Numerical Analysis 2. Final Exam

Due: Thursday, January 8, 2009.

Submit your exam in my postbox in the Euclides building no later than 10.00 in the morning. This exam will account for 1/3 of your grade in the class (same as each of the projects). Problems 1 and 2 are worth 2.5 points each, problem 3 is worth 5 points.

1. Consider the class of *one-stage* Runge-Kutta methods. Prove the maximal order of accuracy p of this class. Recall that for a method of order p, the local error satisfies $le = \mathcal{O}(h^{p+1})$. The subclass of methods of maximal order has a finite number of members. Determine this number.

2. Consider the class of *explicit, two-stage* Runge-Kutta methods. These have Butcher table

$$\begin{array}{c|cc} & & \\ & \alpha & \\ \hline & b_1 & b_2 \end{array}$$

Taking α a free parameter, determine $b_1(\alpha)$ and $b_2(\alpha)$ to satisfy the conditions for second order. Next, determine the stability function of these methods for the scalar test problem $y' = \lambda y$, $\lambda \in \mathbb{C}$. Can you use α to optimize the stability? Otherwise, find α to solve one of the conditions for order three. Finally, the system of differential equations

$$x' = x(x-2) + e^y, \qquad y' = 50y(y-2)$$

has a stable equilibrium at the origin. For your choice of α, determine the maximum stepsize h for which the origin remains a stable fixed point of the numerical method.

3. Occasionally one is presented with differential equations for which there is a natural partitioning of the variables:

$$y' = f(y,z), \quad z' = g(y,z), \quad y, z \in \mathbb{R}^d$$

A generalization of Runge-Kutta methods for such systems is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij}F_j, \qquad Z_i = z_n + h\sum_{j=1}^{s} \alpha_{ij}G_j, \quad i = 1, \ldots, s$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} b_jF_j, \qquad z_{n+1} = z_n + h\sum_{j=1}^{s} \beta_jG_j,$$

where $F_j = f(Y_j, Z_j)$ and $G_j = g(Y_j, Z_j)$. *Derive the order conditions on a_{ij}, b_j, α_{ij}, β_j for a method to be of at least order 2.*

Show that the following method from this class

$$\begin{array}{c|cc} & 1/2 & 0 \\ & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} & 0 & 0 \\ & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

preserves arbitrary first integrals of the form

$$I(y,z) = y^T Az,$$

for A a $d \times d$ matrix.