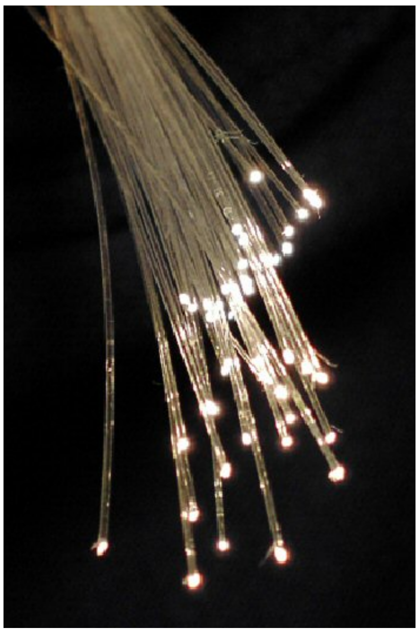
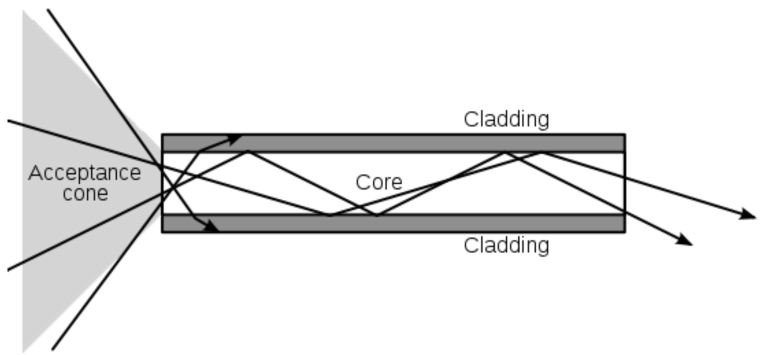
MATHEMATICAL MODELING, PROJECT 2 (2012)

OPTICAL FIBER CABLE WITH INHOMOGENEITY

Throughout much of Amsterdam the copper telephone lines have recently been replaced by optic fiber cables, which carry communication signals encoded in electromagnetic light waves. Optic fiber cables have the advantage of providing fast, high bandwidth communications without risk of electromagnetic interference.





In this project we will study a mathematical model for optic fiber signals. We consider light wave propagation in a medium (e.g. glass), in which the transmission properties of the medium play an important role. Conceptually, optic fibers consist of two layers of material having different wave propagation speeds. Light waves are trapped in the innermost layer (*core*) due to total reflection (i.e. no transmission) at the interface with the outer layer (*cladding*). In this sense, optic fibers are an example of a *wave guide*.

The format of the project will be a short report that addresses *at least* all of the following questions. The report should be well-written and coherent (not just a list of answers) and the report itself will account for 20% of your grade for the project. Depending on the font size and number and size of plots, I would expect it to be under 10 pages in length. You may submit a single report for a group of at most three students. These should be placed in my mailbox, on the 4th floor of Science Park 904, or sent to me directly by e-mail, before 9.00 on 10 December.

Table of symbols:

symbol	dependence	meaning
$\boldsymbol{E} = (E^x, E^y, E^z)$	(t, x, y, z)	Electric field
$\boldsymbol{B} = (B^x, B^y, B^z)$	(t, x, y, z)	Magnetic field
$\boldsymbol{J}$	(t, x, y, z)	current density
ρ	(t, x, y, z)	charge density
ε_0	–	permittivity in vacuum
ε	(x, y, z)	permittivity in medium
μ_0	–	permeability in vacuum
μ	(x, y, z)	permeability in medium
n	(x, y, z)	refraction index
c	–	speed of light in vacuum
ω	–	frequency

The propagation of light (consisting of electromagnetic waves) is described by Maxwell's equations. In the first part of the project, we investigate some general properties of Maxwell's equations in the context of wave mechanics. Next we construct a vastly reduced model that is suitable for studying optic fibers. Finally, we investigate the solutions to the reduced model and its application to optic fibers.

(1) **Maxwell's equations.** Light consists of electric and magnetic fields $\boldsymbol{E}(t, x, y, z)$ and $\boldsymbol{B}(t, x, y, z)$, respectively. Both of these are vector functions in $\mathbb{R}^3$, which is indicated by the bold notation. The evolution of these fields is governed by Maxwell's equations:

$$\mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{B} - \mu_0 \boldsymbol{J}, \tag{ME1}$$

$$0 = \nabla \cdot \boldsymbol{E} - \frac{\rho}{\varepsilon_0}, \tag{ME2}$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \boldsymbol{E}, \tag{ME3}$$

$$0 = \nabla \cdot \boldsymbol{B}, \tag{ME4}$$

where the scalar constants ε_0 and μ_0 are the permittivity and permeability in a vacuum, and the current density $\boldsymbol{J}(t, x, y, z)$ and charge density $\rho(t, x, y, z)$ are given functions. Motivated by (ME3) and (ME4) we introduce the electric potential $\phi(t, x, y, z)$ and magnetic vector potential $\boldsymbol{A}(t, x, y, z)$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}.$$

(a) Using the Helmholtz decomposition from multivariable calculus, show that $\boldsymbol{B}$ and $\boldsymbol{E}$ satisfying (ME3) and (ME4) can be written without loss of generality in terms of such ϕ and $\boldsymbol{A}$ (assuming all functions are sufficiently smooth and decaying at infinity).

(b) Demonstrate that the following Lagrangian gives rise to (ME1) and (ME2) as the extremum of the associated action integral $\mathcal{S}[\phi, \boldsymbol{A}] = \iiiint \mathcal{L}(\phi, \boldsymbol{A}, t, x, y, z)\, dx\, dy\, dz\, dt$:

$$\mathcal{L}(\phi, \boldsymbol{A}, t, x, y, z) = \frac{1}{2\mu_0}\left[\mu_0\, \varepsilon_0 |\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right] - \phi\rho + \boldsymbol{A} \cdot \boldsymbol{J}.$$

(Here, $\boldsymbol{E}$ and $\boldsymbol{B}$ are to be viewed as functions of ϕ and $\boldsymbol{A}$, and $|\cdot|$ denotes the Euclidean norm.) You may assume that all boundary terms vanish when integrating by parts.

(c) Assuming $\rho \equiv 0$ and $\boldsymbol{J} \equiv 0$, apply Noether's theorem to the Lagrangian to determine conservation laws associated to the following symmetries:

(i) Time translations: $(t, x, y, z) \mapsto (t + s, x, y, z)$,

(ii) Space translations: $(t, x, y, z) \mapsto (t, x + s, y, z)$, $(t, x, y, z) \mapsto (t, x, y + s, z)$, $(t, x, y, z) \mapsto (t, x, y, z + s)$,

Hint: *It may be helpful to first generalize the derivation of Noether's theorem as given in the lecture notes (see Handout for Week 41) to the case of a Lagrangian of two independent variables $L(q, q_t, q_x)$. Also note that for a PDE, a conservation law is an equation of the form $\frac{\partial}{\partial t} f + \frac{\partial}{\partial x} g + \frac{\partial}{\partial y} h + \frac{\partial}{\partial z} j = 0$.*

(2) **Reduced model.** For light traveling through a medium such as glass, the permittivity and permeability become functions of the medium. We replace ε_0 and μ_0 in (ME1)–(ME2) with functions $\varepsilon(x, y, z)$ and $\mu(x, y, z)$. To model light waves in an optical fiber, we will make a number of assumptions: (i) the charge density is constant in time $\boldsymbol{J} = \boldsymbol{J}(x, y, z)$, (ii) we work in a planar configuration such that all quantities are independent of the coordinate y, (iii) the light is polarized in the y direction such that $\boldsymbol{E} = (0, E^y, 0)$. In other words, we work in the (x, z) plane, and the light is polarized in the direction normal to the plane.

(a) Derive a reduced Maxwell equation for $E^y(t, x, z)$. You should be able to reduce to a single partial differential equation for this variable.

(b) Construct a Lagrangian for this equation. Which of the conservation laws persist for the reduced model? How does the existence of these conservation laws depend on the inhomogeneity of the medium (i.e. does it matter if ε and μ depend on x and z)?

(c) The index of refraction $n(x, z)$ of a material is defined by $n^2 = \varepsilon\mu c^2$, where c is the speed of light in a vacuum. We next consider periodic solutions with frequency ω by substituting the form $E^y(t, x, z) = e^{-i\omega t} E(x, z)$. Defining $k(x, z) = k_0\, n(x, z)$, where $k_0 = \omega/c$, derive the equation to be satisfied by the amplitude $E(x, z)$. This is known as the *Helmholtz equation.*

(3) **Wave guide.** An optic fiber is a wave guide. It is manufactured in such a way that light waves within a certain range of frequencies are trapped in the medium due to total internal reflection at the interface. To achieve this, two media are used: an internal medium with a large index of refraction and an external medium with a smaller index of refraction. For our simple model, we assume a 2D wave guide (instead of a cylindrical one), that extends infinitely in the z-direction. The internal medium extends from $x = -L$ to $x = L$, and we assume that the external medium extends to infinity in x as well (we will require that there is no wave propagation far from the wave guide, so the extent of the external medium is not important). The index of

refraction is

$$n = \begin{cases} n_1, & |x| \le L, \\ 1, & |x| > L. \end{cases}$$

(a) To solve the Helmholtz equation, make the ansatz that E can be written as a product $E(x,z) = \phi(x)u(z)$. Insert this solution into the Helmholtz equation and manipulate so that all functions of x are on one side of the equality and all functions of z are on the other side. Note that if one has a relation of the form $f(x) = g(z)$ where x and z are independent variables, it must be the case that both of these functions are constant, i.e. $f(x) \equiv g(z) \equiv \lambda = const.$ Use this fact to derive two independent equations for ϕ and u.

(b) Solve the equation for u using the Fourier transform method. What condition must λ satisfy if the solution is to be a (plane) wave?

(c) The material in the cladding ($|x| > L$) is chosen such that $\lim_{x\to\pm\infty} E(x,z) = 0$. Find the general solution for $\phi(x)$ for $x > L$. An analogous solution can be defined for $x < -L$. What condition must hold on λ to satisfy the boundary condition at infinity?

(d) Assuming the above conditions hold on λ, define $k_1 = k_0 n_1$ and write the general solution for $\phi(x)$ in the core $|x| \le L$.

(e) Given that at the interfaces $x = \pm L$, both $\phi(x)$ and $\phi'(x)$ must be continuous, determine a relation that must hold for: k_1, k_0, β, and the amplitudes of the waves in the core and the cladding. It may be helpful to make a plot to illustrate the possible solutions of this relation. Is there a unique solution and what is the physical significance of this?

(4) **Inhomogeneity in z.** In general, the material in the core may have impurities which affect the refraction index. It is important to determine if this will result in a loss of information in the signal. We study the effects of inhomogeneity. For simplicity, we now only consider a 1D model in the z-direction, ignoring the lateral structure.

(a) Write a Lagrangian for the 1D Helmholtz equation for the case $k = k(z)$.

(b) Assume a solution of the form $E = a(z)e^{i\theta(z)}$, where a is the amplitude and θ is the phase function. Substitute this ansatz in the Lagrangian and derive the Euler-Lagrange equations for a and θ.

(c) The phase equation takes the form of a conservation law, i.e. $\frac{\partial}{\partial_z} F(z) = 0$ for some function $F(z)$. It follows that this function is constant, $F(z) = P$ for some constant P. Use this to eliminate θ in the amplitude equation.

(d) An approximate solution can be obtained by assuming the amplitude is well approximated by the steady state of the amplitude equation. Find this steady state and use it to construct an approximate solution for $E(z)$ up to a constant scaling factor. How is the amplitude affected by variations in the material property $k(z)$?