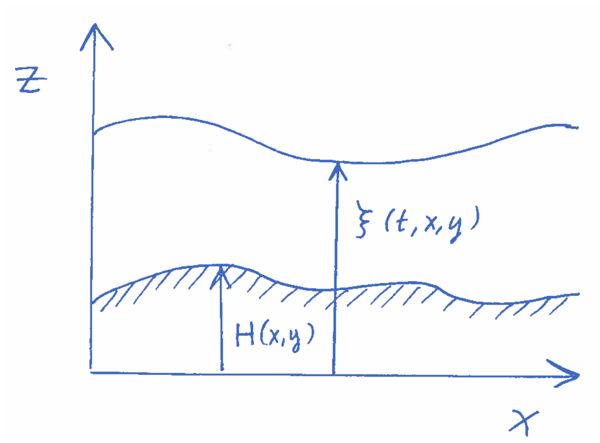
MATHEMATICAL MODELING, PROJECT 3 (2012)

SIMPLE OCEAN MODEL

In this problem we will derive a model for a thin layer of fluid flowing under the influence of gravity over a surface. You can think of this as a simplified ocean model. We will use it to study some characteristics of approximately two-dimensional atmosphere and ocean flows.



The format of the project will be a short report that addresses *at least* all of the following questions. The report should be well-written and coherent (not just a list of answers) and the report itself will account for 20% of your grade for the project. Depending on the font size and number and size of plots, I would expect it to be under 10 pages in length. You may submit a single report for a group of at most three students. These should be placed in my mailbox, on the 4th floor of Science Park 904, or sent to me directly by e-mail, before 9.00 on 14 January.

Table of symbols:

symbol	dependence	meaning
$\boldsymbol{v} = (u, v, w)$	(t, x, y, z)	velocity field
ρ	(t, x, y, z)	mass density
p	(t, x, y, z)	pressure
$\boldsymbol{f}$	(x, y, z)	body force
H	(x, y)	ocean floor topography
ξ	(t, x, y)	height of the ocean surface
U, V	(t, x, y)	depth averaged velocity field
f_0	–	rotation parameter (1/sec)
g	–	gravitational acceleration
ζ	(t, x, y)	2D vorticity
δ	(t, x, y)	2D divergence

(1) **Thin-layer model.** Our starting point is the incompressible Navier-Stokes equations. We assume the fluid to be inviscid ($\mu = 0$), and subject to a gravity force, (i.e. body force $\mathbf{f} = (0, 0, -g)$).

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{f}$$
$$0 = \nabla \cdot \mathbf{v}.$$

The topography is given as a function $z = H(x, y)$. The upper surface of the fluid is a time dependent function $z = \xi(t, x, y)$ which is unknown. To simplify notation we also define the layer height by $h(t, x, y) = \xi(t, x, y) - H(x, y)$.

(a) We assume the domain to be unbounded in x and y. The boundary condition at the free surface is determined as follows: Define functions $F(t, x, y, z) = z - \xi(t, x, y)$ and $G(t, x, y, z) = z - H(t, x, y)$. The free surface and lower topography coincide with the level sets $F \equiv 0$ and $G \equiv 0$. A fluid element once located on the free surface (or topography) remains there for all time. Hence these surfaces are *material surfaces* satisfying $\frac{DF}{Dt}\big|_{z=\xi} = 0$ and $\frac{DG}{Dt}\big|_{z=H} = 0$. In fact, these two statements precisely describe the boundary conditions at the upper ($z = \xi$) and lower ($z = H$) surfaces of the fluid. Rewrite the boundary conditions in terms of partial derivatives ∂_t, ∂_x, ∂_y, and ∂_z of the functions ξ and H and the components of the velocity field $\mathbf{v} = (u, v, w)$. You will need these functions for the rest of the problem.

(b) The *depth-averaged* velocity components (U, V) are defined as

$$U(t, x, y) = \frac{1}{h(t, x, y)} \int_{H(x,y)}^{\xi(t,x,y)} u(t, x, y, z)\, dz,$$
$$V(t, x, y) = \frac{1}{h(t, x, y)} \int_{H(x,y)}^{\xi(t,x,y)} v(t, x, y, z)\, dz.$$

(Note these are *not* the material velocity components.) Determine the depth-averaged continuity equation by integrating the divergence-free condition $\nabla \cdot \mathbf{v} = 0$ with respect to z. Write the continuity equation in terms of the new variables U and V. To do so, notice that the limits of integration ($z = \xi(t, x, y)$, $z = H(x, y)$) are variable: you will have to make use of the Leibnitz rule and the boundary conditions you derived in part (a). It is helpful for the rest of this project to express the continuity in terms of the variable h by replacing ξ in the final result by $\xi = h + H$ wherever it occurs.

(c) We make the assumption that $w \equiv 0$ in the interior of the layer. We also assume that the fluid is homogeneous, so $\rho(x, y, z) = \rho_0$ throughout. What does the momentum equation for $\frac{Dw}{Dt}$ reduce to in this case? Integrate this relation to determine $p(t, x, y, z)$ assuming the pressure has value p_0 uniformly at the free surface (that is, $p(t, x, y, \xi(t, x, y)) = p_0$).

(d) We are going to depth-average the momentum equations. Using the boundary conditions derived in (a), show that the following relations hold (arguments t, x,

y, z suppressed):

$$\int_H^\xi \frac{Du}{Dt}\,dz = \frac{\partial}{\partial t}(hU) + \frac{\partial}{\partial x}\int_H^\xi u^2\,dz + \frac{\partial}{\partial y}\int_H^\xi uv\,dz$$
$$\int_H^\xi \frac{Dv}{Dt}\,dz = \frac{\partial}{\partial t}(hV) + \frac{\partial}{\partial x}\int_H^\xi uv\,dz + \frac{\partial}{\partial y}\int_H^\xi v^2\,dz.$$

(e) Next, we express the velocity field as a perturbation to the depth-averaged field as follows: $u(t,x,y,z) = U(t,x,y) + \tilde{u}(t,x,y,z)$, with a similar relation $v = V + \tilde{v}$. Show that

$$\int_H^\xi u^2\,dz = hU^2 + \sigma^{xx}, \qquad \int_H^\xi v^2\,dz = hV^2 + \sigma^{yy}, \qquad \int_H^\xi uv\,dz = hUV + \sigma^{xy},$$

where σ^{xx}, σ^{yy}, σ^{xy} are integrals depending only on products of $\tilde{u}$ and $\tilde{v}$. We now make the modeling assumption that σ^{xx}, σ^{yy}, σ^{xy} are zero. Under this assumption, write down the depth-averaged momentum equations in terms of U, V, and ξ.

(2) **Properties of the model.** For this and the succeeding questions, we assume for simplicity that $H \equiv 0$ (no topography) so that $\xi = h$. The rotation of the earth has a very significant effect on the motion of the atmosphere and ocean in regions far from the equator. To account for rotation we add new terms to the velocity equations, i.e.

$$\frac{\partial U}{\partial t} = f_0 V + \cdots,$$
$$\frac{\partial V}{\partial t} = -f_0 U + \cdots,$$

where f_0 is related to the rotation rate and $\cdots$ represents the terms you derived in the previous problem. The thin layer model satisfies a number of conservation laws. The three equations themselves express balance laws for total momentum and mass. Additionally the total energy

$$E = \iint \frac{\xi}{2}(U^2 + V^2) + \frac{g}{2}\xi^2\,dx\,dy$$

is conserved. The vorticity is defined as $\zeta = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$, and the *total vorticity* (including the rotation effect) $\omega = \zeta + f_0$ satisfies a conservation law

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(\omega U) + \frac{\partial}{\partial y}(\omega V) = 0.$$

In fact, the ratio of the total vorticity to the layer depth $q = (\zeta + f_0)/\xi$ is conserved along material lines:

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} + V\frac{\partial q}{\partial y} = 0.$$

Verify all three of these conservation properties, and show that the latter implies an infinite family of conserved quantities

$$\frac{dC_F}{dt} = 0, \qquad C_F = \iint \xi(x,y,t)F(q(x,y,t))\,dx\,dy,$$

for well-behaved functions F.

(3) **Small perturbation behavior.** Consider a small disturbance to the thin layer model. Let $\xi(t,x,y) = \bar{\xi}_0 + \eta(t,x,y)$, $U(t,x,y) = u(t,x,y)$ and $V(t,x,y) = v(t,x,y)$, where η, u and v and their derivatives with respect to x and y are small enough that any quadratic terms can be neglected.

(a) Derive equations for the disturbances u, v, and η, neglecting any terms that are quadratic in these variables and their derivatives.

(b) Rewrite these equations in terms of the vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, the divergence $\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, and the layer depth η, and use these to determine a single wave equation for the divergence δ.

(c) Assume the topography is flat. Making the ansatz $\delta(t,x,y) = \exp[i(kx+\ell y-\omega t)]$, determine the dispersion relation $\omega(k,\ell)$, and the group velocity. Consider the limit behavior for the wave number $|\boldsymbol{k}| = (k^2+\ell^2)^{1/2}$. What does this say about the relative propagation speeds of wave groups for $|\boldsymbol{k}| \ll 1$ and $|\boldsymbol{k}| \gg 1$?

(4) **Limit case of slow motion.** We want to understand the slow limit behavior of the thin layer model in the limit of fast rotation. For this we ignore the topography H. To do so, we rescale the rotating thin layer model according to $x = L\bar{x}$, $y = L\bar{y}$, velocity $U = \bar{U}u$, $V = \bar{U}v$, and time scale $\tau = (\bar{U}/L)t$. For the layer depth we assume a mean layer depth of H_0, and rescaled perturbation $\xi(t,x,y) = H_0 + N_0 h(t,x,y)$.

(a) Introduce dimensionless parameters $R = U(Lf_0)^{-1}$, $F = U(gH_0)^{-1/2}$, and $\theta = N_0 H_0^{-1}$, and write the rescaled equations for (u,v,h) in terms of these parameters.

(b) Next we want to consider a distinguished limit in which the parameters R, F, and θ scale as a function of the single small parameter ε:

$$R = \varepsilon, \qquad F = G^{1/2}\varepsilon, \qquad \theta = G\varepsilon,$$

for fixed $G > 0$ and $\varepsilon \ll 1$. Again rescaling the layer depth perturbation to define $\eta = G^{1/2}h$, write the new model in terms of ε.

(c) Our goal is to describe the slow motion in a simple manner. We will do this using the materially conserved quantity q. Make an asymptotic expansion of this variable in ε. We are interested in truncating at the lowest nontrivial terms (a constant term is trivially conserved and can be ignored in the approximation of q). With this approximation of q, our dynamical equation is simply

$$\frac{Dq}{Dt} = 0,$$

where we now need to relate q and $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ to the lowest order dynamics of the scaled problem.

(d) Determine the lowest order approximations of the momentum and layer depth equations u, v, and η. Explain how these relations can be used to relate q to v to obtain a closed system for the slow evolution. Next, show that given any function $q(x,y)$, your approximations for u, v and η define a steady state for the perturbation equations of part (3). What does this imply about the evolution of this approximate model with respect to the fast motion described by the linearized model of part (3)?