# Chapter 7

# Collocation Methods

**Main concepts:** Polynomial interpolation, quadrature, collocation methods

## 7.1 Construction of one-step methods

One-step methods for (1.3) can be constructed in a variety of ways. A natural approach is to integrate both sides of the differential equation over one timestep,

$$y(t+h) - y(t) = \int_t^{t+h} f(y(\tau))d\tau,$$

and then to approximate the integral on the right by a finite sum. For example, the simplest approximation is

$$\int_t^{t+h} f(y(\tau))d\tau \approx hf(y(t)). \tag{7.1}$$

The general idea in discretization is to replace the approximate relation satisfied by the true solution

$$y(t+h) - y(t) \approx hf(y(t))$$

by an exact equation relating the approximate values,

$$y_{n+1} - y_n = hf(y_n).$$

In this case, we see that the choice (7.1) leads to Euler's method (2.1).

A numerical method to approximate a definite integral of one independent variable is known as a *numerical quadrature rule*. We can approximate the integral by any of a variety of different numerical quadrature methods, yielding in this way various methods which may have better accuracy, or other favorable properties. For example, the trapezoidal rule approximation to the integral is

$$\int_t^{t+h} f(y(\tau))d\tau \approx \frac{h}{2}(f(y(t)) + f(y(t+h))),$$

which results in the trapezoidal rule numerical method (2.2).

In this chapter we introduce one-step methods derived using quadrature over polynomial interpolants, using collocation. We begin with a review of polynomial interpolation.

## 7.2 Polynomial interpolation

Let $\mathbb{P}_s$ denote the space of real polynomials of degree $\leq s$. Given and set of $s$ distinct *abscissa points* $c_1 < \cdots < c_s$, $c_i \in \mathbb{R}$, and corresponding *data* $g_1, \ldots, g_s$, there exists a unique polynomial

$P(x) \in \mathbb{P}_{s-1}$ satisfying $P(c_i) = g_i$, $i = 1, \ldots, s$. This polynomial is called the **interpolating polynomial**.

The Lagrange interpolating polynomials $\ell_i$, $i = 1, \ldots, s$ for a set of abscissae are defined by

$$\ell_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{x - c_j}{c_i - c_j}. \tag{7.2}$$

The $i$th Lagrange interpolating polynomial $\ell_i$ is the interpolating polynomial corresponding to the data $\{g_i = 1; g_j = 0, j \neq i\}$, The set of Lagrange interpolating polynomials form a basis for $\mathbb{P}_s$. In this basis, the interpolating polynomial $P(x)$ assumes the simple form

$$P(x) = \sum_{i=1}^{s} g_i \ell_i(x). \tag{7.3}$$

## 7.3 Numerical quadrature

Now consider a smooth function $g$ on the real line. We can approximate the definite integral of $g$ on the interval $[0, 1]$ by exactly integrating the interpolating polynomial of order $s - 1$ based on $s$ points $0 \leq c_1 < c_2 < \ldots < c_s \leq 1$. The points $c_i$ are then known as *quadrature points*. The data are the values of $g$ at the quadrature points $g_i = g(c_i)$, $i = 1, \ldots, s$.

Defining the weights

$$b_i = \int_0^1 \ell_i(x)\, dx,$$

the quadrature formula becomes

$$\int_0^1 g(x)\, dx \approx \int_0^1 P(x)\, dx = \sum_{i=1}^{s} b_i g(c_i). \tag{7.4}$$

To approximate the integral $\int_{t_0}^{t_0+h} g(t)\, dt$, we define a function $t(x) = t_0 + hx$. Then $dt = h\, dx$, and (7.4) becomes

$$\int_{t_0}^{t_0+h} g(t)\, dt = \int_0^1 g(t_0 + hx)\, hdx \approx h \sum_{i=1}^{s} b_i g(t_0 + hc_i). \tag{7.5}$$

By construction, a quadrature formula using $s$ distinct abscissa points will exactly integrate any polynomial in $\mathbb{P}_{s-1}$. However, with a judicious choice of the abscissae we can do even better. We say that a quadrature rule has *order* $p$ if it exactly integrates any polynomial in $\mathbb{P}_{p-1}$. It can be shown that $p \geq s$ always holds, and, for optimal choice of the $c_i$, one has $p = 2s$.

## 7.4 One-step collocation methods

A very elegant approach to constructing one-step methods of given order of accuracy uses the idea of *collocation*. We will construct the method for the first time step interval $[t_0, t_0 + h]$.

Let $0 \leq c_1 < c_2 < \cdots < c_s \leq 1$ be distinct nodes on the unit interval. The *collocation polynomial* $u(t) \in \mathbf{R}^d$ is a (vector-valued) polynomial of degree $s$ satisfying

$$\begin{aligned} &u(t_0) = y_0 \\ &u'(t_0 + c_i h) = f(u(t_0 + c_i h)), \quad i = 1, \ldots, s \end{aligned} \tag{7.6}$$

and the numerical solution of the *collocation method* over the interval $[t_0, t_0 + h]$ is given by $y_1 = u(t_0 + h)$. In other words, we contruct a polynomial that passes through $y_0$ and agrees with

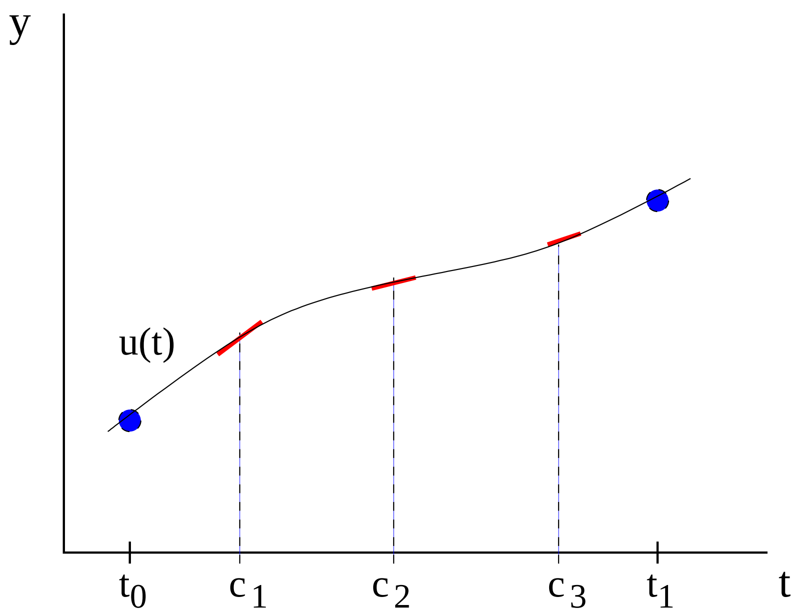


Figure 7.1: The collocation polynomial $u(t)$.

the ODE (1.3) at $s$ nodes on $[t_0, t_1]$. Then we let the numerical solution be the value of this polynomial at $t_1$. See Figure 7.1.

Let $F_i$ be the values of the (as yet undetermined) interpolating polynomial at the nodes:

$$F_i := u'(t_0 + c_i h), \quad i = 1, \ldots, s.$$

Now we use the Lagrange interpolation formula (7.3) to define the polynomial $u'(t)$ passing through these points:

$$u'(t) = \sum_{i=1}^{s} F_i \ell_i\left(\frac{t - t_0}{h}\right) \tag{7.7}$$

Integrating this equation over the intervals $[0, c_i]$ (and introducing the change of variables $x = (t - t_0)/h$, i.e., $dt = h\,dx$) gives

$$u(t_0 + c_i h) = y_0 + h \sum_{j=1}^{s} F_j \int_0^{c_i} \ell_j(x)\,dx, \quad i = 1, \ldots, s. \tag{7.8}$$

Let us denote

$$a_{ij} := \int_0^{c_i} \ell_j(x)\,dx, \qquad b_i := \int_0^1 \ell_i(x)\,dx, \qquad i, j = 1, \ldots, s.$$

Substituting (7.8) into the collocation conditions (7.6) gives

$$F_i = f(y_0 + h \sum_{j=1}^{s} a_{ij} F_j), \qquad i = 1, \ldots, s.$$

Similarly integrating (7.7) over $[0, 1]$ gives

$$y_1 := u(t_0 + h) = y_0 + h \sum_{i=1}^{s} F_i \int_0^1 \ell_j(t)\,dt = y_0 + h \sum_{i=1}^{s} b_i F_i.$$

To summarize, the collocation method is written

$$F_i = f\left(y_n + h\sum_{j=1}^{s} a_{ij}F_j\right), \quad i = 1,\ldots,s, \tag{7.9a}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i F_i. \tag{7.9b}$$

One first solves the coupled, $sd$-dimensional nonlinear system (7.9a). Subsequently, the update (7.9b) is explicit.

**Remark 7.4.1** *Collocation methods have the useful feature that we obtain a continuous approximation of the solution $u(t)$ on each interval $[t_n, t_{n+1}]$.*

**Remark 7.4.2** *In terms of order of accuracy, the optimal choice is attained by using so-called Gauss-Legendre collocation methods and placement of the nodes at the roots of a shifted Legendre polynomial. For $s = 1$, $2$ and $3$, the quadrature points are:*

$$\begin{aligned} c_1 = \tfrac{1}{2}, \qquad & p = 2,\\ c_1 = \tfrac{1}{2} - \tfrac{\sqrt{3}}{6},\ c_2 = \tfrac{1}{2} + \tfrac{\sqrt{3}}{6}, \qquad & p = 4,\\ c_1 = \tfrac{1}{2} - \tfrac{\sqrt{15}}{10},\ c_2 = \tfrac{1}{2},\ c_3 = \tfrac{1}{2} + \tfrac{\sqrt{15}}{10}, \qquad & p = 6. \end{aligned}$$

In the next chapter, we consider the class of Runge-Kutta methods, of which the collocation methods presented here are but a small subclass. For this reason, the convergence analysis of collocation methods is postponed to the next chapter.

## 7.5 Exercises

1. Find the coefficients $a_{ij}$, $b_i$ of the two-stage ($s = 2$) Gauss-Legendre collocation method. (Use Maple to evaluate the integrals).