Chapter 9

Equilibrium Points and Fixed Points

Main concepts: Equilibrium points, fixed points for RK methods, convergence of fixed points for one-step methods

Equilibrium points represent the simplest solutions to differential equations. In terms of the solution operator, they are the fixed points of the flow map. In this chapter we address the question of whether the equilibrium points of differential equation are retained as fixed points of the numerical method.

9.1 Equilibrium Points and Stability for Scalar ODEs

Definition 9.1.1 An **equilibrium point** x^* of the scalar differential equation $dx/dt = f(x)$ is a point for which $f(x^*) = 0$. □

It is a simple fact that the equilibria are themselves solutions of the ordinary differential equation. Assume f is a C' function and suppose all the equilibria $x_1 < x_2 < \ldots$ of an autonomous

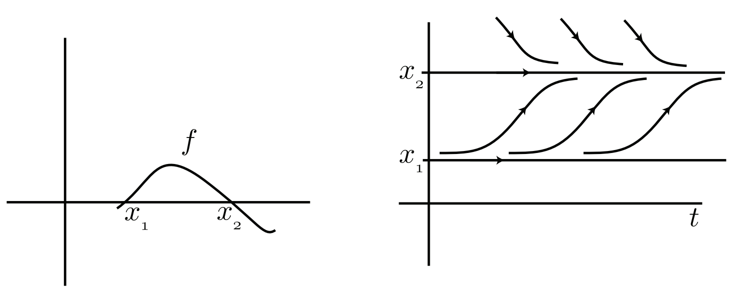

Figure 9.1: The shape of solutions near equilibrium points can be inferred from knowledge of f

scalar equation $dx/dt = f(x)$ are known. Then, as we shall see in the following chapter, we can often understand a great deal about the solutions of the system by evaluating the derivative (or derivatives) of f at the equilibrium points. For example, if we know that $f'(x_1) > 0$, then we know that in a small neighborhood of x_1, the function f is increasing. For x a little below x_1, we must have

$$dx/dt = f(x) < 0$$

and

$$d^2x/dt^2 = f'(x)f(x) < 0$$

This tells us that a little below x_1, x is decreasing and concave down. Similarly, a little above x_1 the graph of x is increasing and concave up. If, moreover, $f'(x_2) < 0$, then, with the assumed smoothness of f, the solutions of the differential equation can be inferred to have the appearance shown on the right of Figure 9.1.

The arrows on the solution curves sketched in Figure 9.1 indicate the direction of time. Notice that curves tend away from x_1 and towards x_2. Loosely speaking, we say that an equilibrium point x^* is *asymptotically stable* if, for all values u sufficiently close to x^*, the solution through u tends asymptotically to x^* as $t \to \infty$. If instead nearby solutions tend away from the equilibrium point we say it is unstable. Thus x_1 in Figure 9.1 would correspond to an unstable equilibrium point and x_2 to an asymptotically stable one. We will define this concept more precisely in the next chapter.

Given an autonomous system (1.13) in $\mathbf{R}^d$, we say that a point $y^* \in \mathbf{R}^d$ is an *equilibrium point* if

$$f(y^*) = 0,$$

in which case the solution $y(t)$ with initial condition $y(t_0) = y^*$ satisfies $y(t) \equiv y^*$ (for all t).

An equilibrium point of the ODE corresponds to a *fixed point* of the flow map, i.e. a point $y^* \in \mathbb{R}^d$ such that

$$\Phi_t(y^*) = y^*, \quad \forall t > 0.$$

We denote the set of fixed points of Φ_t by $\mathcal{F}$:

$$\mathcal{F} = \{y \in \mathbb{R}^d : f(y) = 0\}$$

For example, one can check that the Lotka-Volterra model (1.6) has the following equilibrium points:

$$\mathcal{F} = \{(p = 0, n = 0), (p = 1, n = 1)\}.$$

9.2 Fixed Points of Numerical Methods

For a numerical method with map Ψ_h, the fixed points may depend on h as well as f. We denote the set of fixed points of Ψ_h by $\mathcal{F}_h$:

$$\mathcal{F}_h = \{y \in \mathbb{R}^d : \Psi_h(y) = y\}.$$

An important practical question is whether the sets $\mathcal{F}$ and $\mathcal{F}_h$ coincide.

Consider first Euler's method (2.1)

$$\Psi_h(y) = y + hf(y).$$

Suppose $y \in \mathcal{F}$. Then $f(y) = 0$ and therefore $\Psi_h(y) = y$. Thus, $F \subseteq F_h$. Suppose, conversely, that $y \in \mathcal{F}_h$. Then, $\Psi_h y = y = y + hf(y)$, so $hf(y) = 0$. It follows that for $h > 0$, $F_h \subseteq F$ and therefore, for Euler's method, $F_h = F$.

Similarly, consider the implicit midpoint rule (8.5)

$$\frac{y_{n+1} - y_n}{h} = f\left(\frac{y_{n+1} + y_n}{2}\right).$$

First, suppose y_n and y_{n+1} are such that $(y_{n+1} + y_n)/2$ is an equilibrium. The right hand side evaluates to zero and therefore (for positive h) $y_{n+1} = y_n$, i.e. y_n is a fixed point of the method. Thus, $\mathcal{F} \subseteq \mathcal{F}_h$. Conversely, if y_n is a fixed point of the implicit map Ψ_h, then the left side evaluates to zero, and we are left with $f(y_{n+1}) = f(y_n) = 0$, so again $\mathcal{F}_h \subseteq \mathcal{F}$ and the sets are equivalent.

Next, consider take the explicit Runge-Kutta method with Butcher table

$$\begin{array}{c|cc} 0 & & \\ 1 & & \\ \hline & 0 & 1 \end{array},$$

which we can write simply as

$$y_{n+1} = y_n + hf\left(y_n + hf(y_n)\right).$$

It is clear that if $f(y_n) = 0$, then $y_{n+1} = y_n$, so $\mathcal{F} \subseteq \mathcal{F}_h$ here. The converse is not necessarily true, however, as we illustrate next. We apply this method to the logistic equation (1.4)

$$y' = y(1-y)$$

to obtain the map

$$\Psi_h y = y + h\left[y + hy(1-y)\right]\left[1 - y - hy(1-y)\right].$$

For $y \in \mathcal{F}_h$ we have

$$\left[y + hy(1-y)\right]\left[1 - y - hy(1-y)\right] = 0,$$

so one of the terms in square brackets must equal zero. In the first case, we have

$$y(1 + h(1-y)) = 0 \quad \Rightarrow \quad \{y = 0 \text{ or } y = 1 + \frac{1}{h}\},$$

In the second case,

$$1 - y - hy(1-y) = 0 \quad \Rightarrow \quad \{y = 1 \text{ or } y = \frac{1}{h}\}.$$

Hence, $F_h = \{0, 1, 1 + h^{-1}, h^{-1}\}$. The fixed points $1 + h^{-1}$ and h^{-1}, which are not equilibrium points, are termed *extraneous fixed points*. Note that the extraneous fixed points become very large as the time step is refined.

While the set of fixed points of a numerical method $\mathcal{F}_h = \{y \in \mathbb{R}^d : y = \Psi_h(y)\}$ is not generally the same as the set of fixed points $\mathcal{F} = \{y \in \mathbb{R}^d : f(y) = 0\}$ of the flow map, for many classes of methods, the fixed points of the numerical method are a superset of those of the flow map: $\mathcal{F}_h \supseteq \mathcal{F}$. For example, we can easily show this for Runge-Kutta methods:

Theorem 9.2.1 *For Runge-Kutta methods, $\mathcal{F}_h \supseteq \mathcal{F}$.*

Proof Recall the form of a Runge-Kutta method

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i f(Y_i)$$

where

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j)$$

Let $y_n = y^*$ be an equilibrium point, so $f(y^*) = 0$. If we set $Y_1 = Y_2 = \ldots = Y_s = y^*$, then we see that all the internal equations are satisfied. Moreover, $y_{n+1} = y_n$, implying that $y_n = y^*$ is a fixed point of the numerical method. □

Thus $\mathcal{F}_h$ includes all the fixed points of $\mathcal{F}$ but it may have some extraneous ones. Notice that the existence of extranous fixed points depends not only on the method but on the differential equation we are solving.

You might think that a Runge-Kutta method which admits extraneous fixed points would be of limited interest, since it is then quite possible that a numerical trajectory converges to a point that is nowhere near the stable equlibria of the original dynamical system. Nonetheless, such methods are used frequently in dynamical systems studies. For example, the 4th order Runge-Kutta method (8.4) is arguably the most popular method of all for dynamical studies, and yet you can check that it admits extraneous fixed points.

The reason that we can use methods like this is that in general (as in the example above), as $h \to 0$, all the extraneous fixed points tend to infinity. This gives us a simple way to test for extraneous fixed points: we simply vary the timestep and solve the problem repeatedly. The fixed points which stay put regardless of stepsize are genuine. This is also true of typical methods encountered in practice which are not of Runge-Kutta type.

To show this, we define the ε neighborhoods of the equilibrium and fixed points as follows

$$\mathcal{F}(\varepsilon) := \{y \in \mathbb{R}^d : \|f(y)\| \le \varepsilon\}, \qquad \mathcal{F}_h(\varepsilon) := \{y \in \mathbb{R}^d : \|\Psi_h(y) - y\| \le h\epsilon\}.$$

In the following theorem, we assume that for an arbitrary bounded set B, there is a continuous interval of time steps $h \in (0, h^*)$, where $h^* > 0$ may depend on B, for which the numerical map $\Psi_h(y)$ exists for all $y \in B$.

Theorem 9.2.2 *Let Ψ_h be a consistent method for the the differential equation (1.13), and suppose for any bounded set $B \subset \mathbb{R}^d$ there exists $h^*(B) > 0$ such that $\Psi_h(B)$ exists for all $h \in (0, h^*)$. Then, for $h \in (0, h^*)$,*

$$\mathcal{F}_h(\varepsilon) \cup B \subseteq \mathcal{F}(\varepsilon + Kh). \tag{9.1}$$

In particular,

$$\mathcal{F}_h \cup B \subseteq \mathcal{F}(Kh). \tag{9.2}$$

Proof Given y such that $\|\Psi_h(y) - y\| \le h\varepsilon$, we wish to bound $\|f(y)\|$. Note that[1]

$$\|hf(y)\| \le \|\Psi_h(y) - y\| + \|\Psi_h(y) - y - hf(y)\|.$$

The last term on the right is the difference between a step with Ψ_h and one with Euler's method, which we denote by Ψ_h^E. Since by assumption Ψ_h is a consistent method, we have

$$\|\Psi_h(y) - \Psi_h^E(y)\| = \|\Psi_h(y) - \Phi_h(y) + \Phi_h(y) - \Psi_h^E(y)\| \le \|le(y,h)\| + \|le_E(y,h)\| \le Kh^2.$$

for some positive K dependent on B.

Inserting this into the previous relation we have

$$h\|f(y)\| \le \|\Psi_h(y) - y\| + Kh^2,$$

and since $y \in \mathcal{F}_h(\varepsilon)$,

$$h\|f(y)\| \le h\varepsilon + Kh^2,$$

or,

$$\|f(y)\| \le \varepsilon + Kh.$$

This proves (9.1). Letting $\varepsilon \to 0$ gives (9.2). □

Let $y^*(h)$ denote a continuous branch of fixed points of Ψ_h parameterized by $h \in (0, h^*)$. Either this set is unbounded as $h \to 0$, or there exists a set B containing $y^*(h)$. In the latter case, the theorem shows that $y^*(h)$ converges to a point in $\mathcal{F}(\varepsilon)$.

[1] Triangle inequality $\|b-a\| \le \|b\| + \|a\|$. Let $c = b - a$ to get $\|c\| \le \|b\| + \|b-c\|$. Take $c = hf$ and $b = \Psi_h(y) - y$.