Chapter 4

The Matrix Exponential

Main concepts: In this chapter we solve systems of linear differential equations, introducing the matrix exponential and related functions, and the variation of constants formula.

In general it is possible to exactly solve systems of linear differential equations with constant coefficients using the variation of constant forumula. Because linear systems are of both theoretical and practical importance, will delve into this in some detail. Some numerical techniques for nonlinear problems that we will encounter later require integrating linear systems exactly. To begin, we derive the solution in the scalar case.

4.1 The general scalar, linear equation in the complex plane.

Consider the linear, scalar ODE with constant coefficients in $\mathbb{C}$

$$\dot{x} = \lambda x + c, \tag{4.1}$$

λ, $c \in \mathbb{C}$, $x(t) : \mathbb{R} \to \mathbb{C}$.

We solve this equations as follows:

$$\begin{aligned}
(\dot{x}(s) - \lambda x(s))\, e^{-\lambda s} &= c e^{-\lambda s}, \\
\frac{d}{dt}\left(x(s) e^{-\lambda s}\right) &= c e^{-\lambda s}, \\
\int_0^t x(s) e^{-\lambda s}\, ds &= c \int_0^t e^{-\lambda s}\, ds, \\
x(t) e^{-\lambda t} - x(0) &= -\frac{c}{\lambda}(e^{-\lambda t} - 1),
\end{aligned}$$

yielding

$$x(t) = e^{\lambda t} x(0) + \frac{1}{\lambda}(e^{\lambda t} - 1)c. \tag{4.2}$$

4.2 Systems of linear ODEs

Next we consider linear differential equations of the form

$$\dot{y} = Ay + b \tag{4.3}$$

where $y(t) : \mathbb{R} \to \mathbb{C}^d$, $A \in \mathbb{C}^{d \times d}$, $b \in \mathbb{C}^d$.

4.2.1 Diagonalizable systems

First, let us suppose that the eigenvectors of A span $\mathbb{C}^d$. Thus, there are d linearly independent eigenvectors $z_i \in \mathbb{C}^d$, $i = 1, \ldots, d$, and corresponding eigenvalues $\lambda_i \in \mathbb{C}$ satisfying

$$Az_i = \lambda_i z_i, \qquad i = 1, \ldots, d \tag{4.4}$$

We can construct a square matrix Z whose columns are the z_i:

$$Z = [z_1 | \ldots | z_d]$$

and, since the z_i are linearly independent, Z is invertible. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ be the diagonal matrix with the ordered eigenvalues on the diagonal. Equation (4.4) is equivalent to

$$AZ = Z\Lambda,$$

from which is follows from the invertibility of Z that

$$A = Z\Lambda Z^{-1}.$$

Inserting this into (4.3) gives

$$\dot{y} = Z\Lambda Z^{-1} y + b.$$

Next let us introduce the transformation $x(t) = Z^{-1}y(t)$, $c = Z^{-1}b$. The equation above transforms to

$$\dot{x} = \Lambda x + c.$$

However, since Λ is diagonal, the above differential equation can be written as the folowing system of decoupled scalar equations

$$\dot{x}^{(i)} = \lambda_i x^{(i)} + c^{(i)}, \quad i = 1, \ldots, d.$$

These are of the form (4.1) and so we can make use of the exact solution (4.2) to determine $x^{(i)}(t)$. Given the time evolution of the $x^{(i)}$ we can then reconstruct the solution from the relation $y(t) = Zx(t)$.

The case for general A is not much more difficult. But we need some background on the matrix exponential function first.

4.2.2 The matrix exponential

The exponential of a square matrix $A \in \mathbb{C}^{d\times d}$ is defined as

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \tag{4.5}$$

where $A^0 = I$, the identity matrix on $\mathbb{C}^d$.

The series in (4.5) converges for any matrix A. To see this, let $M > 0$ satisfy

$$|A_{ij}| \le M, \quad i, j = 1, \ldots, d.$$

Note that for all i, j,

$$|(A^2)_{ij}| \le dM^2$$

and in fact

$$|(A^k)_{ij}| \le d^{k-1}M^k \le (dM)^k$$

It follows that

$$|\exp A_{ij}| \le \sum_{k=0}^{\infty} (dM)^k = e^{dM} < \infty.$$

We also use the notation e^A for $\exp A$.

You can check that the matrix exponential satisfies the following properties:

- Commutativity with A: $Ae^{tA} = e^{tA}A$
- For any A, e^A is nonsingular and $(e^A)^{-1} = e^{-A}$.
- For $t \in \mathbb{R}$, $\frac{d}{dt}e^{tA} = Ae^{tA}$.

4.2.3 Variation of constants formula

Now we return to (4.3). Equipped with the matrix exponential, we basically construct the solution as in the scalar case. We can also handle the case of time dependent $b = b(t)$

$$
\begin{aligned}
(\dot{y} - Ay)e^{-As} &= e^{-As}b, \\
\frac{d}{dt}(e^{-As}y) &= e^{-As}b, \\
\int_0^t \frac{d}{dt}(e^{-As}y)\,ds &= (\int_0^t e^{-As}b(s)\,ds), \\
e^{-At}y(t) - y(0) &= (\int_0^t e^{-As}b(s)\,ds),
\end{aligned}
$$

resulting in the *variation of constants formula*

$$y(t) = e^{At}y(0) + (\int_0^t e^{(t-s)A}b(s)\,ds) \tag{4.6}$$

When b is constant, this simplifies to

$$y(t) = e^{At}y(0) + t\theta(At)b, \tag{4.7}$$

where for invertible A and $t \neq 0$ the function $\theta(At)$ can be written

$$\theta(At) = (At)^{-1}(e^{At} - I)$$

For singular A, θ is given by the series expansion

$$\theta(A) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}A^k.$$

You can check that this series also converges for any matrix A.
Example. We calculate $\exp(At)$ for the matrix

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

The eigenvectors of A are $z_1 = (1,1)^T$, $z_2 = (1,-1)^T$, with corresponding eigenvalues $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$. Thus,

$$Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad Z{-}1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and $\Lambda = \text{diag}\{i\omega, -i\omega\}$. We compute

$$\exp(At) = \exp(Z\Lambda Z^{-1}t) = Z\exp(\Lambda t)Z^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos\omega t + i\sin\omega t & 0 \\ 0 & \cos\omega t - i\sin\omega t \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

, or

$$\exp(At) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \tag{4.8}$$

Exercise. Consider the system

$$\begin{pmatrix} \frac{dq}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = A \begin{pmatrix} q \\ p \end{pmatrix} .$$

Show that the exponential of A in this example is

$$\exp tA = \begin{bmatrix} \cos kt & k^{-1} \sin kt \\ -k \sin kt & \cos kt \end{bmatrix} . \tag{4.9}$$

4.3 Computing the exponential of a matrix

Scaled and squared Padé approximation, exercise caution, counterexample ...