Modeling Carbon Market using Forward-Backward SDEs

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Outline

Carbon markets

- Caused mainly by GHG emission, global warming is a challenge for society
- The main part comes from carbon dioxide (CO_2) emission.
- This negative impact of emission has no "priced" cost.
- Emission reduction could be achieved by a right carbon price
- To do so, one can implement of cap-and-trade schemes.
- China, whose carbon emissions make up approximately one quarter of the global total, is considering introducing a national emissions trading scheme (with various pilot schemes already running)
- Since 2005, the EU has had its own emissions trading system (ETS).
  $\hookrightarrow$ This is our main example in this talk.

EU ETS

- The EU ETS is an example of cap and trade scheme:
  - A central authority set a limit on pollutant emission during a given period. Allowances are allocated to participating installations.
  - At the end of the period, each participating installation has to surrender an allowance for each unit of emission. Emissions above the cap leads to a penalty to be paid.
  - During the period, participants can trade the allowances.

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- It has been divided in 4 phases (Phase I: 2005-2007, II: 2008-2012, III: 2013-2020, IV: 2021 - 2028) with various changes in mechanism. Carbon price observed on the market has fluctuated a lot.

EUA price (P. MacDonald 2016 *sandbag.org.uk*)

Related literature & some motivations

Some literature

- Equilibrium approach (discrete time model): Carmona, Fehr, Hinz & Porchet (2010)
- risk neutral pricing, reduced form approach: Carmona & hinz (2011)
- risk neutral pricing, structural approach: Howison & Schwarz (2012)
- FBSDEs related papers: Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013)

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Some motivations

1. Modelisation of allowance price, allows for pricing of emission derivatives (clean spread options: has been done in risk neutral but reduced-form model)
2. Analyse the impact of multiple period setting
3. Quantify the dependency on various model parameter: cap, demand etc.
4. Answer question like: what level of allowance price would reduce the emission with a given probability?

Approach and Results

- We work at an aggregate level and follows risk neutral pricing
- We take into account feedback of the allowance price in the global emission process
  $\hookrightarrow$ this leads to an intricate forward-backward coupling
- We rely on the theory of singular Forward-Backward Stochastic Differential Equation (Carmona & Delarue 2013)

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- To do so, we extend the previous results on singular FBSDE and we study a novel PDE problem for the case of an infinite number of period.
- We are working towards efficient probabilistic numerical methods to compute the allowance price.

Outline

Main features

Three main processes for one period $[0, T]$.

1. The spot allowance price Y: we assume that the market is frictionless and arbitrage-free and that there is a probability such that $(e^{-rt}Y_t)_{0\le t\le T}$ is a martingale, namely

$$\mathrm{d}Y_t = rY_t\mathrm{d}t + Z_t\mathrm{d}W_t$$

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3. Auxiliary process P:

$$\mathrm{d}P_t = b(P_t)\mathrm{d}t + \sigma(P_t)\mathrm{d}W_t$$

   Represent state variables that trigger the emission process (Electricity price or demand & fuel prices etc.) Fundamentals that are linked the good emitting CO_2.

Associated FBSDE

- System of Equations: $0 \leq t \leq T$

$$
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- The terminal condition for the Allowance price Y. There is a cap Λ on the total emission set by the regulator
  1. If non-compliance i.e. $E_T > \Lambda$ then the penalty ρ is paid so $Y_T = \rho$
  2. If compliance i.e. $E_T < \Lambda$ then the Allowance is worth nothing (Emission regulation stops at the end of the period) so $Y_T = 0$

  $\hookrightarrow$ $Y_T = \phi(E_T) := \rho\mathbf{1}_{\{E_T > \Lambda\}}$ and $Y_t = e^{-r(T-t)}\mathbb{E}[Y_T|\mathcal{F}_t]$

Definition

- In a Markovian setting:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s$$

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- Linked with semi-linear PDE: $Y_t = u(t, X_T)$ where u satisfies

$$\partial_t u + b(x)\partial_x u + \frac{1}{2}\sigma^2(x)\partial_{xx} u + f(u) = 0 \text{ and } u(T, \cdot) = g(\cdot)$$

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- The difficult *fully coupled* case:

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Still $Y_t = u(t, X_t)$ *decoupling field* (price function here)

Results for Fully Coupled case

- *Fully coupled* case:

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and associated quasi-linear PDE:

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- Lipschitz coefficients and small time parameter T: existence and uniqueness (via "classical" fixed point argument) and the function u is Lipschitz-continuous
- Extension to arbitrary time if *non-degeneracy* of the noise
- **Here**: no noise in the emission process *and* singularity in the terminal condition!

Results for one-period model

- From Carmona and Delarue (2013), there exists a unique solution to the pricing equations:

$$\begin{aligned} \mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\,\mathrm{d}W_t, \\ \mathrm{d}E_t &= \mu(P_t, Y_t)\,\mathrm{d}t, \\ \mathrm{d}Y_t &= rY_t\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \end{aligned}$$

with terminal condition: $\phi_-(E_T) \le Y_T \le \phi_+(E_T)$!

where $\phi_-(e) = \rho \mathbf{1}_{\{e > \Lambda\}}$ and $\phi_+(e) = \rho \mathbf{1}_{\{e \ge \Lambda\}}$

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- The pricing function is well defined on $[0,T) \times \mathbb{R}^d \times \mathbb{R}$ and satisfies

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- Key ingredients: monotonicity of ϕ and $y \mapsto \mu(e,y)$. Links with scalar conservation law.
- Technical extension: OK for terminal condition $(p,e) \mapsto \phi(p,e)$ (depending on the fundamentals P)

Main step of the proof

- Approximation procedure via regularisation:
  1. Introduce noise into the equation $(\epsilon_n)_n$
  2. Mollify the terminal condition $(\phi_k)_k$
- Get uniform estimates in terms of regularisation parameters:

$$|\partial_p v_{n,k}(t,p,e)| \leq C \quad \text{and} \quad |\partial_e v_{n,k}(t,p,e)| \leq \frac{C}{T-t} \tag{1}$$

- Let $\epsilon_n \to 0$ and $\phi_k \to \phi$: get existence via compactness argument thanks to the previous estimates
- Existence of v with property (1) allows to get existence for (P, E, Y).

Outline

Market mechanisms

1. The one-period model describes quite well the ETS phase I
2. Most cap-and-trade market have many periods:

$$[T_0, T_1], [T_1, T_2] \dots [T_{q-1}, T_q]$$

3. The connection between periods: *Borrowing, Banking and Withdrawal*.
   - *Banking:* allowances that are not used in one period can be carried forward for compliance in the next period.
   - *Withdrawal:* for any $1 \le i \le q-1$, if the cap on emissions is exceeded at T_i, then the regulator removes a quantity of allowances from the $[T_i, T_{i+1}]$ market allocation. The quantity of allowances removed is equal to the level of excess emissions at T_i.
   - *Borrowing:* for any $1 \le i \le q-1$, firms may trade some of the allowances to be released at T_i during $[T_{i-1}, T_i]$. If each trading period represents a year, this means that firms can, in a particular year that is not the final year, use the following year's allowance allocation for compliance.

Example

3 period example

- Suppose that the regulator releases $c_k \geq 0$ allowances at T_{k-1} for $k = 1, 2, 3$
- The cap on emission for the period $[T_0, T_1]$ is $\Gamma_1 = c_1 + c_2$ (borrowing)
- For $[T_1, T_2]$, taking into account banking, borrowing and withdrawal:

$$\Gamma_2(E_{T_1}) = c_1 + c_2 + c_3 - E_{T_1}, \tag{2}$$

- For the last period, $\Gamma_3(E_{T_2}) = c_1 + c_2 + c_3 - E_{T_2}$
- Notation: Cap on emission on $[T_0, T_k]$, $\Lambda_k(E_{T_{k-1}}) = \Gamma_k(E_{T_{k-1}}) + E_{T_{k-1}}$ (constant in the previous example)

Pricing problem

Pricing is given by a backward induction

- On each period $[T_{k-1}, T_k]$, we still have:

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- E_0, P_0 are known and P and E are continuous process (no jump between the period).
  $\hookrightarrow$ But this might not be the case for Y as emission "control" occurs.
- Link between the period: $\{\text{non-compliance}\} = \{E_{T_k} - E_{T_{k-1}} \geq \Gamma_k(E_{T_{k-1}})\}$

$$
Y_{T_k-} = \rho \mathbf{1}_{\{\text{non-compliance}\}} + Y_{T_k+}\mathbf{1}_{\{\text{compliance}\}}
$$

and for the last period: $Y_{T_q} = \rho \mathbf{1}_{\{\mathrm{non-compliance}\}}$

Main results

- There exists a unique solution to the pricing system (P, E, Y, Z) on $[0, T]$
- For $t \in [T_{k-1}, T_k]$ (period k), $Y_t = v_k(t, P_t, E_t)$

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- On each period, $(t, p, e) \mapsto v_k(t, p, e)$ solution to

$$\partial_t v + \mu(p, v)\partial_e v + b(p)\partial_p v + \frac{1}{2}\sigma^2(p)\partial_{pp} v = rv$$

with terminal condition $\phi_k(p, e) = \rho \mathbf{1}_{\{e \geq \Lambda_k\}} + v_{k+1}(t, p, e)\mathbf{1}_{\{e < \Lambda_k\}}$

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- **Proof**: main point is to make sure we can recursively use the results of the one period model.
  $\hookrightarrow$ Prove in particular that $e \mapsto \phi_k(p, e)$ is non-decreasing.

Setting

- We consider a scheme running without specified end date:
  $T_{k+1} - T_k = \Delta T$, the cap at each period is $\Lambda_k = k \times \Delta\Lambda$
  $\hookrightarrow$ At T_k, we have $\{\text{non-compliance}\} = \{E_{T_k} \geq k \times \Delta\Lambda\}$

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- In a stationary regime, we expect that the market is just a repetition of similar pricing period
  $\hookrightarrow$ Decoupling between the period for the pricing function but...
- Focus on the first period: $[T_0, T_1]$
  1. Within the period, usual system of equation.
  2. At the end

$$Y_{T_1-} = \rho \mathbf{1}_{\{\text{non-compliance}\}} + v_2(T_1, P_{T_1}, E_{T_1})\mathbf{1}_{\{\text{compliance}\}}$$

  3. Note: v_2 is almost the same as v_1, the cap sequence has increased by $\Delta\Lambda$.
     $\hookrightarrow v_2(T_1, p, e) = v_1(T_0, p, e - \Delta\Lambda)$

Main results

- On each period $[T_k, T_{k+1}]$, the pricing function satisfies:

$$\partial_t v + \mu(p, v)\partial_e v + b(p)\partial_p v + \frac{1}{2}\sigma^2(p)\partial_{pp} v = rv$$

with terminal condition $\phi(p, e) = \rho \mathbf{1}_{\{e \geq k\Delta\Lambda\}} + v(T_k, p, e - \Delta\Lambda)\mathbf{1}_{\{e < k\Delta\Lambda\}}$

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- The processes (P, E, Y) exist on $[0, \infty)$ with $Y_t = v(t, P_t, E_t)$, for $t \neq T_k$, $k \geq 1$.

Main results

- On each period $[T_k, T_{k+1}]$, the pricing function satisfies:

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- If $r > 0$, the function v is unique
- The function v is given by the monotone limit of the q-period model when $q \to \infty$.

Setting

- One period Toy model ($r = 0$):

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\begin{aligned}
\mathrm{d}P_t &= \mathrm{d}W_t, & P_0 &= p, \\
\mathrm{d}E_t &= (P_t - Y_t)\mathrm{d}t, & E_0 &= e, \\
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- Two probabilistic methods
  1. Bender & Zhang method (generic)
  2. Particle system method (for this model)
- Note: there are other probabilistic methods that could be considered (and in this low dimensional example deterministic method as well, see Howison & Schwarz (2010))

Bender & Zhang method

- "Nothing to do" for P. Picard iteration for the coupled system (E, Y) (start with $E_t^0 = e$)

$$\mathrm{d}Y_t^j = Z_t^j \mathrm{d}W_t, \qquad Y_T^j = \mathbf{1}_{[\Lambda,\infty)}(E_T^{j-1}), \quad t \in [0, T],$$
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- Discretisation in time on grid $(t_i)_{1 \le i \le N}$:

$$Y_{t_i}^j = \tilde{v}^j(t_i, P_{t_i}, E_{t_i}^{j-1}) = \mathbb{E}\Big[Y_{t_{i+1}}^j | P_{t_i}, E_{t_i}^{j-1}\Big] \text{ (we need } \tilde{v}^j)$$
$$E_{t_{i+1}}^j = E_{t_i}^j + (P_{t_i} - \tilde{v}^j(t_i, P_{t_i}, E_{t_i}^j))(t_{i+1} - t_i)$$

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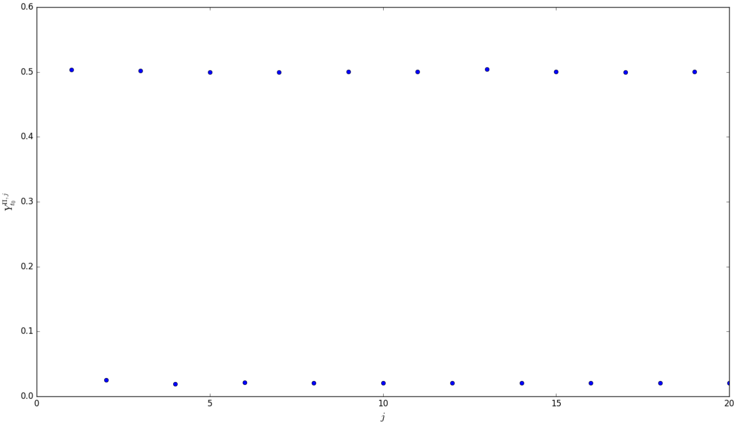
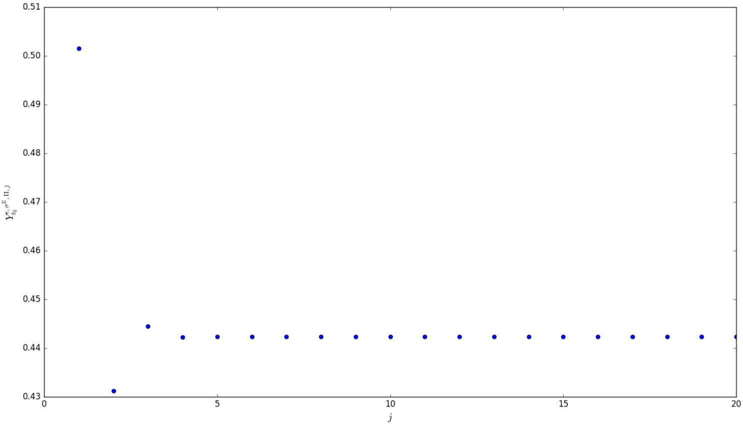
- Compute the conditional expectation on some basis function via regression method e.g.

Numerical results for Bender & Zhang (FAIL)



Bender & Zhang + Regularisation (OK)



Particle system approach (Bossy & Talay 97)

- Previous method requires regularisation, difficult to tune...
- Here, one can simplify the model by considering $\bar{E}_t = E_t + (T-t)P_t$:

$$\mathrm{d}\bar{E}_t = -Y_t \mathrm{d}t + (T-t)\mathrm{d}W_t$$
$$\mathrm{d}Y_t = Z_t \mathrm{d}W_t$$

- Decoupling field $\bar{v}$ satisfies:

$$\partial_t \bar{v} - \bar{v}\partial_e \bar{v} + \frac{(T-t)^2}{2}\partial_{ee}\bar{v} = 0 \quad \text{and} \quad \bar{v}(T,e) = \mathbf{1}_{e \geq \Lambda}$$

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- Set $w(t,e) = \partial_e \bar{v}(T-t,e)$ then w solves the (forward) Fokker-Planck equation:

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- Associated McKean-Vlasov equation: $\bar{v}(t,e) = \int \mathbf{1}_{\{y \leq e\}} \mathrm{d}w(t,y)$

$$\mathrm{d}X_t = \bar{v}(t,X_t)\mathrm{d}t + t\mathrm{d}W_t \quad \text{where} \quad X_t \text{ has density } w(t,\cdot)$$

Numerical result for infinite period model

From 2 to 4000 periods... (left:price function at t=0 / right: zoom to check monotonicity)