# Numerical methods for FBSDEs

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# Agenda

- FBSDEs, derivation of discrete scheme
- BSDEs, BCOS method (joint work with Marjon Ruijter)
- BSDEs, SGBM method (joint work with Ki Wai Chau)

# Semilinear PDE and BSDEs

- The semilinear partial differential equation:

$$v_t(t,x) + \mathcal{L}v(t,x) + g(t,x,v,\sigma(x)v_x(t,x)) = 0, \; v(T,x) = \xi(x),$$

We can solve this PDE by means of the FSDE:

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s, \; X_t = x.$$

and the BSDE:

$$dY_s = -g(s, X_s, Y_s, Z_s)ds + Z_s d\omega_s, \; Y_T = \xi(X_T).$$

- Theorem:

$$Y_t = v(t, X_t), \; Z_t = \sigma(X_t)v_x(t, X_t).$$

is the solution to the BSDE.

# Backward SDE

$$dY_t = -g(t, Y_t, Z_t)dt + Z_t d\omega_t, \quad Y_T = \xi.$$

$g$ is the *driver* function. $\xi$ is $\mathcal{F}_T$-measurable random variable.

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$g$ is the *driver* function. $\xi$ is $\mathcal{F}_T$-measurable random variable.
A solution is a *pair* adapted processes $(Y, Z)$ satisfying

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s d\omega_s.$$

$Y$ is adapted if and only if, for every realization and every $t$, $Y_t$ is known at time $t$. Adapted process cannot "see into the future".

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A BSDE is *not a time-revered* FSDE, because at time $t$ the pair $(Y_t, Z_t)$ is $\mathcal{F}_t$-measurable and the process does not "know" the terminal condition yet.

# $\theta$-method

ODE Euler method: $\displaystyle\int_{t_m}^{t_{m+1}} h(s)ds \approx h(t_m)\Delta t \quad \Rightarrow \mathcal{O}(\Delta t)$

$\theta$-method: $\displaystyle\int_{t_m}^{t_{m+1}} h(s)ds \approx \theta h(t_m)\Delta t + (1-\theta)h(t_{m+1})\Delta t,$

$$\theta \in [0,\,1], \quad \theta = \tfrac{1}{2} \Rightarrow \mathcal{O}((\Delta t)^2)$$

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$$\theta \in [0, 1], \quad \theta = \frac{1}{2} \Rightarrow \mathcal{O}((\Delta t)^2)$$

# $Y$ component

$$Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s, \tag{1}$$

$\mathcal{X}_t := (X_t, Y_t, Z_t)$.

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Taking conditional expectation $\mathbb{E}_m[.] = \mathbb{E}[.|\mathcal{F}_{t_m}]$:

$$Y_m = \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)] ds - \mathbb{E}_m \left[ \int_{t_m}^{t_{m+1}} Z_s d\omega_s \right] \quad (2)$$

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$$\begin{aligned} Y_m &= \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)]ds \\ &\approx \mathbb{E}_m[Y_{m+1}] \\ &+ \Delta t \theta g(t_m, \mathcal{X}_m) + \Delta t(1-\theta)\mathbb{E}_m[g(t_{m+1}, \mathcal{X}_{m+1})], \end{aligned} \quad (2)$$

$\theta \in [0, 1]$, theta-time discretization method to approximate the time-integral.

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For $\theta = 1$ we get

$$Y_m \approx \mathbb{E}_m[Y_{m+1}] + \Delta t g(t_m, \mathcal{X}_m). \tag{3}$$

# $Z$ component

$$Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \quad (4)$$

Multiplying by $\Delta\omega_{m+1}$, taking the conditional expectation gives

$$\mathbb{E}_m[Y_m \Delta\omega_{m+1}] = \mathbb{E}_m[Y_{m+1}\Delta\omega_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)\Delta\omega_{m+1}] ds - \mathbb{E}_m\left[\int_{t_m}^{t_{m+1}} Z_s d\omega_s \Delta\omega_{m+1}\right]$$

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$$- \int_{t_m}^{t_{m+1}} \mathbb{E}_m[Z_s] ds$$

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&- \int_{t_m}^{t_{m+1}} \mathbb{E}_m[Z_s] ds \\
&\approx \mathbb{E}_m[Y_{m+1}\Delta\omega_{m+1}] + \Delta t(1-\theta)\mathbb{E}_m[g(t_{m+1}, \mathcal{X}_{m+1})\Delta\omega_{m+1}] \\
&- \Delta t\theta Z_m - \Delta t(1-\theta)\mathbb{E}_m[Z_{m+1}], \quad \theta \in [0,1].
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$$\begin{aligned}
\Rightarrow Z_m \approx & -\tfrac{1-\theta}{\theta}\mathbb{E}_m[Z_{m+1}] + \tfrac{1}{\Delta t\theta}\mathbb{E}_m[Y_{m+1}\Delta\omega_{m+1}] \\
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$$\begin{aligned}
\Rightarrow Z_m &\approx -\tfrac{1-\theta}{\theta}\mathbb{E}_m[Z_{m+1}] + \tfrac{1}{\Delta t \theta}\mathbb{E}_m[Y_{m+1}\Delta\omega_{m+1}] \\
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\end{aligned} \tag{5}$$

$$\text{For } \theta = 1 \text{ we get } Z_m \approx \tfrac{1}{\Delta t}\mathbb{E}_m[Y_{m+1}\Delta\omega_{m+1}]. \tag{6}$$

# Discretization scheme FBSDE

$$
\begin{aligned}
X_0^\Delta &= x_0, \\
&\text{for } m = 0, \ldots, M-1: \\
X_{m+1}^\Delta &= X_m^\Delta + \mu(X_m^\Delta)\Delta t + \sigma(X_m^\Delta)\Delta\omega_{m+1}, \\
Y_M^\Delta &= \xi(X_M^\Delta),\ Z_M^\Delta = \sigma(X_M^\Delta)\xi_x(X_M^\Delta), \\
&\text{for } m = M-1, \ldots, 0: \\
Z_m^\Delta &= -\tfrac{1-\theta}{\theta}\mathbb{E}_m[Z_{m+1}^\Delta] + \tfrac{1}{\Delta t\theta}\mathbb{E}_m[Y_{m+1}^\Delta \Delta\omega_{m+1}] \\
&\quad + \tfrac{1-\theta}{\theta}\mathbb{E}_m[g(t_{m+1}, \mathcal{X}_{m+1}^\Delta)\Delta\omega_{m+1}], \\
Y_m^\Delta &= \mathbb{E}_m[Y_{m+1}^\Delta] + \Delta t\theta g(t_m, \mathcal{X}_m^\Delta) \\
&\quad + \Delta t(1-\theta)\mathbb{E}_m[g(t_{m+1}, \mathcal{X}_{m+1}^\Delta)].
\end{aligned}
$$

All processes "go forward" in time but the BSDE is approximated backwards in time.
We use the COS method to approximate the expectations.

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All processes “go forward” in time but the BSDE is approximated backwards in time.
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# Numerical Pricing Approach

- One can apply different numerical techniques to calculate conditional expectations:
  - Numerical integration,
  - Monte Carlo simulation,
- Each of the methods has its merits and demerits.
- Numerical challenges:
  - The problem's dimensionality
  - Speed of solution methods

# COS method

COS method is based on Fourier cosine series expansions.
Approximate expected values.

- European options (F. Fang, CWO, 2008),
- Bermudan and American options (F. Fang, CWO, 2009).

Fourier cosine series expansions of $h(t_m, x)$ on $[a, b]$

$$h(t_m, x) = {\sum_{k=0}^{N-1}}' \cos\left(k\pi \frac{x-a}{b-a}\right) \mathcal{H}_k(t_m),$$

with series coefficients

$$\mathcal{H}_k(t_m) = \frac{2}{b-a} \int_a^b h(t_m, x) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

# COS formula

$$I := \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^{\Delta})] = \int_{\mathbb{R}} h(t_{m+1}, \zeta) f(\zeta|x) d\zeta$$
$$\approx \int_a^b h(t_{m+1}, \zeta) f(\zeta|x) d\zeta. \quad (7)$$

Replace the density function and function $h$ by their Fourier cosine series expansions. Truncation of the series summations gives us the approximation

$$I \approx \frac{b-a}{2} {\sum_{k=0}^{N-1}}' \mathcal{H}_k(t_{m+1}) \mathcal{P}_k(x). \quad (8)$$

The Fourier cosine coefficients of a density function

$$\begin{aligned}\mathcal{P}_k &\approx \frac{2}{b-a}\int_{\mathbb{R}} f(\zeta)\cos\left(k\pi\frac{\zeta-a}{b-a}\right)d\zeta \\ &= \frac{2}{b-a}\Re\left(\int_{\mathbb{R}} f(\zeta)\exp\left(ik\pi\frac{\zeta-a}{b-a}\right)d\zeta\right) \\ &= \frac{2}{b-a}\Re\left(\varphi\left(\frac{k\pi}{b-a}\right)e^{ik\pi\frac{-a}{b-a}}\right),\end{aligned} \tag{9}$$

$\varphi(.)$ is the *characteristic function*.

The COS formula:

$$\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^{\Delta})] \approx {\sum_{k=0}^{N-1}}' \mathcal{H}_k(t_{m+1})\Re\left(\varphi\left(\frac{k\pi}{b-a}|X_m^{\Delta}=x\right)e^{ik\pi\frac{-a}{b-a}}\right).$$

# BCOS method

Ingredients:

- Characteristic function $X$, $X_m^\Delta = x$
- $\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)]$
- $\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)\Delta\omega_{m+1}]$
- Recover Fourier cosine coefficients backward in time
- FFT algorithm

# Example: European Call Option - GBM - $\mathbb{P}$-Measure

Asset price: $dX_t = \bar{\mu} X_t dt + \bar{\sigma} X_t d\omega_t$.

Hedge portfolio $Y_t$ with: $a_t$ assets $X_t$ and $Y_t - a_t X_t$ bonds

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Hedge portfolio $Y_t$ with: $a_t$ assets $X_t$ and $Y_t - a_t X_t$ bonds

$$dY_t = r(Y_t - a_t X_t)dt + a_t dX_t$$

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Asset price: $dX_t = \bar{\mu} X_t dt + \bar{\sigma} X_t d\omega_t.$

Hedge portfolio $Y_t$ with: $a_t$ assets $X_t$ and $Y_t - a_t X_t$ bonds

$$\begin{aligned} dY_t &= r(Y_t - a_t X_t)dt + a_t dX_t \\ &= \left( rY_t + \frac{\bar{\mu} - r}{\bar{\sigma}} \bar{\sigma} a_t X_t \right) dt + \bar{\sigma} a_t X_t d\omega_t, \end{aligned}$$
$$Y_T = \xi(X_T) = \max(X_T - K, 0).$$

# Example: European Call Option - GBM - $\mathbb{P}$-Measure

$$\text{Asset price: } dX_t = \bar{\mu} X_t dt + \bar{\sigma} X_t d\omega_t.$$

Hedge portfolio $Y_t$ with: $a_t$ assets $X_t$ and $Y_t - a_t X_t$ bonds

$$\begin{aligned} dY_t &= r(Y_t - a_t X_t)dt + a_t dX_t \\ &= \left( rY_t + \frac{\bar{\mu} - r}{\bar{\sigma}} \bar{\sigma} a_t X_t \right) dt + \bar{\sigma} a_t X_t d\omega_t, \\ Y_T &= \xi(X_T) = \max(X_T - K, 0). \end{aligned}$$

If we set $Z_t = \bar{\sigma} a_t X_t$, then $(Y, Z)$ solves a BSDE with driver,

$$g(t, x, y, z) = -ry - \frac{\bar{\mu} - r}{\bar{\sigma}} z.$$

$Y_t$ corresponds to the value of the portfolio and $Z_t$ is related to the hedging strategy. The option value is given by $v(t, X_t) = Y_t$ and $\sigma(X_t) v_x(t, X_t) = Z_t$.

# Results European call option

Exact solutions $Y_0 = v(0, X_0) = 3.66$ and $Z_0 = \bar{\sigma} X_0 v_x(0, X_0) = 14.15$.

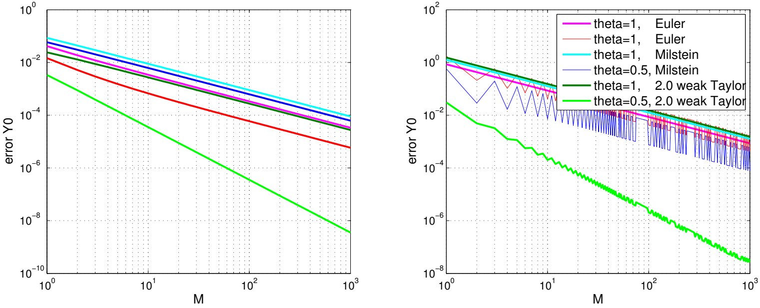


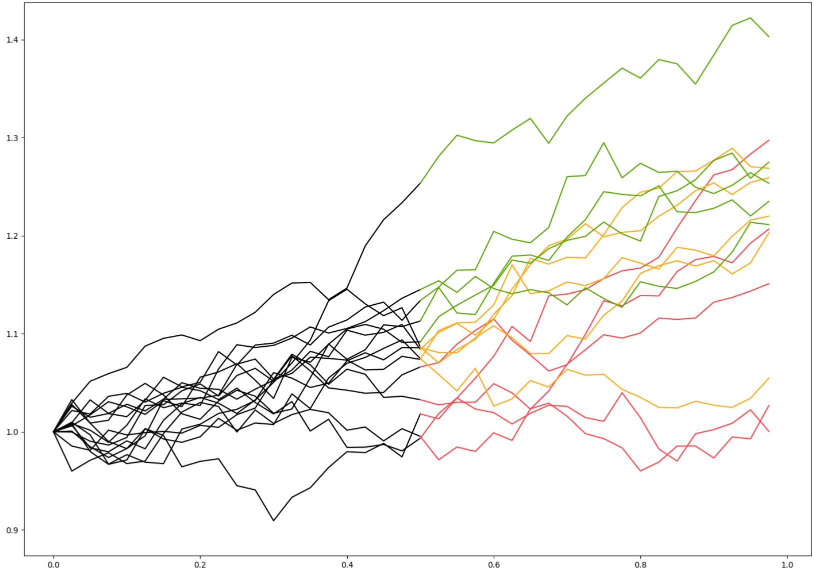

Figure: Euler, Milstein and order 2.0 weak Taylor scheme, $\theta = 1$ and $\theta = 0.5$.

# SGBM

A Monte-Carlo scheme in a high-dimensional setting does not suffer from the curse of dimensionality.
The Stochastic Grid Bundling Method (SGBM) is a Monte Carlo scheme that combines:

- Bundling
  - At each time point, all paths are bundled into equal-size, non-overlapping partitions based on the result of $X_m^\Delta$.
  - We perform the approximation separately at each bundle.
- Regress-later
  - The least-squares regression technique is performed on the random variable $X_{m+1}^\Delta$.
  - Then we use the (analytic) expectation of the resulting approximation in our algorithm.

# Bundling



# Stochastic Grid Bundling Method, with Shashi Jain, Ki Wai Chau

Step 1: The grid points in SGBM are generated by simulation, $\{\mathbf{X}_{t_0}(n), \ldots, \mathbf{X}_{t_M}(n)\}, \; n = 1, \ldots, N,$

Step 2: Compute the option value at terminal time.

Step 3: Bundle the grid points at $t_{m-1}$ into $\mathcal{B}_{t_{m-1}}(1), \ldots, \mathcal{B}_{t_{m-1}}(\nu)$ non-overlapping bundles.

Step 4: For $\mathcal{B}_{t_{m-1}}(\beta), \; \beta = 1, \ldots, \nu,$ compute $h(\mathbf{X}_{t_m}, \alpha^{\beta}_{t_m})$.
$h : \mathbb{R}^d \times \mathbb{R}^K \mapsto \mathbb{R},$ is a parametrized function which assigns values to states $\mathbf{X}_{t_m}$.

Step 5: The cond. expectations for grid points in $\mathcal{B}_{t_{m-1}}(\beta), \; \beta = 1, \ldots, \nu,$ are approximated by

$$\mathbb{E}[h(\mathbf{X}_{t_m}, \alpha^{\beta}_{t_m}) | \mathbf{X}_{t_{m-1}}(n)]$$

# Intuition

- The objective is to choose, corresponding to each bundle $\beta$ at $t_{m-1}$, a parameter vector $\alpha^{\beta}_{t_m}$ so that, $Y_{t_m}(\mathbf{X}_{t_m}) \approx h(\mathbf{X}_{t_m}, \alpha^{\beta}_{t_m})$.
- We use OLS, to define

$$h(\mathbf{X}_{t_m}, \widehat{\alpha}^{\beta}_{t_m}) = \sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_m}(k) p_k(\mathbf{X}_{t_m}). \tag{10}$$

# Intuition

- The objective is to choose, corresponding to each bundle $\beta$ at $t_{m-1}$, a parameter vector $\alpha^{\beta}_{t_m}$ so that, $Y_{t_m}(\mathbf{X}_{t_m}) \approx h(\mathbf{X}_{t_m}, \alpha^{\beta}_{t_m})$.
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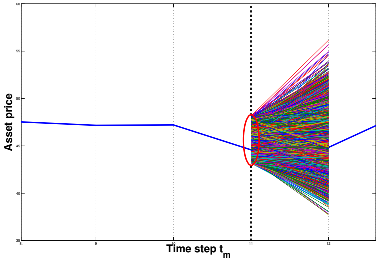


- In the limiting case, of number of bundles $\nu$, and paths $N$, the distribution would be similar to a point source.

# Computing the cond. expectation

- The cond. expectation is computed as:

$$\mathbb{E}[h(\mathbf{X}_{t_m}, \widehat{\alpha}^{\beta}_{t_m})|\mathbf{X}_{t_{m-1}} = \mathbf{X}_{t_{m-1}}(n)], \tag{11}$$

  where $\mathbf{X}_{t_{m-1}}(n) \in \mathcal{B}_{t_{m-1}}(\beta)$.

- This can be written as:

$$\mathbb{E}\left[\left(\sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_m}(k) p_k(\mathbf{X}_{t_m})\right) |\mathbf{X}_{t_{m-1}} = \mathbf{X}_{t_{m-1}}(n)\right]$$
$$= \sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_m}(k) \mathbb{E}\left[p_k(\mathbf{X}_{t_m})|\mathbf{X}_{t_{m-1}} = \mathbf{X}_{t_{m-1}}(n)\right].$$

- Choose basis functions $p_k$ so that $\mathbb{E}\left[p_k(\mathbf{X}_{t_m})|\mathbf{X}_{t_{m-1}} = X\right]$ has an analytic solution.

# SGBM for BSDEs

To recap, SGBM requires the following ingredients:

- Equal Partition Bundling
- Regression basis $p$
- OLS algorithm to find regression parameters
- $\mathbb{E}_m^x\left[p(X_{m+1}^\Delta)\right]$
- $\mathbb{E}_m^x\left[\Delta\omega_{r,m+1}p(X_{m+1}^\Delta)\right]$

# Example: European Basket Put - GBM - $\mathbb{Q}$-Measure I

Asset price for higher-dimensional cases:

$$dX_{i,t} = rX_{i,t}dt + \bar{\sigma}_i X_{i,t} dW_{i,t},\ 1 \leq i \leq q$$

with $W_t$ being a correlated Wiener process and $dW_{i,t}dW_{j,t} = \rho_{ij}dt$.
The parameters $\rho_{ij}$ form a symmetric matrix $\rho$.
$W_t$ relates to standard Brownian motion with $W_t = L\omega_t$, where $LL^T = \rho$ from a Cholesky decomposition.
We can determine $Y_0$ by solving the BSDE to price a European option with terminal payoff $\xi(X_t)$.

$$\begin{cases} dY_t = rY_t dt + Z_t d\omega_t; \\ Y_T = \xi(X_T). \end{cases}$$

# Example: European Basket Put - GBM - $\mathbb{Q}$-Measure II

Weights: $(w_1, w_2, w_3, w_4, w_5) = (38.1, 6.5, 5.7, 27.0, 22.7)$.
Reference price from another (more intensive) Monte-Carlo algorithm: 0.175866
Bundling: equal-partitioning and sorting the paths with $\sum_{i=1}^{5} w_i X_{t_p,i}^m$.

Regression basis: $p_k(x) = \left(\sum_{i=1}^{5} w_i x_i\right)^{k-1}$ for $k = 1, \ldots, K$.
Test Case:

| | $\theta_1$ | $\theta_2$ | no. of paths | no. of steps | no. of bundles | K |
|---|---|---|---|---|---|---|
| Explicit | 0 | 1 | $2^{11}$ | 10 | $2^{2J}$ | 2 |

Result of $|Y_0 - y_0^{(\theta_1,\theta_2),R}(x_0)|$:

| J | 0 | 1 | 2 |
|---|---|---|---|
| Explicit | 0.00293 | 0.00189 | 0.000222 |

# Summary and conclusion

- BCOS method
- $\theta$ discretization $\Rightarrow$ conditional expectations
- Higher-dimensional processes: SGBM for BSDEs
- Our papers:
  - M.J. Ruijter, CWO, 'A Fourier-cosine method for an efficient computation of solutions to BSDEs'. *SIAM J. Sci. Comput.* **37(2),** (2015).
  - M.J. Ruijter, CWO, 'Numerical Fourier method and second-order Taylor scheme for Backward SDEs in finance'. *Applied Num. Math.* **103,** (2016).
  - T.P. Huijskens, M.J. Ruijter, CWO, 'Efficient numerical Fourier methods for coupled forward-backward SDEs.' *J. Comp. Appl. Math.*, **296**, (2016).
  - K.W. Chau and CWO, On the wavelets-based SWIFT method for backward stochastic differential equations. *IMA J. Numerical Analysis*, **38(2)**, (2018).
  - A. Borovykh, A. Pascucci, CWO, Efficient Computation of Various Valuation Adjustments Under Local Lévy Models *SIAM J. Finan. Math.* **9(1),** (2018).
  - K.W. Chau and CWO, Stochastic grid bundling method for backward stochastic differential equations. To appear in Int. J. Computer Math. 2019.