduate-level discussion of analytic real valued functions in [20]). Here this means that all solutions must be artificially joined at the point $x = \gamma$

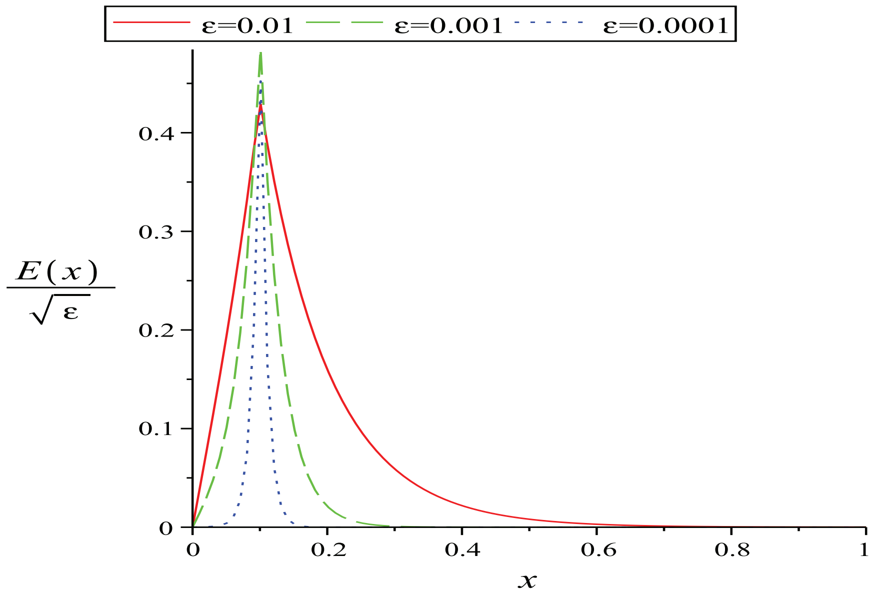

Fig. 6 *Plot of $E(x)/\sqrt{\varepsilon}$, where $E(x)$ is the error in the leading order asymptotic solution* (15) *compared to the numerical solution. The parameter values are* $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$, $\mathrm{Pe} = 1$, $\gamma = 0.1$.

by demanding continuity of u, c. This approach is quite similar to the continuity requirements imposed at inflection points in sessile drops in [2]. The discontinuity in $N(x) = c(x)$ as in (16) and the corner in its integral $(u(x))$ cannot be completely resolved without a boundary layer analysis in the vicinity of $x = \gamma$. The diffusive terms $O(\varepsilon/\mathrm{Pe})$ of (9) are significant in this region. There are a couple of ways of dealing with this issue. The obvious attempt is to develop a matched asymptotic expansions solution with an inner layer [8] near $x = \gamma$, matching this to outer solutions to the left and right as given by (15) and (16). In fact, the boundary layer gives rise to exponentially small terms which are the reason that the leading order approximation has the correct basic features. In the appendix, we discuss a model problem with the same features. These terms are an essential component of the solution in the vicinity of the corner: ordinarily they would be a stumbling block for the complete resolution of the problem using matched asymptotic expansions. In the present problem the additional continuity requirement allows these terms to be accommodated.

We now examine two distinct scenarios using matched asymptotics [6]. The first is designed to improve on the nonsmooth solutions of the leading order approximations (15) and (16) by the insertion of appropriate corner layers. Classical corner layers [8] often have an analytic structure (i.e., the exact solution across the corner has a single functional representation). That is not the case with the present problem: it is necessary to insert a boundary layer on either side of the corner at $x = \gamma$ and to impose appropriate continuity requirements at this point. The second scenario considers the case where the smallness of γ is taken into consideration. This is achieved by choosing an appropriate distinguished limit based on the numerical values in Table 1. Here we choose $\gamma = O(\sqrt{\varepsilon})$ as being appropriate (and giving a rich balance), though other choices are possible. Each choice will give reasonable solutions provided the asymptotic limiting process reflects the actual numerical parameters.

4.2.1. Solutions Based on $\varepsilon \ll 1, \gamma = O(1), \mathrm{Pe} = O(1)$. First we construct a smoothed version of (15) and (16). Referring to (9) and (10), it is simplest to reduce to a single third order equation for $u(x)$. Referring to Figure 5, we will thus construct a solution made up of a left outer $u_l(x)$, a left inner (corner layer) $\bar{u}_l(\bar{x})$, a right inner $\bar{u}_r(\bar{x})$, and a right outer solution $u_r(x)$. It is necessary to proceed beyond the leading order in order to capture the details of the corner layers.

The left outer problem has the same scaling as (9) and (10). Noting that $N(x) = 1$, $x < \gamma$, we have

$$u_l' + \varepsilon(u_l u_l')' = \frac{\varepsilon}{\mathrm{Pe}} u_l''' + 1, \tag{17}$$

$$u_l(x = 0) = 0, \tag{18}$$

$$u_l''(x = 0) = 0, \tag{19}$$

and the extra information will be provided via asymptotic matching. In the vicinity of the corner, we rescale via

$$x = \gamma + \sqrt{\varepsilon}\bar{x},$$
$$u(x) = \bar{u}_l(\bar{x}),$$

and the left inner layer problem is

$$\bar{u}_l' + \sqrt{\varepsilon}(\bar{u}_l \bar{u}_l')' = \frac{1}{\mathrm{Pe}} \bar{u}_l''' + \sqrt{\varepsilon}. \tag{20}$$

The right inner problem is

$$\bar{u}_r' + \sqrt{\varepsilon}(\bar{u}_r \bar{u}_r')' = \frac{1}{\mathrm{Pe}} \bar{u}_r'''. \tag{21}$$

At $x = \gamma$, i.e., $\bar{x} = 0$, we will demand continuity of u and u', i.e.,

$$\bar{u}_l(\bar{x} = 0) = \bar{u}_r(\bar{x} = 0),$$
$$\bar{u}_l'(\bar{x} = 0) = \bar{u}_r'(\bar{x} = 0),$$

and $\bar{u}_r$ will then be matched with right outer solution $u_r(x)$ governed by

$$u_r' + \varepsilon(u_r u_r')' = \frac{\varepsilon}{\mathrm{Pe}} u_r''',$$
$$u_r'(x = 1) = 0. \tag{22}$$

Solutions. Proceeding first with the left outer problem (17), we seek a solution in the form

$$u_l \sim u_{l0} + \varepsilon u_{l1} + \cdots$$

and we find that

$$u_{l0} = x,$$
$$u_{l1} = -x, \tag{23}$$

satisfying the leading order version of the two boundary conditions (17), so

$$u_l \sim x - \varepsilon x \ldots. \tag{24}$$

The left inner problem (20) has the solution

$$\bar{u}_l \sim \bar{u}_{l0} + \sqrt{\varepsilon}\bar{u}_{l1}$$

with

$$\bar{u}_{l0} = d_1 + d_2 e^{\sqrt{\mathrm{Pe}}\bar{x}} + d_3 e^{-\sqrt{\mathrm{Pe}}\bar{x}} \tag{25}$$

and

$$\begin{aligned}\bar{u}_{l1} = {}& \frac{d_5\, e^{\sqrt{\mathrm{Pe}}\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{d_4\, e^{-\sqrt{\mathrm{Pe}}\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{3}{4}\sqrt{\mathrm{Pe}}\, d_1\, d_2\, e^{\sqrt{\mathrm{Pe}}\bar{x}} + \frac{1}{2}\mathrm{Pe}\, d_1\, d_2\, \bar{x} e^{\sqrt{\mathrm{Pe}}\bar{x}} \\ & + \bar{x} + \frac{1}{3}\sqrt{\mathrm{Pe}}\, {d_2}^2 e^{2\sqrt{\mathrm{Pe}}\bar{x}} + \frac{1}{2}\mathrm{Pe}\, d_3\, d_1\, \bar{x} e^{-\sqrt{\mathrm{Pe}}\bar{x}} + \frac{3}{4}\sqrt{\mathrm{Pe}}\, d_3\, d_1\, e^{-\sqrt{\mathrm{Pe}}\bar{x}} \\ & - \frac{1}{3}\sqrt{\mathrm{Pe}}\, {d_3}^2 e^{-2\sqrt{\mathrm{Pe}}\bar{x}} + d_6.\end{aligned} \tag{26}$$

Using extended van Dyke matching [23], we write the outer solution (24) in terms of the inner variable $\bar{x}$ and expand with $\bar{x} = O(1)$, $\varepsilon \ll 1$, truncating after $O(\sqrt{\varepsilon})$. This yields

$$(u_l)_{\mathrm{inner}} \sim x + o(\sqrt{\varepsilon}).$$

Using the analogous procedure with the left inner solution (but ultimately aiming to truncate after $O(\varepsilon)$), we obtain

$$\begin{aligned}(\bar{u}_l)_{\mathrm{outer}} = {}& d_1 + d_2\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} + d_3\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & + \sqrt{\varepsilon}\Bigg(d_5\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\mathrm{Pe}}} - d_4\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\mathrm{Pe}}} - \frac{3}{4}\sqrt{\mathrm{Pe}}\, d_2\, d_1\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad + \frac{1}{2}\mathrm{Pe}\, d_2\, d_1\, (x-\gamma)\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\varepsilon}} + \frac{x-\gamma}{\sqrt{\varepsilon}} + \frac{1}{3}\sqrt{\mathrm{Pe}}\, {d_2}^2 e^{2\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad + \frac{1}{2}\mathrm{Pe}\, d_3\, d_1\, (x-\gamma)\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\varepsilon}} + \frac{3}{4}\sqrt{\mathrm{Pe}}\, d_3\, d_1\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad - \frac{1}{3}\sqrt{\mathrm{Pe}}\, {d_3}^2 e^{-2\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} + d_6 \Bigg).\end{aligned} \tag{27}$$

We examine this expression assuming that $x = O(1)$, $\varepsilon \ll 1$, and noting that $x < \gamma$ in the matching. Using modified van Dyke matching [23], we expand as far as $O(\varepsilon)$, noting the presence of both exponentially small and large terms. The latter can all be eliminated by setting

$$d_3 = d_4 = 0,$$

and matching with the outer solution now yields

$$d_1 = \gamma,\ d_6 = 0.$$

We note that the coefficients of the exponentially small terms cannot be determined at this point: d_2, d_5 are still to be determined.

The solution of the right inner problem (21) is

$$\bar{u}_r \sim \bar{u}_{r0} + \sqrt{\varepsilon}\bar{u}_{r1}$$

with

$$\bar{u}_{r0} = c_1 + c_2 e^{\sqrt{\mathrm{Pe}}\bar{x}} + c_3 e^{-\sqrt{\mathrm{Pe}}\bar{x}}, \tag{28}$$

$$\begin{aligned}\bar{u}_{r1} = {} & \frac{c_5\, e^{\sqrt{\mathrm{Pe}}\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{c_4\, e^{-\sqrt{\mathrm{Pe}}\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{3}{4}\sqrt{\mathrm{Pe}}\, c_1\, c_2\, e^{\sqrt{\mathrm{Pe}}\bar{x}} + \frac{1}{2}\,\mathrm{Pe}\, c_1\, c_2\, \bar{x} e^{\sqrt{\mathrm{Pe}}\bar{x}} \\ & + \frac{1}{3}\sqrt{\mathrm{Pe}}\, {c_2}^2 e^{2\sqrt{\mathrm{Pe}}\bar{x}} + \frac{1}{2}\,\mathrm{Pe}\, c_3\, c_1\, \bar{x} e^{-\sqrt{\mathrm{Pe}}\bar{x}} + \frac{3}{4}\sqrt{\mathrm{Pe}}\, c_3\, c_1\, e^{-\sqrt{\mathrm{Pe}}\bar{x}} \\ & - \frac{1}{3}\sqrt{\mathrm{Pe}}\, {c_3}^2 e^{-2\sqrt{\mathrm{Pe}}\bar{x}} + c_6.\end{aligned} \tag{29}$$

The constants c_i are to be determined by matching with the right outer solution and by the continuity requirement at $x = \gamma$. The outer solution of (22) is simply

$$u_r = b_1 + \varepsilon b_2. \tag{30}$$

Writing the outer solution (30) in terms of the inner variable $\bar{x}$ and expanding with $\bar{x} = O(1)$, $\varepsilon \ll 1$, truncating after $O(\sqrt{\varepsilon})$ yields

$$(u_r)_{\mathrm{inner}} \sim b_1 + o(\sqrt{\varepsilon}).$$

Using the analogous procedure with the right inner solution we obtain

$$\begin{aligned}(\bar{u}_r)_{\mathrm{outer}} = {} & c_1 + c_2\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} + c_3\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & + \sqrt{\varepsilon}\Bigg(c_5\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\mathrm{Pe}}} - c_4\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\mathrm{Pe}}} - \frac{3}{4}\sqrt{\mathrm{Pe}}\, c_2\, c_1\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad + \frac{1}{2}\,\mathrm{Pe}\, c_2\, c_1\, (x-\gamma)\, e^{\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\varepsilon}} + \frac{1}{3}\sqrt{\mathrm{Pe}}\, {c_2}^2 e^{2\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad + \frac{1}{2}\,\mathrm{Pe}\, c_3\, c_1\, (x-\gamma)\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \frac{1}{\sqrt{\varepsilon}} + \frac{3}{4}\sqrt{\mathrm{Pe}}\, c_3\, c_1\, e^{-\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} \\ & \quad - \frac{1}{3}\sqrt{\mathrm{Pe}}\, {c_3}^2 e^{-2\frac{\sqrt{\mathrm{Pe}}(x-\gamma)}{\sqrt{\varepsilon}}} + c_6 \Bigg).\end{aligned} \tag{31}$$

In this case the matching is for $x > \gamma$. We remove the exponentially large terms by setting

$$c_2 = c_5 = 0$$

and matching requires

$$\begin{aligned} c_1 &= b_1, \\ c_6 &= 0, \end{aligned}$$

and b_2 cannot be determined at this level of approximation. At this point c_3 and c_4, both of which multiply exponentially small terms in the matching, are indeterminate. To complete the solutions it remains to impose continuity of u, u_x at $x = \gamma$. This leads to

$$\begin{aligned} c_3 &= d_2 = 0, \\ c_1 &= b_1 = \gamma, \\ c_4 &= \frac{1}{2}, \\ d_5 &= -\frac{1}{2}. \end{aligned}$$

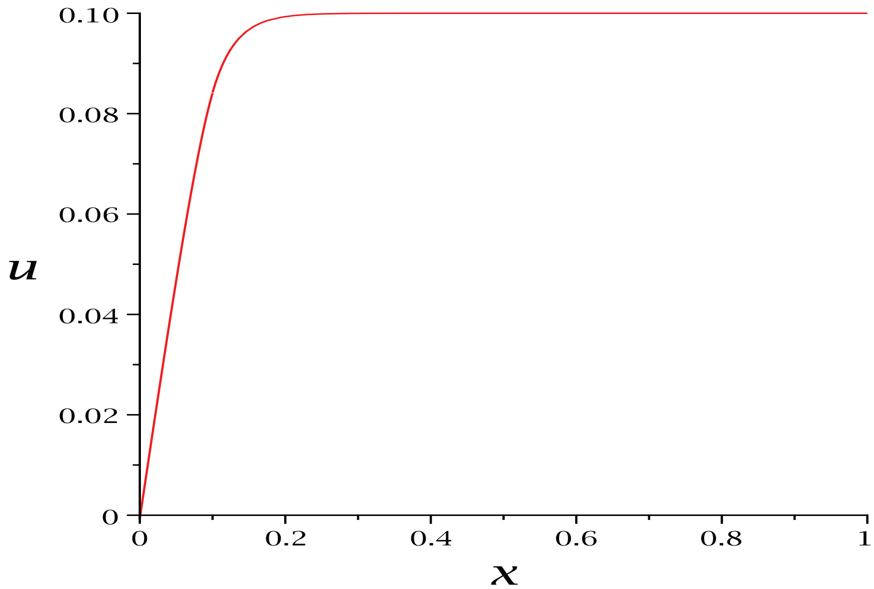

Fig. 7 *Composite solution for* $u(x)$ *based on* (32). $\varepsilon = 0.001$, $\mathrm{Pe} = 1$.

We can construct a composite solution (in two parts) in the usual way and this leads to

$$\begin{aligned} u^c &\sim u_l + \bar{u}_l - x, \quad 0 \le x \le \gamma, \\ &\sim u_r + \bar{u}_r - \gamma, \quad \gamma < x \le 1. \end{aligned} \tag{32}$$

For the "typical" parameter values in Table 1, this solution is not strictly applicable. In the next section, we describe how to modify our solutions to take account of the smallness of γ. To illustrate the basic correctness of the approach, we provide a plot of the composite solution for $u(x)$ in Figure 7 for a case where $\varepsilon \ll \gamma$ so that the smallness of γ does not interfere with the asymptotic structure of the solutions.

4.2.2. Solutions for $\gamma \ll 1, \varepsilon \ll 1$. The asymptotic development above was based on $\varepsilon \ll 1$, $\gamma = O(1)$. But, in fact, choosing the distinguished limit $\gamma = O(\sqrt{\varepsilon})$, the boundary layer thickness will also be $O(\sqrt{\varepsilon})$. For $\varepsilon = 1/18$, we note that $\sqrt{\varepsilon} = 0.23$, and this is comparable to $\gamma = 0.1$. Thus the boundary layer located at $x = \gamma$ is of the same order of magnitude as the left outer region. Hence, we should not use the solutions just obtained for the "typical" parameter values of Table 1. To consider the case of small γ, a sensible choice of distinguished limit is to put $\gamma = \gamma_0\sqrt{\varepsilon}$ (with $\gamma_0 = 1/2.3 = O(1)$). Referring to Figure 4, we will thus construct a solution made up of a left boundary layer $\bar{u}_l(\bar{x})$, a right inner $\bar{u}_r(\bar{x})$, and a right outer solution $u_r(x)$. There is no left outer solution in this case as the left layer solution overlaps the left boundary at $x = 0$. As before, it is necessary to proceed beyond the leading order in order to capture the details of the corner layers.

The left boundary layer scales are

$$\begin{aligned} x &= \sqrt{\varepsilon}\bar{x}, \\ u(x) &= \bar{u}_l(\bar{x}), \end{aligned}$$

and the left inner layer problem is

$$\bar{u}_l' + \sqrt{\varepsilon}(\bar{u}_l\bar{u}_l')' = \frac{1}{\mathrm{Pe}}\bar{u}_l''' + \sqrt{\varepsilon},$$
$$\bar{u}_l(\bar{x}=0) = 0,$$
$$\bar{u}_l''(\bar{x}=0) = 0. \tag{33}$$

The missing boundary information is provided by matching with the right inner problem:

$$\bar{u}_r' + \sqrt{\varepsilon}(\bar{u}_r\bar{u}_r')' = \frac{1}{\mathrm{Pe}}\bar{u}_r'''. \tag{34}$$

At $\bar{x} = \gamma_0$, i.e., $x = \gamma = \gamma_0\sqrt{\varepsilon}$, we will demand continuity of u and u', i.e.,

$$\bar{u}_l(\bar{x}=\gamma_0) = \bar{u}_r(\bar{x}=\gamma_0),$$
$$\bar{u}_l'(\bar{x}=\gamma_0) = \bar{u}_r'(\bar{x}=\gamma_0).$$

$\bar{u}_r$ will then be matched with right outer solution $u_r(x)$ governed by

$$u_r' + \varepsilon(u_r u_r')' = \frac{\varepsilon}{\mathrm{Pe}}u_r''', \tag{35}$$

$$u_r'(x=1) = 0. \tag{36}$$

Solutions. We find first on setting

$$\bar{u}_l = \bar{u}_{l0}(\bar{x}) + \sqrt{\varepsilon}\bar{u}_{l1}(\bar{x})$$

that (33) has solutions

$$\bar{u}_{l0} = -f_3\, e^{\sqrt{\mathrm{Pe}}\,\bar{x}} + f_3\, e^{-\sqrt{\mathrm{Pe}}\,\bar{x}},$$
$$\bar{u}_{l1} = \frac{f_5\, e^{\sqrt{\mathrm{Pe}}\,\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{f_5\, e^{-\sqrt{\mathrm{Pe}}\,\bar{x}}}{\sqrt{\mathrm{Pe}}} + \bar{x}$$
$$+ \frac{1}{3}\, {f_3}^2\sqrt{\mathrm{Pe}}\, e^{2\sqrt{\mathrm{Pe}}\,\bar{x}} - \frac{1}{3}\, {f_3}^2\sqrt{\mathrm{Pe}}\, e^{-2\sqrt{\mathrm{Pe}}\,\bar{x}}.$$

Similarly, (34) has the solution

$$\bar{u}_r \sim \bar{u}_{r0} + \sqrt{\varepsilon}\bar{u}_{r1}$$

with

$$\bar{u}_{r0} = g_1 + g_2\, e^{\sqrt{P}\bar{x}} + g_3\, e^{-\sqrt{P}\bar{x}},$$
$$\bar{u}_{r1} = \frac{g_5\, e^{\sqrt{P}\bar{x}}}{\sqrt{\mathrm{Pe}}} - \frac{g_4\, e^{-\sqrt{\mathrm{Pe}}\,\bar{x}}}{\sqrt{\mathrm{Pe}}} + \frac{1}{2}\,\mathrm{Pe}\, g_2\, g_1\, \bar{x}e^{\sqrt{\mathrm{Pe}}\,\bar{x}} - \frac{3}{4}\,\sqrt{\mathrm{Pe}}\, g_2\, g_1\, e^{\sqrt{\mathrm{Pe}}\,\bar{x}}$$
$$+ \frac{1}{3}\,\sqrt{\mathrm{Pe}}\, {g_2}^2 e^{2\sqrt{\mathrm{Pe}}\,\bar{x}} + \frac{3}{4}\,\sqrt{\mathrm{Pe}}\, g_3\, g_1\, e^{-\sqrt{\mathrm{Pe}}\,\bar{x}} + \frac{1}{2}\,\mathrm{Pe}\, g_3\, g_1\, \bar{x}e^{-\sqrt{\mathrm{Pe}}\,\bar{x}}$$
$$- \frac{1}{3}\, {g_3}^2\sqrt{\mathrm{Pe}}\, e^{-2\sqrt{\mathrm{Pe}}\,\bar{x}} + g_6.$$

The outer solution of (35) is apparently

$$u_r = h_1 + \varepsilon h_2. \tag{37}$$

However, without reproducing the details, on matching the right inner with the right outer solution it becomes clear that the form of the right outer solution is not quite right. In order to achieve matching it is necessary to include a switchback term [11]. This is a well-known device for including terms missing from an asymptotic expansion. Here we have assumed that the relevant gauge functions for the solution are $1, \varepsilon \ldots$.

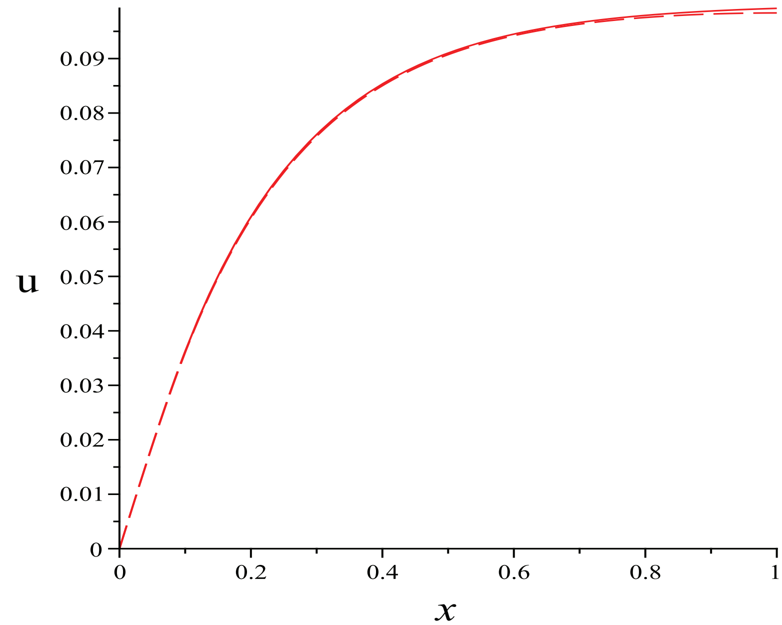

Fig. 8 *Comparison of numerical (solid line) and second asymptotic solution of section* 4.2.2 *using the same typical parameter values* $\varepsilon = 1/18$, $\mathrm{Pe} = 4/3$, $\gamma = 0.1$ *as for the leading order solution in Figure* 4.

In fact, the right outer solution should have been written in the form

$$u_r = h_1 + \sqrt{\varepsilon} h_s \cdots ,$$

as (35) ensures that the solution at each order (i.e., h_1, h_s) must be a constant. Here h_s is referred to as a switchback term. In reality, it reflects the fact that we made an incorrect initial assumption and this was flagged during the matching process. With this adjustment, and imposing continuity requirements at $x = \gamma = \gamma_0\sqrt{\varepsilon}$, $\bar{x} = \gamma_0$, i.e.,

$$\bar{u}_l(\bar{x} = \gamma_0) = \bar{u}_r(\bar{x} = \gamma_0),$$
$$\bar{u}_l'(\bar{x} = \gamma_0) = \bar{u}_r'(\bar{x} = \gamma_0),$$

we find that

$$g_1 = g_2 = g_5 = h_1 = 0,$$
$$g_6 = h_s = \gamma_0,$$

and

$$f_3 = g_3 = 0,$$
$$g_4 = -\frac{1}{2}(-e^{\sqrt{\mathrm{Pe}}\gamma_0} + e^{-\sqrt{\mathrm{Pe}}\,\gamma_0}),$$
$$f_5 = -\frac{1}{2e^{\sqrt{\mathrm{Pe}}\,\gamma_0}}.$$

This new asymptotic solution is graphed against the exact (numerical) solution in Figure 8. The success of the matched asymptotics approach is clear. This solution is an obvious improvement on the leading order solution (15) as graphed in Figure 4.

Preferential Promotion. In this problem (see also the appendix), it is also possible to adopt an ad hoc approach and preferentially promote the diffusion terms, i.e., to treat the problem (9) and (10) as though the $O(\varepsilon/\mathrm{Pe})$ terms are $O(1)$. It is well known in applied and numerical circles that diffusion terms have the effect of smoothing solutions, which is precisely what is required in this problem in the vicinity of the discontinuity. The other point is that in this example the slightly more complicated equations can still be solved analytically. The idea of preferential promotion has often been exploited in thin film theory [17], [18]; the justification is that if the terms really are small, then the formal asymptotic error is unchanged. However, if these terms play a significant role, we can obtain an improved leading order solution without resorting to matched asymptotic expansions. Note that technically the error in this approximate solution is $o(1)$: the error in the matched asymptotic solutions is $o(\sqrt{\varepsilon})$.

An Improved Leading Order Approximation. We rewrite the leading order problem in the following form:

$$\frac{du}{dx} = \frac{1}{\alpha^2}\frac{d^2c}{dx^2} + N(x), \tag{38}$$

$$\frac{du}{dx} = c, \tag{39}$$

on defining $\frac{1}{\alpha^2} \equiv \frac{\varepsilon}{\mathrm{Pe}}$. The exact solutions are

$$c = A(e^{\alpha x} + e^{-\alpha x}) + 1, \quad 0 \le x \le \gamma, \tag{40}$$

$$\quad = E(-e^{-2\alpha}e^{\alpha x} + e^{-\alpha x}), \quad \gamma < x \le 1, \tag{41}$$

$$u = A/\alpha(e^{\alpha x} - e^{-\alpha x}) + x, \quad 0 \le x \le \gamma, \tag{42}$$

$$\quad = E/\alpha(-e^{-2\alpha}e^{\alpha x} - e^{-\alpha x}) + \gamma, \quad \gamma < x \le 1, \tag{43}$$

where

$$A = -\frac{1}{2}\frac{e^{\alpha(-2+\gamma)} + e^{-\alpha\gamma}}{1 + e^{-2\alpha}},$$
$$E = \frac{1}{2}\frac{e^{\alpha\gamma} - e^{-\alpha\gamma}}{1 + e^{-2\alpha}}.$$

The improved approximations for $c(x)$ and $u(x)$ are compared with the numerical solutions in Figure 9. The success of the simple approximation is obvious. By inspection, it is even better than the matched asymptotic solution, but this is an artifact of the particular problem. It is not difficult to understand why this happens: if we single out the term $\varepsilon(uc)' \equiv \varepsilon(uu')'$ in (9), we note that away from the boundary layer $u \sim x$ or $u \sim 1$ and so this term makes a negligible contribution at $O(\varepsilon)$ outside the layer. In the layer the diffusion terms play the dominant role. Thus (in this example) the preferential promotion approach, which includes the diffusion terms but excludes other formally $O(\varepsilon)$ terms, plays no price for doing so. If preferential promotion is possible, it can give impressive results if the resulting equations are integrable in closed form. However, one must not lose sight of the fact that matched asymptotics is a much more general and flexible tool.

5. Summary. We revisit the standing gradient problem of Lin and Segel [12]. The problem was originally presented as an exercise in regular perturbation methods. Using an alternative nondimensionalization we show that the problem can be analyzed

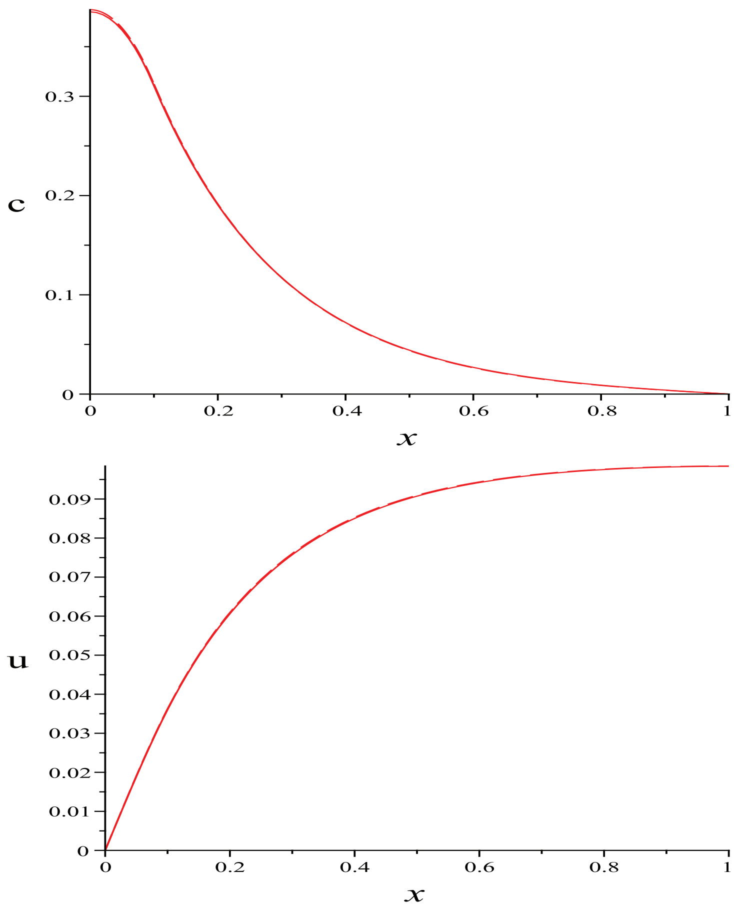

Fig. 9 *Comparison of numerical and improved leading order asymptotic solution (via preferential promotion) using typical parameter values* $\varepsilon = 1/18$, $\mathrm{Pe} = 4/3$, $\gamma = 0.1$.

using a singular perturbation approach. In doing so we demonstrate some of the tools of modern mathematical modeling: asymptotic simplification, development of a simple leading order solution as a regular perturbation with a corner, smoothing layers and matched asymptotics, and the concept of preferential promotion of small terms. The asymptotic solutions were validated by comparison with numerical solutions obtained using the MATLAB routines ode45 and ode15s with a tolerance of 10^{-6}.

Appendix. A Model Problem. A model linear problem with a similar structure to the problem in the paper is

$$\begin{aligned} &\varepsilon c_{xx} - c_x = -H(1-x), \\ &c(0) = 0,\ c_x(2) = 0, \quad 0 < x < 2, \end{aligned} \tag{A.1}$$

where H is the Heaviside step function. The leading order solution of this problem is clearly

$$\begin{aligned} c(x) &= x, \quad 0 \le x \le 1, \\ &= 1, \quad 1 < x \le 2. \end{aligned} \tag{A.2}$$

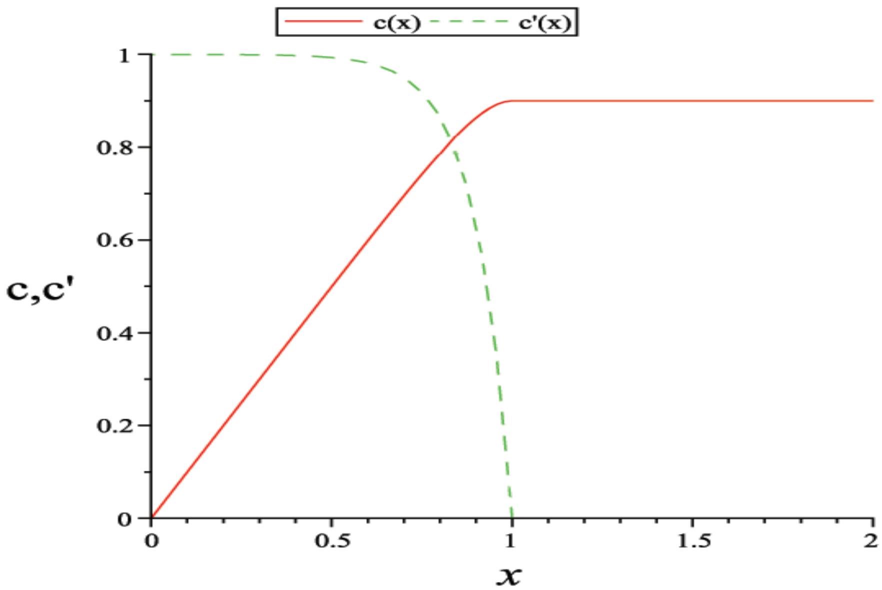

Fig. 10 *Solution of model problem for* $c(x)$ *and* $c'(x)$. $\varepsilon = 0.1$.

This has a corner [8] at $x = 1$ which can be smoothed via a boundary layer analysis using the rescaling

$$x = 1 + \varepsilon\bar{x}$$

to reintroduce the higher derivative terms. Similar to the body of the paper we must construct a solution in four pieces: left and right outer ($c_l(x)$, $c_r(x)$) and left and right smoothing inner solutions ($\bar{c}_l(\bar{x})$, $\bar{c}_r(\bar{x})$) with continuity requirements at the point $x = 1$ ($\bar{x} = 0$). Without reproducing the details, we obtain the following solutions:

$$\begin{aligned} c_l &\sim x, \\ \bar{c} &\sim 1 + \varepsilon(\bar{x} - e^{\bar{x}}), \\ \bar{c}_r = c_r &\sim 1 - \varepsilon. \end{aligned} \tag{A.3}$$

Here it is also possible to preferentially promote the εc_{xx} terms, i.e., to solve (A.1) exactly. We find that the exact solution is

$$\begin{aligned} c(x) &= x + \varepsilon e^{-\frac{1}{\varepsilon}} - \varepsilon e^{\frac{x-1}{\varepsilon}}, \quad 0 \le x \le 1, \\ &= 1 - \varepsilon + \varepsilon e^{-\frac{1}{\varepsilon}}, \quad 1 < x \le 2. \end{aligned} \tag{A.4}$$

It is easy to see that this is approximated by (A.3) by using the appropriate limiting processes (i.e., $\varepsilon \to 0$ with x or $\bar{x}$ fixed). We also verify that the solution has been smoothed and c and c_x are continuous at $x = 1$. Note that c_x is now a smoothed version of the Heaviside function $H(1 - x)$ (illustrated in Figure 10).

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