

Chapter 10

Stability of Runge-Kutta Methods

Main concepts: Stability of equilibrium points, stability of maps, Runge-Kutta stability function, stability domain.

In the previous chapter we studied equilibrium points and their discrete counterpart, fixed points. A lot can be said about the qualitative behavior of dynamical systems by looking at the local solution behavior in the neighborhood of equilibrium points. In this chapter we study *stability* of these points and the related stability of fixed points of Runge-Kutta methods.

10.1 Stability of Equilibrium Points

We now define the stability of an equilibrium point. In general, the stability concerns the behavior of solutions near an equilibrium point in the long term. Given an autonomous system of differential equations (1.13), denote the solution of the system with the initial condition $y(0) = y_0$ by $y(t; y_0)$.

(1) The equilibrium point y^* is *stable in the sense of Lyapunov* (or just *stable*) for the given differential equation if $\forall \epsilon > 0, \exists \delta > 0, T > 0$ such that, if $\|y_0 - y^*\| < \delta$ then

$$\|y(t; y_0) - y^*\| < \epsilon, \quad t > T.$$

(2) The equilibrium point y^* is *asymptotically stable* if there exists $\gamma > 0$ such that, for any initial condition y_0 such that $\|y_0 - y^*\| < \gamma$,

$$\lim_{t \rightarrow \infty} \|y(t; y_0) - y^*\| = 0$$

(3) The equilibrium point y^* is *unstable* if it is not stable (in the sense of Lyapunov).

Note: any asymptotically stable equilibrium point is also stable.

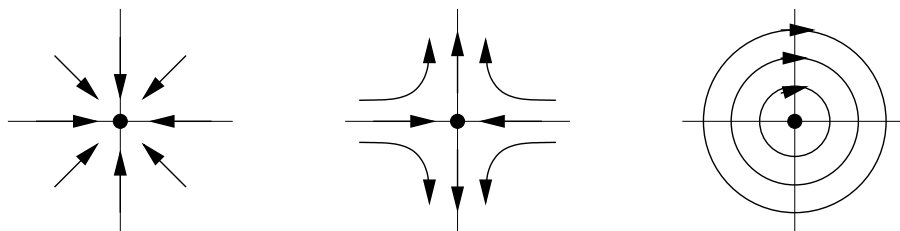
The simplest way to illustrate the types of situations that can arise is to consider the linear systems

$$\frac{d}{dt}y = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} y, \quad \frac{d}{dt}y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y, \quad \frac{d}{dt}y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y.$$

The origin is the only equilibrium point in each case. For the first system it is asymptotically stable (and also stable), for the second it is unstable, and for the third it is stable but not asymptotically stable. See Figure 10.1.

One sometimes hears it said in engineering or science that a certain *system* is stable. This use of the term may be the same as that used above but suggesting that the equilibrium point of interest is understood, or it may be a different usage of the term entirely. (Some caution is urged.)

There are a number of methods for analyzing stability in general. One of these is the Poincaré-Lyapunov Theorem and is based on looking at the eigenvalues of the linear part of the problem. We state a simplified version here, without proof, which also is sometimes called the Linearization Theorem.

Figure 10.1: Some phase diagrams for linear systems in \mathbb{R}^2 .

Theorem 10.1.1 (Linearization Theorem) Consider the equation in \mathbf{R}^d :

$$dy/dt = Ay + F(y) \quad (10.1)$$

subject to initial condition $y(0) = y_0$. Assume A is a constant $d \times d$ matrix with eigenvalues having all negative real parts and the function F is C^1 in a neighborhood of $y = 0$, with $F'(0) = F(0) = 0$, where $F'(y)$ is the Jacobian matrix of F . Then $y = 0$ is an asymptotically stable equilibrium point. If A has any eigenvalues with positive real part, then $y = 0$ is an unstable equilibrium point.

Although the conditions on A and F may at first appear restrictive, Theorem 10.1.1 is actually quite powerful. First, if we have an equilibrium point y^* of a differential equation (1.13) which is not at the origin, we can always shift it to the origin by introducing a translation. Define $\tilde{y} = y - y^*$ and then we have

$$\frac{d\tilde{y}}{dt} = \frac{dy}{dt} = f(\tilde{y} + y^*) \equiv \tilde{f}(\tilde{y}),$$

which is in the form (1.13) but has an equilibrium point at $\tilde{y} = 0$.

Second, consider an arbitrary differential equation (1.13) with C^2 vector field and an equilibrium point at the origin. We may proceed as follows to use Theorem 10.1.1. First expand f in a Taylor series expansion about 0:

$$f(y) = f(0) + f'(0)y + R(y).$$

Because 0 is an equilibrium point, we have $f(0) = 0$, and we set $A = f'(0)$. Define $F(y) = R(y)$. We see that we can write our equation in the form

$$\frac{dy}{dt} = Ay + F(y),$$

and it is easy to verify that $F(0) = F'(0) = 0$. If the eigenvalues of A have negative real parts, then we can conclude that the origin is asymptotically stable. Let us state an alternative form of the Linearization Theorem based on this:

Theorem 10.1.2 (Linearization Theorem II) Suppose that f in (1.13) is C^2 and has an equilibrium point y^* . If the eigenvalues of

$$J = f'(y^*)$$

all lie strictly in the left complex half-plane, then the equilibrium point y^* is asymptotically stable. If J has any eigenvalue in the right complex half-plane, then y^* is an unstable point.

10.2 Stability of Fixed Points of Maps

We have seen in the introduction two versions of stability for equilibrium points of dynamical systems. These concepts have natural analogues for fixed points of maps.

Definition 10.2.1 Consider a map Ψ on \mathbf{R}^d and a fixed point y^* : $y^* = \Psi(y^*)$. Define $y^n(y_0)$ as the iteration of the map n times applied to y_0 , so $y^1(y_0) = \Psi(y_0)$, $y^2(y_0) = \Psi(\Psi(y_0))$, etc. We say that (1) y^* is *stable in the sense of Lyapunov* (or just *stable*) if $\forall \epsilon > 0, \exists \delta > 0, N > 0$ such that, if $\|y_0 - y^*\| < \delta$ then

$$\|y^n(y_0) - y^*\| < \epsilon, \quad n > N.$$

(2) y^* is *asymptotically stable* if there exists $\gamma > 0$ such that, for any initial condition y_0 such that $\|y_0 - y^*\| < \gamma$,

$$\lim_{n \rightarrow \infty} \|y^n(y_0) - y^*\| = 0$$

(3) y^* is *unstable* if it is not stable (in the sense of Lyapunov). \square

It is easy to see that these definitions agree with the previous definitions for continuous dynamics. That is, let us take $\Psi \equiv \Phi_t$, the time- t exact flow map of the differential equation. Then, if an equilibrium point of the continuous dynamics is stable, then it is also a stable fixed point for Ψ .

Next consider a more general map Ψ , not necessarily the flow map. What are the conditions for stability of a fixed point y^* ? In other words: *when does the sequence of iterates obtained by successively applying Ψ starting from a point near y^* eventually converge to y^* ?*

First consider a scalar linear iteration, $\Psi(y) = ay$, $a \in \mathbf{R}$, started from a point $y_0 \neq 0$. The iteration of Ψ yields the sequence $y_n = a^n y_0$. Then we have (i) $|y_n| \rightarrow 0$, if $|a| < 1$, (ii) $|y_n| \equiv |y_0|$, if $|a| = 1$, (iii) $|y_n| \rightarrow \infty$, $|a| > 1$. The fixed point is at the origin, so we see that we have (i) asymptotical stability, (ii) stability, and (iii) instability.

Now consider a linear iteration in \mathbf{R}^d . Then we have $\Psi(y) = Ky$, $K \in \mathbf{R}^{d \times d}$. We consider the convergence to the fixed point at the origin of the sequence of points y_0, y_1, \dots generated by

$$y_n = Ky_{n-1}.$$

We have $y_1 = Ky_0$, $y_2 = Ky_1 = K^2 y_0$, etc. The question concerns the norms of powers of K :

$$\|K^n y_0\|.$$

Let $\rho(K)$ denote the *spectral radius* of the matrix K , i.e. the radius of the smallest circle centered at the origin enclosing all eigenvalues of K . We have a standard theorem of linear iterations to apply to this case:

Theorem 10.2.1 *Let $z_n = \|K^n y_0\|$. Then (i) $z_n \rightarrow 0$, as $n \rightarrow \infty$, if and only if,*

$$\rho(K) < 1$$

Moreover, (ii) $z_n \rightarrow \infty$ for some y_0 , if

$$\rho(K) > 1$$

Finally, (iii) if $\rho(K) = 1$ and eigenvalues on the unit circle are semisimple, then $\{z_n\}$ remains bounded as $n \rightarrow \infty$.

When the eigenvectors of K form a basis of \mathbf{R}^d , then all parts are easy to prove by diagonalizing the matrix. When K does not have a basis of eigenvectors, we use the Jordan canonical form for this purpose. Recall that an eigenvalue is semisimple if the dimension of its largest Jordan block is 1.

Finally we turn our attention to a more general iteration of the form

$$y_n = \Psi(y_{n-1}), \tag{10.2}$$

where Ψ is a map on \mathbf{R}^d .

Note that

$$y^* = \Psi(y^*). \tag{10.3}$$

Take the difference of (10.2) and (10.3) to obtain

$$y_n - y^* = \Psi(y_{n-1}) - \Psi(y^*).$$

Next, we use Taylor's Theorem for functions from \mathbf{R}^d to \mathbf{R}^d to obtain

$$y_n - y^* = \Psi'(y^*)(y_{n-1} - y^*) + R(y_{n-1} - y^*).$$

$\Psi'(y^*)$ here refers to the Jacobian matrix of Ψ . $R(\Delta)$ is a remainder term which goes to zero quadratically:

$$R(\Delta) = O(\|\Delta\|^2)$$

When $\Delta = y_{n-1} - y^*$ is small in norm, its squared norm is very small. It is natural to think of simply neglecting the remainder term $R(y_{n-1} - y^*)$, i.e.,

$$y_n - y^* \approx \Psi'(y^*)(y_{n-1} - y^*).$$

in which case the convergence would appear to be related to convergence of a linear iteration.

Whether we can neglect the remainder depends on the eigenvalues of Ψ' . It is possible to show the following theorem:

Theorem 10.2.2 *Given a smooth (C^2) map Ψ , (i) the fixed point y^* is asymptotically stable for the iteration $y_{n+1} = \Psi(y_n)$ if*

$$\rho(\Psi'(y^*)) < 1.$$

(ii) the fixed point y^ is unstable if $\rho(\Psi'(y^*)) > 1$.*

The marginal case $\rho(\Psi') = 1$ is delicate and must be considered on a case-by-case basis.

10.3 Stability of Numerical Methods: Linear Case

For a linear system of ODEs

$$dy/dt = Ay,$$

where A is a $d \times d$ matrix with a basis of eigenvectors, it can easily be shown that the general solution can be written in the compact form

$$y(t) = \sum_{i=1}^d C_i e^{\lambda_i t} u_i,$$

where $\lambda_1, \lambda_2, \dots, \lambda_d$ are the eigenvalues, u_1, u_2, \dots, u_d are the corresponding eigenvectors, and C_1, C_2, \dots, C_d are coefficients. It is easy to see that this means that stability is determined by the eigenvalues. For example, if all the eigenvalues lie in the left half plane, then the origin is stable in the sense of Lyapunov. Also, if all the eigenvalues have negative real part, then the origin is asymptotically stable.

A related statement can be shown to hold for many of the numerical methods in common use. For example, consider Euler's method applied to the linear problem $dy/dt = Ay$:

$$y_{n+1} = y_n + hAy_n = (I + hA)y_n.$$

If we let y_n be expanded in the eigenbasis (say u_1, u_2, \dots, u_d), we may write

$$y_n = \alpha_1^{(n)} u_1 + \alpha_2^{(n)} u_2 + \dots + \alpha_d^{(n)} u_d.$$

If we now apply Euler's method, we find

$$\begin{aligned} y_{n+1} &= (I + hA)(\alpha_1^{(n)} u_1 + \alpha_2^{(n)} u_2 + \dots + \alpha_d^{(n)} u_d), \\ &= \alpha_1^{(n)} (I + hA)u_1 + \alpha_2^{(n)} (I + hA)u_2 + \dots + \alpha_d^{(n)} (I + hA)u_d \end{aligned}$$

Now, as u_i is an eigenvector of A , we have

$$(I + hA)u_i = u_i + hAu_i = u_i + h\lambda_i u_i = (1 + h\lambda_i)u_i$$

and this implies

$$y_{n+1} = \sum_{i=1}^d \alpha_i^{(n)} (1 + h\lambda_i) u_i.$$

We may also write

$$y_{n+1} = \sum_{i=1}^s \alpha_i^{(n+1)} u_i,$$

and, comparing these last two equations and using the uniqueness of the basis representation, we must have

$$\alpha_i^{(n+1)} = (1 + h\lambda_i) \alpha_i^{(n)}.$$

It follows from this that the origin is a stable point if $|1 + h\lambda_i| \leq 1$, $i = 1, 2, \dots, d$, and is asymptotically stable if $|1 + h\lambda_i| < 1$, $i = 1, 2, \dots, d$. The condition for stability can be stated equivalently as requiring that for every eigenvalue λ of A , $h\lambda$ must lie inside a disk of radius 1 centred at $z = -1$ in the complex plane. This region is sketched below. We call it the *region of absolute stability* of Euler's method.

Thus, given the set of eigenvalues of A , this condition implies a restriction on the maximum stepsize h that can be used if the origin is to remain stable for the numerical map. This is illustrated in Figure 10.2.

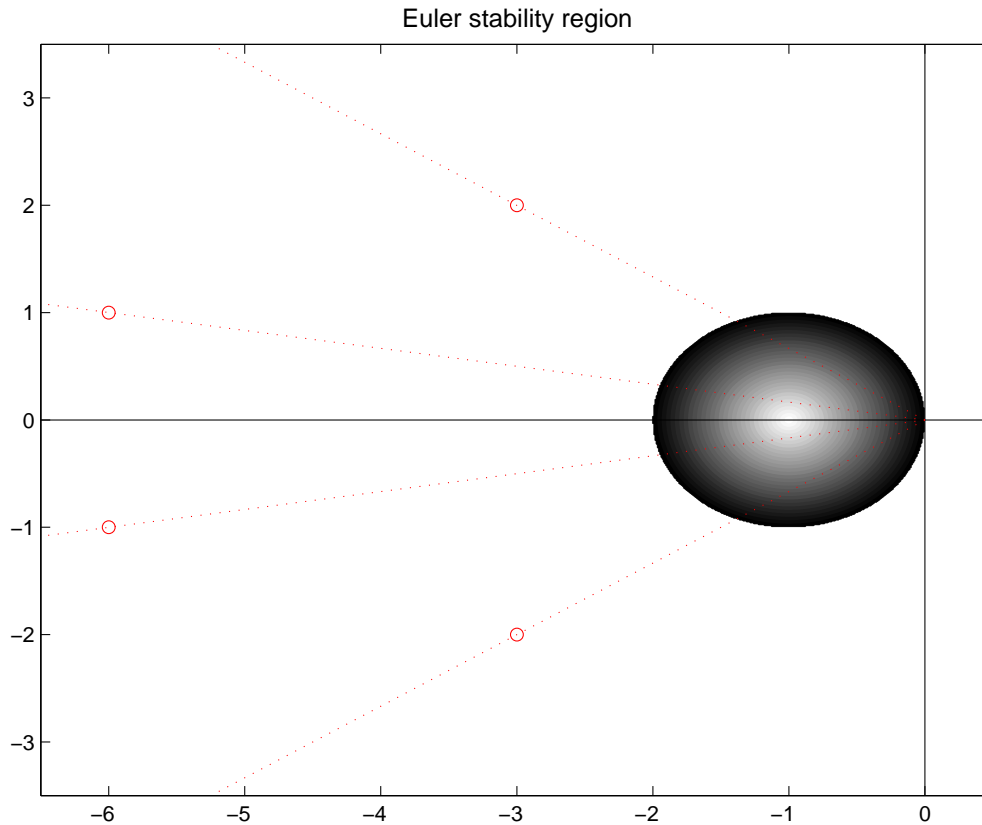


Figure 10.2: The spectrum of A is scaled by h . Stability of the origin is recovered if $h\lambda$ is in the region of absolute stability $|1 + z| < 1$ in the complex plane.

10.3.1 Stability functions

To develop a general theory, let us consider first the scalar case. When applied to $y' = \lambda y$, Euler's method has the form

$$y_{n+1} = P(h\lambda)y_n,$$

where $P(\mu) = I + \mu$. Many methods, including all *explicit* Runge-Kutta methods, have the form $y_{n+1} = P(\lambda)y_n$, for some polynomial $P(x)$, when applied $y' = \lambda y$. More generally, *all* Runge-Kutta methods, including the implicit ones, when applied to $y' = \lambda y$ can be written in the form

$$y_{n+1} = R(h\lambda)y_n,$$

where $R(\mu)$ is a rational function of μ , i.e. the ratio of two polynomials

$$R(\mu) = P(\mu)/Q(\mu),$$

where P and Q are two polynomials. R is called the *stability function* of the RK method.

To see this, apply a general Runge-Kutta method to the linear test problem $y' = \lambda y$. We get for the internal stages the relations

$$Y_i = y_n + \mu \sum_{j=1}^s a_{ij}Y_j, \quad i = 1, \dots, s.$$

Introducing the vector $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^s$, the above system can be cast in matrix form (with $Y = (Y_1, \dots, Y_s)^T$ and A the matrix with entries a_{ij}),

$$Y = y_n \mathbf{1} - \mu AY,$$

which means $Y = y_n(I - \mu A)^{-1} \mathbf{1}$. Similarly we can write

$$y_{n+1} = y_n + \mu \sum_{j=1}^s b_j Y_j = y_n + \mu b^T Y.$$

Combining these expression yields the desired stability function R :

$$y_{n+1} = R(\mu)y_n, \quad R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} \mathbf{1} \quad (10.4)$$

That this function is a rational function in μ follows from Kramer's rule for computing the inverse of a matrix.

Next, Consider applying the general RK method to $y' = Ay$:

$$y_{n+1} = y_n + h \sum_{i=1}^s b_i AY_i,$$

where

$$Y_i = y_n + h \sum_{j=1}^s a_{ij} AY_j.$$

Now expand y_n , y_{n+1} and Y_i , $i = 1, \dots, s$ in the eigenbasis (a linearly independent set of d eigenvectors) of A . For simplicity, we write the eigenvectors as columns of a matrix U , so that $AU = U\Lambda$, where Λ is the diagonal matrix of eigenvalues. So $U^{-1}AU = \Lambda$ is the diagonalization of A . We define z_n and Z_i , $i = 1, \dots, s$ by

$$y_n = Uz_n, \quad Y_i = UZ_i, \quad i = 1, \dots, s,$$

then we can rewrite our Runge-Kutta method as

$$Uz_{n+1} = Uz_n + h \sum_{i=1}^s b_i AUZ_i,$$

or

$$z_{n+1} = z_n + h \sum_{i=1}^s b_i \Lambda Z_i.$$

Now for the internal variables, we have

$$UZ_i = Uz_n + h \sum_{j=1}^s a_{ij} AUZ_j,$$

or

$$Z_i = z_n + h \sum_{j=1}^s a_{ij} \Lambda Z_j.$$

We may observe that, since Λ is a diagonal matrix, the system completely decouples into d independent scalar iterations: this is equivalent to the application of the same Runge-Kutta method to the d scalar (complex) differential equations

$$\frac{dz^{(i)}}{dt} = \lambda_i z^{(i)},$$

where $y = Uz$ and $z^T = (z^{(1)}, z^{(2)}, \dots, z^{(d)})$.

The result of all this is that we can analyze the stability of the origin for a given numerical method by just considering the scalar problem $dx/dt = \lambda x$. We state this in a theorem:

Theorem 10.3.1 *Given the differential equation $y' = Ay$, where we suppose that the $d \times d$ matrix A has a basis of eigenvectors and the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$, consider applying a particular Runge-Kutta method. The RK method has a stable (asymptotically stable) fixed point at the origin when applied to*

$$\frac{dy}{dt} = Ay,$$

if and only if the same method has a stable (asymptotically stable) equilibrium at the origin when applied to each of the scalar differential equations

$$\frac{dx}{dt} = \lambda x,$$

where λ is an eigenvalue of A .

Since we know that a Runge-Kutta method applied to $\frac{dx}{dt} = \lambda x$ can be written

$$x_{n+1} = R(h\lambda)x_n$$

we have the following corollary:

Corollary 10.3.1 *Consider a linear differential equation $dy/dt = Ay$ with diagonalizable matrix A . Let an RK method be given with stability function R . The origin is stable for the numerical method applied to $dy/dt = Ay$ (at stepsize h) if and only if*

$$|R(\mu)| \leq 1$$

for all $\mu = h\lambda$, where $\lambda \in \sigma(A)$.

10.3.2 Stability regions, A-stability and L-stability

Evidently the key issue for understanding the long-term dynamics of Runge-Kutta methods near fixed points concerns the region where $\hat{R}(\mu) = |R(\mu)| \leq 1$. This is what we call the *stability region* of the numerical method. Let us examine a few stability functions and regions:

Euler's Method:

$$\hat{R}(\mu) = |1 + \mu|$$

The stability region is the set of points such that $\hat{R}(\mu) \leq 1$. The condition

$$|1 + \mu| \leq 1$$

means μ lies inside of a disc of radius 1, centred at the point -1 .

Trapezoidal rule: the stability region is the region where:

$$\hat{R}(\mu) = \left| \frac{1 + \mu/2}{1 - \mu/2} \right| \leq 1.$$

This occurs when

$$|1 + \mu/2| \leq |1 - \mu/2|,$$

or, when $\mu/2$ is closer to -1 than to 1 , which is just the entire left complex half-plane.

Another popular method: *Implicit Euler*,

$$x_{n+1} = x_n + h\lambda x_{n+1}$$

$$\hat{R}(\mu) = |1 - \mu|^{-1}.$$

which means the stability region is the *exterior* of the disk of radius 1 centred at 1 in the complex plane. These are some simple examples. All three of these are graphed in Figure 10.3.

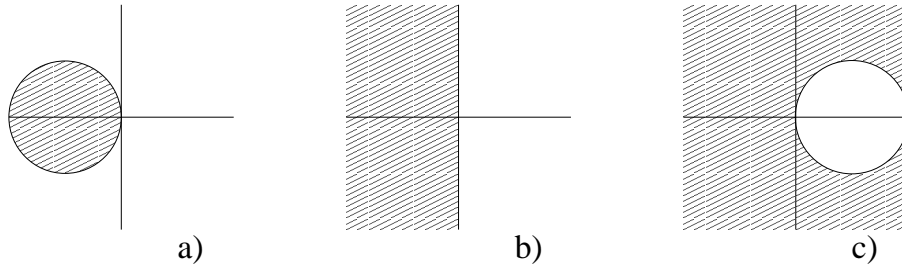


Figure 10.3: Stability Regions: (a) Euler's method, (b) trapezoidal rule, (c) implicit Euler

For the fourth-order Runge-Kutta method (8.4), the stability function is found to be:

$$R(\mu) = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4.$$

Note that as we would expect, $R(h\lambda)$ agrees with the Taylor Series expansion of $\exp(h\lambda)$ through fourth order; the latter gives the exact flow map for $dx/dt = \lambda x$. To graph R we could use the following trick. For each value of μ , R is a complex number. The boundary of the stability region is the set of all μ such that $R(\mu)$ is on the unit circle. That means

$$R(\mu) = e^{i\theta},$$

for some $\theta \in [0, 2\pi]$. One way to proceed is to solve the equation $R(\mu) = e^{i\theta}$ for various values of θ and plot the points. There are four roots of this quartic polynomial equation, and in theory we can obtain them in a closed form expression. Unfortunately they are a bit tedious to work out in practice. Certainly we will need some help from Maple.

A second approach, more suited to MATLAB than Maple, is to just plot some level curves of \hat{R} viewed as a function of x and y , the real and imaginary parts of μ . In fact, the single level curve $\hat{R} = 1$ is just precisely the boundary of the stability region! It is useful to have the nearby ones for a range of values near 1. In Matlab, we can achieve this as follows:

```
>> clear i;
>> [X,Y] = meshgrid(-5:0.01:5,-5:0.01:5);
>> Mu = X+i*Y;
>> R = 1 + Mu + .5*Mu.^2 + (1/6)*Mu.^3 + (1/24)*Mu.^4;
>> Rhat = abs(R);
>> contourplot(X,Y,Rhat,[1]);
```

Figure 10.4 shows stability regions for some Runge-Kutta methods up to order 4. The shading in the figure indicates the magnitude $|R(z)|$ within the stability region.

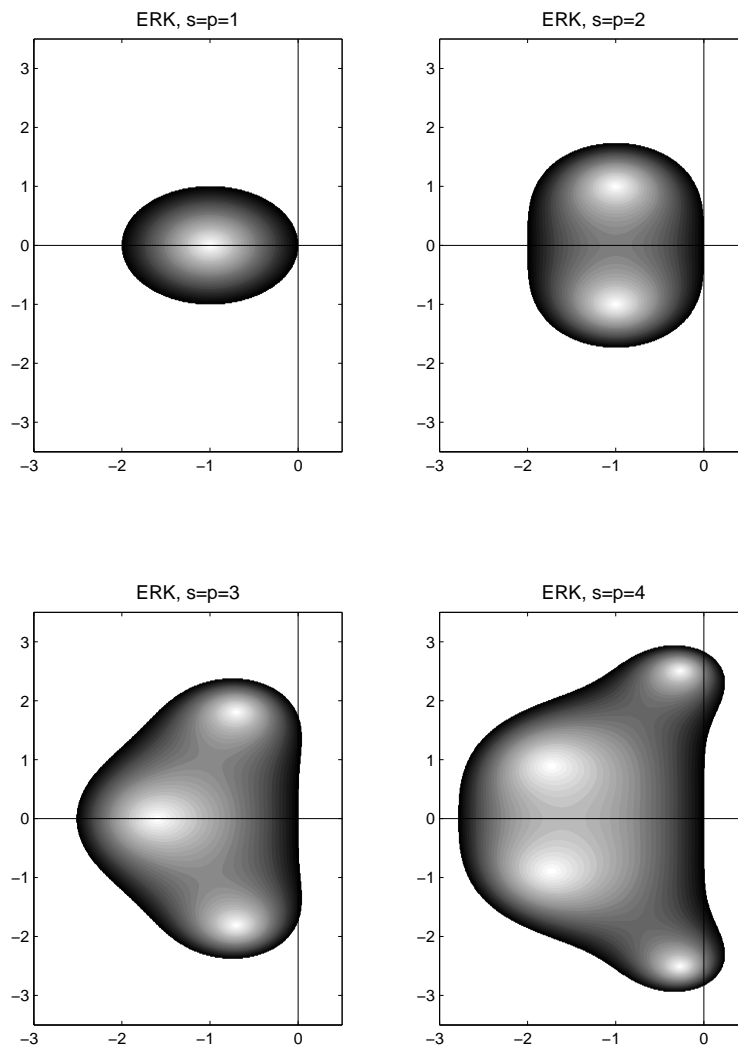


Figure 10.4: Explicit Runge-Kutta Stability Regions

What is the meaning of these funny diagrams? They tell us a huge amount. Consider first a scalar differential equation $dx/dt = \lambda x$, with possibly complex λ . We know that for the differential equation, the origin is stable for λ lying in the left half plane, or, if we think of the map $\Phi_h = e^{h\lambda}$ as defining a discrete dynamics, the origin is stable independent of h if λ is in the left half plane.

For a numerical method, the origin is stable if $h\lambda$ lies in the stability region. This is not usually going to happen independent of h , even if λ lies in the left half plane. For the methods seen above, this is only true of trapezoidal rule and implicit Euler. For Euler's method and for the 4th order RK method the origin is only stable for the numerical method provided h is suitably restricted. Note that, in the case of the Euler method, if λ lies strictly in the left half plane, you can see that for h sufficiently small, we will have $\hat{R}(h\lambda) < 1$, since the rescaling of λ by a smaller value of h moves the point $h\lambda$ towards the origin.

On the other hand, observe that this will be impossible to achieve if λ lies on the imaginary axis. Thus Euler's method is a very poor choice for integrating a problem like $dx/dt = i\omega x$, where ω is real. A very interesting question is whether the 4th order Runge-Kutta method has an unstable point at the origin for λ on the imaginary axis and $h \rightarrow 0$. The operative question is: does the RK-4 stability region of 10.4 contain a segment on the imaginary axis containing 0? The answer is not obvious from the picture, although it certainly appears to be plausible.

The fact that a linear system can be decomposed into scalar linear problems (involving the eigenvalues as coefficients) means that we can apply the same approach to analyze the stability of a system by applying the scalar argument to each of the eigenvalues.

A very valuable feature if we are interested in preserving the asymptotic stability of the origin under discretization is if the stability region includes the entire left half plane. In this case we say that the method is *A-stable*. An A-stable numerical method has the property that the origin is stable regardless of the stepsize.

Another important issue is whether the numerical method matches the asymptotic behavior as $\mu \rightarrow \infty$. If, in the scalar differential equation $dx/dt = \lambda x$, the coefficient λ tends to $-\infty$, then we know that the solution will decay increasingly fast. We could ask that the numerical method have a similar property. We say that a method is L-stable if it has the properties (1) it is A-stable, and (2) $R(\mu) \rightarrow 0$, as $\mu \rightarrow \infty$.

10.4 Oscillatory Systems

An important special case for stability analysis is that of oscillatory linear systems, which have all their eigenvalues on the imaginary axis. For the moment, we will consider just a harmonic oscillator

$$\frac{dx}{dt} = u, \quad (10.5)$$

$$\frac{du}{dt} = -\omega^2 x. \quad (10.6)$$

To understand the asymptotic behavior of, for example, Runge-Kutta methods for the harmonic oscillator we need only compute the eigenvalues of this 2nd order problem (converting first to a 2-dimensional first order system) and we find that these are $\pm i\omega$. Thus the behavior of an RK method is just determined by the approximation of the stability region to the imaginary axis.

However, it is also of interest to consider some methods constructed in alternate ways and for which the stability region must be defined differently. One such method is the *Störmer-Verlet* or *leapfrog* method which is applicable to problems of the form $dx/dt = u$; $du/dt = f(x)$:

$$x_{n+1} = x_n + hu_{n+1/2}, \quad (10.7)$$

$$u_{n+1/2} = u_n + \frac{h}{2}f(x_n), \quad (10.8)$$

$$u_{n+1} = u_{n+1/2} + \frac{h}{2}f(x_{n+1}). \quad (10.9)$$

We will therefore consider various numerical methods applied directly to (10.5)-(10.6) and analyze their asymptotic behavior directly.

Applying the leapfrog method to the harmonic oscillator results in the propagator

$$\Psi_h \left(\begin{bmatrix} x \\ u \end{bmatrix} \right) = \begin{bmatrix} 1 - \frac{h^2\omega^2}{2} & h \\ -h\omega^2 \left(1 - \frac{h^2\omega^2}{4}\right) & 1 - \frac{h^2\omega^2}{2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \equiv M(\theta) \begin{bmatrix} x \\ u \end{bmatrix}, \quad \theta = h\omega. \quad (10.10)$$

The matrix $M(\theta)$ has eigenvalues

$$\lambda_{\pm} = 1 - \frac{1}{2}\theta^2 \pm \sqrt{\theta^2\left(\frac{1}{4}\theta^2 - 1\right)},$$

$\lambda_+\lambda_- = 1$ for all $\mu \geq 0$, hence $\det M(\theta)$ is equal to one. For $\theta^2 < 4$, the eigenvalues are complex, and

$$|\lambda_{\pm}|^2 = \left(1 - \frac{1}{2}\theta^2\right)^2 + \theta^2\left(1 - \frac{1}{4}\theta^2\right) = 1.$$

When $\theta^2 > 4$, both of the eigenvalues are real and one always has modulus greater than unity. This implies that, for the leapfrog method applied to the harmonic oscillator, the origin is a stable equilibrium point if

$$h^2\omega^2 < 4. \quad (10.11)$$

As h increases from $h = 0$, the eigenvalues of the leapfrog method move around the unit circle until $h = 2/\omega$, at which point the eigenvalues are both at -1 . As we continue to increase h , one eigenvalue tends toward the origin, while the other increases without bound along the negative real axis. The situation is diagrammed in Figure 10.5, below.

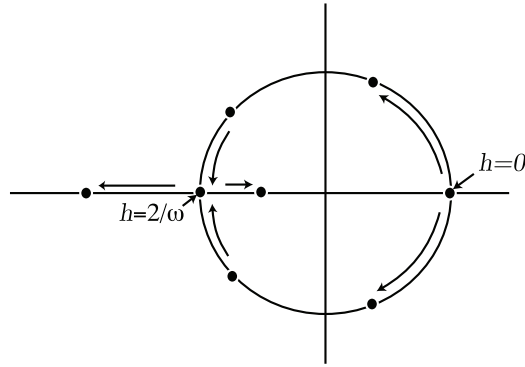


Figure 10.5: Eigenvalues of the leapfrog method for the harmonic oscillator: as h increases, the complex conjugate pair moves around the unit circle, separating into two real eigenvalues at $h\omega = 2$.

This discussion can be extended to a $2d$ -dimensional linear systems of differential equations of the form

$$\frac{d}{dt}q = v, \quad (10.12)$$

$$\frac{d}{dt}v = -Kq. \quad (10.13)$$

Here K is an $d \times d$ symmetric matrix. The stability of the equilibrium point at the origin in a linear system like this depends on the eigenvalue–eigenvector structure of the matrix K . When the

dimension d is large, there will be many eigenvalues ($2d$ of them, to be precise) but the stability of the origin under discretization is determined by the highest frequency term. Linear systems like (10.12)-(10.13) arise naturally through linearization of nonlinear systems of the form

$$M \frac{d^2 x}{dt^2} = -\nabla V(x)$$

near equilibrium points.

10.5 Exercises

1. Use Theorem 10.1.1 to show that the pendulum equation with damping

$$\frac{d^2 \theta}{dt^2} + \mu \frac{d\theta}{dt} + (g/L) \sin \theta = 0$$

has an asymptotically stable equilibrium at the origin in the phase plane. What does the theorem tell us about the undamped case $\mu \equiv 0$?

2. Consider the following system of differential equations expressing damped mechanical motion in a double-well potential:

$$\dot{q} = p, \quad \dot{p} = -q(q^2 - 1) - 2p.$$

Determine the equilibrium points and their stability.

3. Compute the stability functions and sketch the stability regions for (a) Heun's Method (8.3), (b) the implicit midpoint method

$$y_{n+1} = y_n + hf((y_n + y_{n+1})/2),$$

- (c) the RK method with Butcher diagram:

$$\begin{array}{c|cc} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 0 \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array},$$

Which of these three methods (a), (b), (c) are A-stable? L-stable?

4. Prove that an A-stable Runge-Kutta method is necessarily implicit.
5. Why does it not matter in which direction μ tends to ∞ in the definition of L-stability?
6. Show that the following family of Runge-Kutta methods all are second order ($\alpha > 0$ is a free parameter). Compute the stability function and graph the stability regions for $\alpha = 1/2, 3/4$.

$$\begin{array}{c|cc} 0 & & \\ (2\alpha)^{-1} & (2\alpha)^{-1} & \\ \hline & 1 - \alpha & \alpha \end{array},$$