# Chapter 6

# Convergence of One-Step Methods

**Main concepts:** In this chapter we prove the convergence of a large class of one-step methods to the exact flow map. Keywords are: Taylor series, local error, convergence theory for one-step methods.

## 6.1 Approximation

A numerical method is an approximation of the exact flow map of a differential equation. As such, there is always an element of error in the numerically generated solution. If the method is to be useful, we must be able to control this error, at least on short enough time intervals. In this chapter we consider convergence of the numerical flow map $\Psi_h$ to the exact flow map $\Phi_h$ in the approximation limit $h \to 0$.

We define the *global error* after $n$ time steps to be the difference between the discrete approximation and the exact solution

$$e_n := y_n - y(t_n). \tag{6.1}$$

For an approximation we want the error to be small in norm at each step of simulation, that is, we would like to satisfy

$$\max_{n=0,1,\ldots N} \|e_n\| \leq \delta,$$

for some user-specified tolerance $\delta$. For a given vector field $f$, initial value $y_0$ and time interval $T$, we have only one free parameter, the timestep $h = T/N$, which we can vary to make sure the norm of the error meets our tolerance. If the method is going to be useful, we must be able to vary $h$ to meet any tolerance we choose.

Given a Lipschitz vector field $f$, a method is said to be *convergent* if, for every $T$,

$$\lim_{\substack{h\to 0,\\ h=T/N}} \max_{n=0,1,\ldots,N} \|e_n\| = 0.$$

(Note that this definition considers only discrete values of $h$ which are integral fractions of the time interval. Equivalently we could take the limit as $N \to \infty$ with $h = T/N$.) In this chapter we establish conditions for the convergence of a large class of one-step methods.

A tool of singular importance in numerical analysis is Taylor's series, for which the relevant form here is

$$y(t+\tau) = y(t) + \tau y'(t) + \frac{\tau^2}{2} y''(t) + \cdots \tag{6.2}$$

for a perturbation $\tau > 0$ around $t$. For a *scalar function* ($d = 1$), assuming $y(t)$ is $\nu$-times continuously differentiable, Taylor's theorem says there is a point $t^* \in [t, t+\tau]$ such that

$$y(t+\tau) = \sum_{i=0}^{\nu-1} \frac{\tau^i}{i!} \frac{d^i y}{dt^i}(t) + \frac{\tau^\nu}{\nu!} \frac{d^\nu y}{dt^\nu}(t^*).$$

For a vector function ($d > 1$), such a statement holds for each component, but in general the mean value will be attained at a different $t^*$ for each component. Nonetheless the norm of the last (remainder) term is bounded on $[t, t+\tau]$, and we have

$$\|\frac{\tau^\nu}{\nu!}\frac{d^\nu y}{dt^\nu}(t)\| \leq C\tau^\nu$$

so we write

$$y(t+\tau) = \sum_{i=0}^{\nu-1} \frac{\tau^i}{i!}\frac{d^i y}{dt^i}(t) + \mathcal{O}(\tau^\nu). \tag{6.3}$$

in general.

## 6.2 Convergence of generalized one-step methods

Define the **local error** of a numerical method as the difference between the flow-map and its discrete approximation:

$$le(y,h) = \Psi_h(y) - \Phi_h(y).$$

The local error measures just how much error is introduced in a single timestep of size $h$. Let us assume that, on our (invariant) domain of interest $D$, we can expand $le$ in powers of $h$ (typically using Taylor series), and that it satisfies

$$\|le(y,h)\| \leq Ch^{p+1}, \tag{6.4}$$

where $C$ is a constant that depends on $y(t)$ and its derivatives, and $p \geq 1$. A method that meets this criterion is said to be **consistent**.

We will further suppose that $f$ is Lipschitz with constant $L$.

The error can be viewed as the difference between $n$ iterations of $\Psi_h$ and $n$ iterations of $\Phi_h$, thus we define it to be

$$e_n = y_n - y(t_n),$$

so

$$e_{n+1} = y_{n+1} - y(t_{n+1}) = \Psi_h(y_n) - \Phi_h(y(t_n)).$$

To this expression we subtract and add $\Phi_h(y_n)$, which is the exact solution started from a point on the numerical trajectory, then take norms to obtain

$$\begin{aligned} \|e_{n+1}\| &= \|\Psi_h(y_n) - \Phi_h(y_n) + \Phi_h(y_n) - \Phi_h(y(t_n))\| \\ &\leq \|\Psi_h(y_n) - \Phi_h(y_n)\| + \|\Phi_h(y_n) - \Phi_h(y(t_n))\|. \end{aligned}$$

Now we use assumption (6.4) to bound the first term, and the Gronwall Lemma (3.2.1) to bound the second term (cf. 3.5), obtaining

$$\begin{aligned} \|e_{n+1}\| &\leq Ch^{p+1} + e^{hL}\|y_n - y(t_n)\| \\ &= e^{hL}\|e_n\| + Ch^{p+1}. \end{aligned}$$

Next applying the discrete Gronwall Lemma (3.2.2) yields

$$\|e_n\| \leq e^{Lhn}\|e_0\| + \frac{Ch^{p+1}}{e^{Lh}-1}(e^{Lhn}-1).$$

Note that for positive $h$ and $L$,

$$\frac{1}{e^{hL}-1} \leq \frac{1}{hL}.$$

Furthermore, on our integration interval $0 \leq nh \leq T$ implies $e^{Lnh} \leq e^{LT}$. If we assume the initial condition is exact $e_0 = 0$, we get the uniform bound

$$\|e_n\| \leq h^p \frac{C}{L}(e^{LT} - 1), \tag{6.5}$$

which proves convergence at order $p$. We summarize this result in an important convergence theorem:

**Theorem 6.2.1 (Convergence of One-Step Methods)** *Given a differential equation (1.13) with Lipschitz vector field $f$ and a consistent one-step method $\Psi_h$, the global error satisfies*

$$\max_{n=0,\ldots,N=T/h} \|\Psi_h^n(y_0) - \Phi_{hn}(y_0)\| = \mathcal{O}(h^p).$$

This theorem is powerful. Without specifying anything about the structure of the method, it guarantees the convergence of any consistent one-step method.

As an example, let us use Theorem 6.2.1 to prove the convergence of Euler's method (2.1) for smooth vector fields $f$. Consider a compact domain $D \subset \mathbb{R}^d$ and suppose $f$ is smooth on $D$ and has Lipschitz constant $L$ on $D$ (since $f$ is smooth, we can take $L = \max_D \|\frac{\partial f}{\partial y}\|$).

The exact solution satisfies

$$\Phi_h(y) = y + h\frac{dy}{dt} + \frac{h^2}{2}\frac{d^2y}{dt^2} + \mathcal{O}(h^3) = y + hf(y) + \frac{h^2}{2}\frac{d^2y}{dt^2} + \mathcal{O}(h^3)$$

Therefore the local error is

$$le(y,h) = y + hf(y) - \left[y + hf(y) + \frac{h^2}{2}\frac{d^2y}{dt^2} + \mathcal{O}(h^3)\right] = \mathcal{O}(h^2),$$

and we can apply Theorem 6.2.1 with $C = \max_D \|\frac{d^2y}{dt^2}\|$ to show that Euler's method is convergent with order $p = 1$.

In the proof of the Theorem 6.2.1, the relation (6.5) indicates that the magnitude of the global error bound will be reduced in proportion to $h^p$. For example, when using Euler's method in practice, we typically observe that halving the stepsize reduces the error by a factor of two. We say for this reason that Euler's method is *1st order accurate*. The error incurred in each time step is $\mathcal{O}(h^{p+1})$, and in fact this bound holds for any fixed number of time steps. The loss of one order of $h$ occurs because the number of time steps needed to cover a fixed interval of length $T$ increases as $h \to 0$ at a rate proportional to $1/h$.

Note the proof suggests that—although in the approximation limit $h \to 0$, $T$ fixed, the error can be made as small as possible—in the dynamics limit $T \to \infty$, $h$ fixed, the global error may grow at an exponential rate.

## 6.3 Exercises

1. Use Theorem 6.2.1 to establish convergence of the trapezoidal rule (2.2) and the explicit trapezoidal rule:
$$y_{n+1} = y_n + \frac{h}{2}\left[f(y_n) + f\left(y_n + hf(y_n)\right)\right]$$
for a vector field $f$ with Lipschitz constant $L$.