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Gauß and Beyond: The Making of Easter Algorithms

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Abstract

It is amazing to see how many webpages are devoted to the art of finding the date of Easter Sunday. Just for illustration, the reader may search for terms such as *Gregorian calendar*, *date of Easter*, or *Easter algorithm*. Sophisticated essays as well as less enlightening contributions are presented, and many a doubt is expressed about the reliability of some results obtained with some Easter algorithms. In short, there is still a great interest in those problems.

Gregorian Easter algorithms exist for two centuries (or more?), but most of their history is rather obscure. Some reasons may be that some important sources are written in Latin or in the German of Goethe's time, or they are difficult to discover. Without being complete, the following paper is intended to shed light on how those techniques emerged and evolved.[1] Like a microcosm, the history of Easter algorithms resembles the history of any science: it is a story of trials, errors, and successes, and, last but not least, a story of offended pride.

1. Definitions

Easter is the first Sunday strictly after the *Paschal Full Moon*, i. e. *Luna XIV paschalis*. Such a full moon does not denote an astronomical phenomenon that occurs at a specific moment, but is an ecclesiastical term that names a specific day, usually defined by tables such as Table 2. What those tables are based on and how they can be compiled will later be shown.

Intentionally, this definition leaves the role of the date of the equinox somewhat vague. What interests me here is the interval in which the Paschal Full Moon is allowed to fall. Dionysius Exiguus writes in his *Epistola prima scripta anno Christi 525*,

> Thus the heavenly Authority decreed that Easter should be celebrated from the evening of the 14th day to the 21st of the first [lunar] month [of spring]. But nothing is said about the beginning and end of this month. The 318 bishops [of the Council of Nicea] therefore investigated the tradition of Moses in greater detail. [...] They came to the conclusion that it was the new crescent moon from and including 8 March up to and including 5 April which should decide the beginning of the first month. We must therefore carefully

[1] A number of articles, published before 1910, are cited in: A. Fraenkel, "Die Berechnung des Osterfestes." *Journal für die reine und angewandte Mathematik*, Volume 138 (1910), 133–146.

investigate the days from and including 21 March up to and including 18 April to find Luna 14 paschalis.[2]

In the following, the term *Easter* means always Gregorian or Western Easter, i. e. the date of Easter is determined according to the Calendar Reform of 1582. *Moon* and *sun* mean always the ecclesiastical terms. If not otherwise stated, all translations are made by myself. As above, textual explanations and omissions are enclosed by brackets.

The *floor function* $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to a given real number x. Thus $\lfloor 62/19 \rfloor = 3$, i.e. the integer part of the fraction 62/19, but $\lfloor -62/19 \rfloor = -4$. For integers a, $c (c > 0)$, we define $a \bmod c = a - \lfloor a/c \rfloor \times c$. For instance, $62 \bmod 19 = 5$, i. e. the integer remainder. The function mod c has some useful properties:

(1) If $b = a \mod c$, then $(b - a)/c$ is an integer. The converse is false: $(24 - 62)/19$ is an integer, but $24 \neq 62 \mod 19$.
(2) If $b = a \mod c$, then $0 \leq b \leq c - 1$.
(3) If z is an arbitrary integer, then $a \bmod c = (a + z \times c) \bmod c$.

2. Gauß' Osterformel

In August 1800 C. F. Gauß published an algorithm for determining the date of Easter day for any given year, usually called *Osterformel* ("Easter formula").[3] His aim, he claimed, was to present a purely analytical solution where chronological terms such as *Golden Number* or *Epact* were not applied, and since higher arithmetic was not supposed, every enthousiast could do the calculations; moreover, his algorithm would please each connoisseur by its simplicity and elegance. Three months before he recorded concisely in his diary, "Around the same days (16 May) we elegantly solved the chronological problem of the Paschal feast [i. e. Easter]."[4]

It is often said that Gauß was the first scholar who worked out such a rule. This is not quite true. In 1776 J. H. Lambert published some lengthy and clumsy remarks under the title *Einige Anmerkungen über die Kirchenrechnung* where he demonstrated a rule for finding the Eastern Easter only.[5] He did not succeed in mastering the western counterpart and ended up in the remarkable sentence, "A general formula is partly too detailed by itself partly no longer general, because *Clauius* [Clavius], who designed the Gregorian calendar, differed several times from the general principles which it [the calendar] is based upon." It is also often said that Gauß had invented his rule in order to

[2] The English translation is taken from G. Teres, "Time computation and Dionysius Exiguus." *Journal for the history of astronomy*, Volume 15 (1984), 177–188.

[3] C. F. Gauß, "Berechnung des Osterfestes," *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde*, Aug. 1800 (= Werke VI, 73–79) (Note: Werke = *Gauß' collected work*, Göttingen 1863).

[4] C. F. Gauß, Werke X, Abt. 1, 547-548. The original text is in Latin; verbatim, Gauß writes, "Iisdem diebus circa (Mai. 16.) problema chronologicum de festo paschali eleganter resolvimus. (Promulg[atum] in Zachii Comm. liter. Aug. 1800, p. 121, 223.)".

[5] J. H. Lambert, "Einige Anmerkungen über die Kirchenrechnung," *Astronomisches Jahrbuch oder Ephemeriden für das Jahr 1778*, Berlin 1776, 210–226.

Table 1. Gauß' final Osterformel. The condition $(11M + 11) \bmod 30 < 19$ can be replaced by $a > 10$

	yr	=	year	$yr = 1777$
(1)	a	=	$yr \mod 19$	$a = 10$
(2)	b	=	$yr \mod 4$	$b = 1$
(3)	c	=	$yr \mod 7$	$c = 6$
(4)	k	=	$\lfloor yr/100 \rfloor$	$k = 17$
(5)	p	=	$\lfloor (13 + 8k)/25 \rfloor$	$p = 5$
(6)	q	=	$\lfloor k/4 \rfloor$	$q = 4$
(7)	M	=	$(15 - p + k - q) \mod 30$	$M = 23$
(8)	N	=	$(4 + k - q) \mod 7$	$N = 3$
(9)	d	=	$(19a + M) \mod 30$	$d = 3$
(10)	e	=	$(2b + 4c + 6d + N) \mod 7$	$e = 5$
(11)			Easter is $22 + d + e$ March, or $d + e - 9$ April	30 March
			(12.1) If $d = 29$ and $e = 6$, replace 26 April by 19 April.	
			(12.2) If $d = 28$, $e = 6$, and $(11M + 11) \mod 30 < 19$, replace 25 April by 18 April.	

find out precisely the date of his birth; the only information he got from his mother was that he was born on a Wednesday in 1777, eight days before Ascension Day.[6] This might be a pretty legend. Gauß could be quite as well somewhat disappointed from a reading of Lambert's article and decided to search for a proper solution. In fact, Gauß' method can be employed too to find the Orthodox date of Easter. Unfortunately, the Osterformel was erroneous, and Gauß created much confusion during the following 16 years. After the article of 1800, already mentioned, he published a simplified version in 1807[7], a modified version for the period 1700–1900 in 1811[8] and a final version[9], correcting a mistake found by his student P. Tittel in 1816[10]; there is also an undated and unpublished note by Gauß on the matter, probably from the period between 1807 and 1811[11].

In order to understand the flaws in the earlier versions we compare these with the final version of 1816 on which Table 1 is based.

– 1800: In his first paper (cited above) Gauß proposes $p = \lfloor k/3 \rfloor$ instead of $p = \lfloor 13 + 8\,k)/25 \rfloor$, which is correct; see Table 1, (5). As a consequence, results become wrong from 4200 onwards. In 4213 for instance, Easter will not fall on 25 April but on 28 March.

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[6] Ph. Maennchen, "Gauß als Zahlenrechner," ch. X, "Chronologische Arbeiten," 49–63. In: C F. Gauß, Werke X, Abt. 2.

[7] C. F. Gauß, "Noch etwas über die Bestimmung des Osterfestes," *Braunschweigisches Magazin* (1807) (= Werke VI, 82–86).

[8] C. F. Gauß, "Eine leichte Methode den Ostersonntag zu finden," *Astronomisches Jahrbuch für das Jahr1814*, Berlin 1811 (= Werke XI, Abt. 1, 199–200).

[9] C. F. Gauß, "Berichtigung zu dem Aufsatze: Berechnung des Osterfestes Mon. Corr. 1800 Aug. S. 121," *Zeitschrift für Astronomie und verwandte Wissenschaften*, Volume 1 (1816) (= Werke XI, Abt. 1, 201–202).

[10] P. Tittel, "Methodus technica, brevis, perfacilis ac perpetua construendi Calendarium ecclesiasticum stylo tam novo quam vetere, Goettingae 1816" (= Werke XI, Abt. 1, 204–205).

[11] C. F. Gauß, "Praecepta universalia ad computandum diem paschatis anni cuiuslibet dati secundum calendarium tum Gregorianum tum Iulianum," (= Werke XI, Abt. 1, 211–214).

– 1807: Gauß does not present any expressions for p and q, but a table of M and N up to 2499. The condition $(11M + 11) \mod 30 < 19$ in (12.2) is replaced by the equivalent but more convenient condition $a > 10$.
– 18??: After Gauß' death a Latin manuscript was found, written by himself in his personal copy of C. Wolf's textbook *Elementa matheseos universae*. There, Gauß proposes the condition $11M + 11 \mod 30 < 8$ instead of <19. Otherwise, the exceptions are the same as in 1800.
– 1811: This paper is even worse than better. Gauß presents his algorithm for the 18th and 19th century only, and the two exceptions (12.1) and (12.2) are replaced by the rule that 19 April should always be substituted for 26 April and 18 April for 25 April. This proposal fails for the years 1734 and 1886.
– 1816: Finally, Gauß' student P. Tittel found out that p is wrong and Gauß corrected for it, admitting, "I owe this remark to Herr Dr Tittel, who is at present in Göttingen, and is devoting himself to the study of the astronomical sciences with excellent eagerness." In the same year Tittel himself published a work in which he gave a full account of the ecclesiastical computus.

In 1815 J. B. Delambre critized Gauß sharply,

> Monsieur Gauss a donné dans la Correspondance Astronomique de M. de Zach, année 1800, deuxième partie, page 129, des formules très-curieuses pour trouver le jour de Pâques, sans avoir besoin ni de l'Epacte, ni du Nombre d'or, ni de la Lettre Dominicale. Il n'a fait qu'en indiquer la démonstration, en avertissant qu'elle suppose ses principes encore inédits d'Arithmétique transcendante. [...] J'ai reconnu depuis qu'on pouvait [...] donner ainsi une solution complète et directe des problèmes du Calendrier Grégorien. Ces formules sont aussi simples au moins que celles de M. Gauss, et la démonstration en découle des principes de l'arithmétique ordinaire.[12]

Delambre seems to be angry with a 23-year-old man (in 1800!) who claims to have solved the problem on the grounds of "higher arithmetic" while simple divisions are needed only. It is interesting to note that Gauß never reviewed this specific issue of a journal he has reviewed repeatedly. Ironically, Delambre's algorithm is erroneous too.

Gauß' Osterformel looks undoubtedly odd, and in particular the two exceptions (12.1) and (12.2) appear artificial. A better understanding of Gauß' Osterformel, however, requires some discussion of the Gregorian Easter reckoning.

3. The Gregorian computus paschalis

The Gregorian computus is explained in a voluminous and rare book by Chr. Clavius.[13] A more easily accessible source is *The Catholic Encyclopedia*.[14]

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[12] J. B. Delambre, "Formules pour calculer la Lettre Dominicale, le Nombre d'Or, l'Epacte et la fête de Pâques, pour une année Grégorienne ou Julienne quelconque," *Connaissance des tems pour l'an1817*, Paris 1815, 307–317.

[13] Chr. Clavius, "Romani Calendarii a Gregorio XIII P. M. restituti explicatio," Roma 1603.

[14] "The Catholic Encyclopedia," Online Edition 1999.

The term *Epact* is the key to the Gregorian Easter reckoning. To every year yr larger than 1582 a *Golden Number*

$$gn = (yr \bmod 19) + 1$$

is assigned. The *Restored Epact*

$$epr = (11 \times gn - 10) \bmod 30$$

improves the antegregorian epact $(11 \times gn - 11) \bmod 30$ and is supposed to measure reliably the moon's age on 0 January for the years from 1583 up to 1699. For the following centuries

$$cy = \lfloor yr/100 \rfloor + 1.$$

However, two corrections are applied, the (accumulated) *solar equation*

$$sol = \lfloor (3 \times cy)/4 \rfloor - 12,$$

omitting three leap days in every four centuries, and the (accumulated) *lunar equation*

$$lun = \lfloor (8 \times cy + 5)/25 \rfloor - 5$$

by which eight extra days in 2500 years are added to compensate for the deficiency of the 19-year lunar (or *Metonic*) cycle. In this way, the moon is kept fairly well synchronized with the sun. We call

$$epp = (-sol + lun) \bmod 30$$

the *Principal Epact*. The *Gregorian Epact*, or simply *Epact*, is then defined by

$$epg = (epr - sol + lun) \bmod 30 = (epr + epp) \bmod 30.$$

The Gregorian Epact is a function of the Golden Number and thus changes from year to year, whereas the Principal Epact is constant over one century at least. To each Principal Epact belongs a 19-year cycle of Gregorian Epacts, repeating itself as long as the Principal Epact is not changed. Once the Gregorian Epact is known, all new and full moons of a specific year are known following a simple scheme of alternating 29-day and 30-day months. The Paschal Full Moon falls on

$$pfm = 44 - epg \text{ March, or on } pfm + 30 \text{ March if } pfm < 21.$$

The papal calendar commission made two additional decisions.

(1) In deviating from tradition, Epact 24 makes possible that Easter Sunday can fall on 26 April, a date not possible before. For the sake of consistency the Paschal Full Moon of Epact 24 is replaced by 18 April, the Paschal Full Moon of Epact 25.
(2) Within the same 19-year cycle of Golden Numbers the full moons must not fall twice on the same date. Such a coincidence happens with Epacts 24 and 25 in the centuries 20, 21 and 22, for instance. Therefore, the commission introduced a "second" Epact 25, denoted here by 25*, which is used as if it were Epact 26.

Table 2. Relationship between Epacts and Paschal Full Moons according to the Gregorian reform

Epact	Full Moon	Epact	Full Moon	Epact	Full Moon
0	13 April	10	3 April	20	24 March
1	12 April	11	2 April	21	23 March
2	11 April	12	1 April	22	22 March
3	10 April	13	31 March	23	21 March
4	9 April	14	30 March	24	18 April
5	8 April	15	29 March	25	18 April
6	7 April	16	28 March	25*	17 April
7	6 April	17	27 March	26	17 April
8	5 April	18	26 March	27	16 April
9	4 April	19	25 March	28	15 April
				29	14 April

In short, both rules read as follows:

$$\text{if } epg = 24,\ \text{ or } epg = 25 \text{ and } gn > 11,\ \text{ then } epg \text{ should be replaced by } epg + 1.$$

An overview is given in Table 2. It should be noted that Epacts 24 and 25, and likewise Epacts 25* and 26, can never both occur during the same 19-year cycle.

The problem remains to determine the Sunday after the Paschal Full Moon. This can be done by using the Dominical Letter which is the letter of the first Sunday of a year if the first seven days are designated with the first seven letters of the alphabet. By a cyclic succession the Dominical Letter marks all Sundays of a year, and in particular Easter Sunday. For the sake of brevity I take here a somewhat different approach.

In the year 1583, 6 March was the first Sunday of March, because 15 October 1582 was a Friday. In every year after 1583 the first Sunday of March falls either one or two days earlier than in the year before, depending on the Gregorian leap year rule: 4 March 1584, 3 March 1585, 2 March 1586, 1 March 1587. –1 March 1588, i. e. 28 February 1588, is a Sunday again, and 6 March 1588 is the first Sunday of March. In general,

$$\begin{aligned} fsd &= (6 - (\lfloor (5 \times yr)/4 \rfloor - \lfloor (5 \times 1583)/4 \rfloor) + sol) \bmod 7 \\ &= 6 - (\lfloor (5 \times yr)/4 \rfloor - 1978) + sol) \bmod 7 \\ &= (6 - (\lfloor (5 \times yr)/4 \rfloor - 4) + sol) \bmod 7 \\ &= (10 - \lfloor (5 \times yr)/4 \rfloor + sol) \bmod 7 \end{aligned}$$

is the first Sunday of March in the year *yr*, provided that $fsd = 7$ is taken whenever $fsd = 0$. Easter day is then

$$pfm + 7 - (pfm + 7 - fsd) \bmod 7 = ed \text{ March.}$$

Using $x = 10 - \lfloor (5 \times yr)/4 \rfloor + sol$, one may write more elegantly

$$pfm + 7 - (pfm - x) \bmod 7 = ed \text{ March.}$$

Table 3. The Gregorian computus summarized

gn	=	$(yr \mod 19) + 1$
epr	=	$(11 \times gn - 10) \mod 30$
cy	=	$\lfloor yr/100 \rfloor + 1$
sol	=	$\lfloor (3 \times cy)/4 \rfloor - 12$
lun	=	$\lfloor (8 \times cy + 5)/25 \rfloor - 5$
epg	=	$(epr - sol + lun) \mod 30$
if epg	$= 24$, or $epg = 25$ and $gn > 11$, then epg is replaced by $epg + 1$	
pfm	=	$44 - epg$
if $pfm < 21$, then pfm is replaced by $pfm + 30$		
fsd	=	$(10 - \lfloor (5 \times yr)/4 \rfloor + sol) \mod 7$
ed	=	$pfm + 7 - (pfm + 7 - fsd) \mod 7$
if $ed > 31$, Easter Day is $(ed - 31)$ April, otherwise ed March		

In other words,

$$\text{if } ed > 31, \text{ Easter day is } (ed - 31) \text{ April, otherwise } ed \text{ March.}$$

A summary is given in Table 3.

The exposition above is a slightly modified unfolding of D. Knuth's algorithm, first published some 40 years ago.[15]

4. Gauß' Osterformel continued

One may agree that Gauß was rather secretive. In his paper from 1800 however, he gives some deeper insight in his thinking processes. These processes, as I understand them, are the subject of the present section.

Table 4 shows a complete 19-year cycle of $a = yr \mod 19$ for the years 1710–1728. In column 4 $pfm - 21$, the number of days after the earliest Paschal Full Moon, is expressed as a linear combination of 11 and 19. This shows nicely that the date of the Paschal Full Moon is either 11 days earlier or 19 days later than in the year before. The transition from $a = 18$ to $a = 0$ involves 18 days only, i. e. an omission of one day out of 19 days known as *saltuslunae*, the moon's jump. In general, one may write

$$pfm - 21 = 23 - \mu \times 11 + \nu \times 19, \text{ where } \mu + \nu = a$$

and one finds finally

$$pfm - 21 = 23 - 30\mu + 19a = (19a + 23) \mod 30$$

which is equal to Gauß' d for 1700–1899, see (9) in Table 1.

[15] D. Knuth, "The Calculation of Easter...," *Communications of the Association for Computing Machinery*, Volume 5, Number 4, April, 1962, 209–210.

Table 4. The Pascal Full Moons of a complete 19-year cycle for 1700-1899

year	a	pfm	$pfm-21=$		epg
1767	0	44	$23=23$	$-0\times 11+0\times 19$	0
1768	1	33	$12=23$	$-1\times 11+0\times 19$	11
1769	2	22	$1=23$	$-2\times 11+0\times 19$	22
1770	3	41	$20=23$	$-2\times 11+1\times 19$	3
1771	4	30	$9=23$	$-3\times 11+1\times 19$	14
1772	5	49	$28=23$	$-3\times 11+2\times 19$	25
1773	6	38	$17=23$	$-4\times 11+2\times 19$	6
1774	7	27	$6=23$	$-5\times 11+2\times 19$	17
1775	8	46	$25=23$	$-5\times 11+3\times 19$	28
1776	9	35	$14=23$	$-6\times 11+3\times 19$	9
1777	10	24	$3=23$	$-7\times 11+3\times 19$	20
1778	11	43	$22=23$	$-7\times 11+4\times 19$	1
1779	12	32	$11=23$	$-8\times 11+4\times 19$	12
1780	13	21	$0=23$	$-9\times 11+4\times 19$	23
1781	14	40	$19=23$	$-9\times 11+5\times 19$	4
1782	15	29	$8=23$	$-10\times 11+5\times 19$	15
1783	16	48	$27=23$	$-10\times 11+6\times 19$	26
1784	17	37	$16=23$	$-11\times 11+6\times 19$	7
1785	18	26	$5=23$	$-12\times 11+6\times 19$	18
		March			

Gauß knew that 21 March 1700 was a Sunday. If 22 $+d+e$ March is the date of Easter in the year yr, where $0\le e\le 6$, the number of days between both dates is

$$nod = 1+d+e+i+365\times(yr-1700).$$

Here, i denotes the number of leap days and is given by

$$i=\begin{cases}(yr-b-1700)/4 & \text{for } 1700\le yr\le 1799\\ (yr-b-1700)/4-1 & \text{for } 1800\le yr\le 1899\end{cases}$$

Since nod is divisible by 7, it follows that

$$(1+d+i+365\times(yr-1700)) \bmod 7 = (7-e) \bmod 7.$$

After some simple, but cumbersome transformations Gauß finally ends up with the expression

$$e=(2b+4c+6d+\left\{\begin{matrix}3 \text{ for } 1700\le yr\le 1799\\ 4 \text{ for } 1800\le yr\le 1899\end{matrix}\right\}) \bmod 7$$

For convenience, the whole algorithm is summarized in Table 5.

Table 5. Gauß' Osterformel for 1700–1899

$$a = yr \bmod 19$$
$$b = yr \bmod 4$$
$$c = yr \bmod 7$$
$$d = (19a + 23) \bmod 19$$
$$e = \left(2b + 4c + 6d + \begin{Bmatrix} 3 \text{ for } 1700 \leq yr \leq 1799 \\ 4 \text{ for } 1800 \leq yr \leq 1899 \end{Bmatrix}\right) \bmod 7$$

Easter falls on $22 + d + e$ March or $d + e - 9$ April

Table 6. The Paschal Full Moons for a complete 19-year cycle for 1900–2199

year	a	pfm	$pfm - 21 =$		epg
1995	0	45	$24 = 24$	$-0 \times 11 + 0 \times 19$	29
1996	1	34	$13 = 24$	$-1 \times 11 + 0 \times 19$	10
1997	2	23	$2 = 24$	$-2 \times 11 + 0 \times 19$	21
1998	3	42	$21 = 24$	$-2 \times 11 + 1 \times 19$	2
1999	4	31	$10 = 24$	$-3 \times 11 + 1 \times 19$	13
2000	5	49	$28 = 24$	$-3 \times 11 + 2 \times 19 - 1$	24
2001	6	39	$18 = 24$	$-4 \times 11 + 2 \times 19$	5
2002	7	28	$7 = 24$	$-5 \times 11 + 2 \times 19$	16
2003	8	47	$26 = 24$	$-5 \times 11 + 3 \times 19$	27
2004	9	36	$15 = 24$	$-6 \times 11 + 3 \times 19$	8
2005	10	25	$4 = 24$	$-7 \times 11 + 3 \times 19$	19
2006	11	44	$23 = 24$	$-7 \times 11 + 4 \times 19$	0
2007	12	33	$12 = 24$	$-8 \times 11 + 4 \times 19$	11
2008	13	22	$1 = 24$	$-9 \times 11 + 4 \times 19$	22
2009	14	41	$20 = 24$	$-9 \times 11 + 5 \times 19$	3
2010	15	30	$9 = 24$	$-10 \times 11 + 5 \times 19$	14
2011	16	48	$27 = 24$	$-10 \times 11 + 6 \times 19 - 1$	25*
2012	17	38	$17 = 24$	$-11 \times 11 + 6 \times 19$	6
2013	18	27	$6 = 24$	$-12 \times 11 + 6 \times 19$	17
		March			

Indeed, the problem is elegantly solved. Some difficulties, however, arise when extending the limit beyond 1899, see Table 6. This table is valid for the entire period 1900–2199.

Together, Tables 4 and 6 contain all possible Paschal Full Moons. For $a = 5$ and $a = 16$, correspondig to $epg = 24$ and $epg = 25^*$, the scheme is disturbed according to the exception rules of the calendar commission. Instead of correcting for d, and leaving unchanged the date of Easter day as in Table 7, Gauß changed the date. Both strategies yield the same results, but, in a sense, Gauß' strategy is more efficient because corrections are only made when necessary.

Table 7. Gauß' Osterformel; the exceptions are redefined. General expressions for d and e are given in Table 1

$$a = yr \mod 19$$
$$b = yr \mod 4$$
$$c = yr \mod 7$$
$$\mathrm{d} = (19a + \begin{Bmatrix} 23 \text{ for } 1700 \le yr \le 1899 \\ 24 \text{ for } 1900 \le yr \le 2199 \end{Bmatrix}) \mod 30$$

if $d = 29$, or $d = 28$ and $a > 10$, then replace d by $d - 1$

$$\mathrm{e} = (2b + 4c + 6d + \begin{Bmatrix} 3 \text{ for } 1700 \le yr \le 1799 \\ 4 \text{ for } 1800 \le yr \le 1899 \\ 5 \text{ for } 1900 \le yr \le 2099 \\ 6 \text{ for } 2100 \le yr \le 2199 \end{Bmatrix}) \mod 7$$

Easter falls on $22 + d + e$ March or $d + e - 9$ April

In particular in Germany, Gauß' Osterformel has occasioned a number of variants, e. g. by J. Hartmann[16,17,18,19], Joh. Bach[20], F. W. Ristenpart[21], C. Wortelboer[22] and, recently, H. Lichtenberg[23].

Hartmann, director of the Göttingen observatory, provides another example of how Easter algorithms may cause deeper emotions; the controversy was aired in the *Astronomische Nachrichten*. In 1912 he published an "Easter formula", in which the exceptions were rearranged in a similar way as in Table 7. He was proud of making them nearly "not noticeable". Dr Bach, a gymnasialdirektor (director of a secondary school), was not pleased. He let Hartmann know, that he was the author of a book in which twelve Easter algorithms were listed; and Gauß' Osterformel was less complicated than Hartmann's formula. Then Ristenpart, director of the observatory at Santiago de Chile, realized that Hartmann's algorithm is not quite suitable to mental arithmetic. Hartmann found him-

[16] J. Hartmann, "Osterformel," *Astronomische Nachrichten*, Volume 187, Number 4473 (1910), 129–134.

[17] J. Hartmann, "Antwort auf Herrn J. Bachs Bemerkungen zu meiner Osterformel," *Astronomische Nachrichten*, Volume 190, Number 4541 (1912), 81–84.

[18] J. Hartmann, "Über den Zweck einer Osterformel," *Astronomische Nachrichten*, Volume 190, Number 4560, 1912, 451–454.

[19] J. Hartmann, "Notiz zur Osterformel," *Astronomische Nachrichten*, Volume 204, Number 4879 (1917), 123–126.

[20] Jos. Bach, "Drei Osterformeln," *Astronomische Nachrichten*, Volume 189, Number 4517 (1911), 73–80.

[21] F. W. Ristenpart, "Osterformel mit kleinen Zahlen," *Astronomische Nachrichten*, Volume 190, Number 4548 (1912), 211–216.

[22] G. Wortelboer, "Eine Osterformel," *Astronomische Nachrichten*, Volume 262, Number 6269 (1937), 71–74.

[23] H. Lichtenberg, "Zur Interpretation der Gaußschen Osterformel und ihrer Ausnahmeregeln," *Historia Mathematica*, Volume 24 (1997), 441–444.

self in an unpleasant situation and made a reply to both scholars. He came to questions such as what the use of Easter algorithms was, and why those rules were so similar; and finally, it has remained a grin without a cat.

5. Preliminary conclusion

It becomes clear, so far, that any Easter algorithm should consist of two basic steps,

(1) finding the Paschal Full Moon by the Gregorian Epact or a substitute, e. g. by $M = (22 - epp) \bmod 30$,
(2) finding the Sunday that follows the Paschal Full Moon.

The Gregorian exceptions can be applied either in step (1) or in step (2). They may be camouflaged as Chr. Zeller[24] proposed, following H. Kinkelin.[25] For instance, both conditions

$$\text{“if } d = 29, \text{ or } d = 28 \text{ and } a > 10, \text{ then replace } d \text{ by } d - 1\text{”}$$

(see Table 7), and

$$\text{“replace in any case } d \text{ by } d - \lfloor (d + \lfloor a/11 \rfloor)/29 \rfloor\text{”}$$

have the very same effect.

Keeping in mind the two construction steps, the final discussion can be shortened without any loss of understanding.

6. Oudin’s work

In his article from 1815, Delambre nicely translated Clavius’ tables and rules into arithmetical formulae, giving the Dominical Letter, the Epact, and the date of Easter Day.[26] He failed when describing the Gregorian exceptions.

In 1940 J.-M. Oudin published a treatise, admirable for its lucidity of exposition.[27] In its first part, Gauß’ and Delambre’s errors are discussed in great detail, because he found (see p. 391), “Les deux grands géomètres Gauss et Delambre, dans les solutions qu’ils ont données de ce problème, ont commis certaines erreurs que des auteurs modernes reproduisent encore [...].” Oudin defends the Osterformel against Delambre’s accusations, for instance, that no real proof was presented. Unfortunately, Gauß himself somewhat cryptically said, “The investigation by which the formula [...] is found is based on higher arithmetic, for which I presently cannot refer to any publication; and certainly, it cannot

[24] Chr. Zeller, “Kalender-Formeln,” *Acta Mathematica*, Volume 9 (1886), 131–136.
[25] H. Kinkelin, “Berechnung des christlichen Osterfests,” *Zeitschrift für Mathematik und Physik*, Volume 15 (1870), 217–228.
[26] See reference 12.
[27] J.-M. Oudin, “Étude sur la date de Pâques,” *Bulletin astronomique*, deuxième série, Volume 12 (1940), 391–410.

Table 8. Oudin's algorithm

$m = \text{year}$
$c = \lfloor m/100 \rfloor$
$k = \lfloor (c-17)/25 \rfloor$
$r = (c - \lfloor c/4 \rfloor - \lfloor (c-k)/3 \rfloor + 19 \times (m \bmod 19) + 15) \bmod 30$
$R = \begin{cases} r-1, & \text{if } r = 29 \\ r-1, & \text{if } r = 28, \text{ and } m \bmod 19 > 10 \\ r & \text{else} \end{cases}$
$J = (m + \lfloor m/4 \rfloor + R + 2 - c + \lfloor c/4 \rfloor) \bmod 7$
$P = 28 + R - J$ March

be demonstrated here in its full simplicity."[28] Subsequently, Ph. Maennchen[29], R. Wolf[30] and others continued in disseminating the legend of the insufficient proof. Simply, the proof lies in the fact that Clavius' tables and rules are precisely translated: all possible 30 Paschal Full Moons and their secular variations are discussed, and a general formula indicates Easter Sunday. By the same token, the validity of any other Easter algorithm can be proven.

In the second part, Oudin demonstrates, that

- the period of the Principal Epacts *epp* is 300,000 years,
- the period of Gauß' quantity M is 300,000 years,
- the period of the Gregorian Epacts *epg* is 5,700,000 years, and
- the period of the dates of Easter is 5,700,000 years, too.

The Easter period turns out to be extremely long; *Homo habilis* appeared on earth about 2 million years ago. The antegregorian Easter period amounts to 532 years only.

In the last part, Oudin presents his own algorithm, see Table 8. The quantity r is nothing else but Gauß' quantity d. As in section 5, the expression for R may be replaced by

$$R = r - \lfloor (r + \lfloor (m \bmod 19)/11 \rfloor /29 \rfloor$$

In his modification of Oudin's algorithm L. E. Doggett[31] employs

$$R = r - \lfloor r/28 \rfloor \times (1 - \lfloor r/28 \rfloor \times \lfloor 29/(r+1) \rfloor \times \lfloor (21 - (m \bmod 19))/11 \rfloor)$$

It should be mentioned that, in his original version, Doggett did not use the floor function, but simply discarded the noninteger part of a fraction.

[28] See reference 3.
[29] See reference 6.
[30] R. Wolf, "Handbuch der Astronomie, zweiter Halbband," Zürich 1891, 622–625.
[31] L. E. Doggett, "Calendars," In *Explanatory Supplement to the Astronomical Almanac*, edited by P. K. Seidelmann, Hill Valley 1992.

Table 9. Algorithm of an anonymous correspondent

Divide	By	And call the	
		Quotient	Remainder
The year of our Lord	19	–	a
” ” ” ” ”	100	b	c
b	4	d	e
$b+8$	25	f	–
$b-f+1$	3	g	–
$19a+b-d-g+15$	30	–	h
c	4	i	k
$32+2e+2i-h-k$	7	–	l
$a+11h+22l$	451	m	–
$h+l-7m+114$	31	n	o

n is the number of the month of the year and $o+1$ is the number of the day of the month on which Easter falls.

7. An anonymous correspondent

It seems that in Britain, and in its former colonies as well, people were not inclined to get involved in those debates. Probably, they determined the date of Easter by using the tables appended in their Common Prayer Books. It is worth to note that R. W. Mallen[32] presents a translation of the official tables into a BASIC programme, valid for 1583 to 4099.

It only seems so. In 1818 Easter Sunday fell on the first full moon in spring, in contradiction to the Calendar Act of 1750 (24 Geo. II cap. 23), which defines Easter as “the first Suaday [sic] after the first full moon which happens [on or] next after the 21th day of March, and if the full moon happens upon a Sunday, Easter-day is the Sunday after.”

Some discussion arose. A. de Morgan, known today mainly for his contribution to mathematical logic, finally cut the knot. He explained – in accordance with Clavius’ statements – that *moon* is neither an astronomical nor a mean moon, but a fiction based on cyclic reckoning, and presented a general recipe to find Easter for any year. The whole story can be found in de Morgan’s collection *A Budget of Paradoxes* which is still a source of great intellectual pleasure.[33]

In 1876 *Nature* published an algorithm submitted by “an anonymous correspondent from New York”, see Table 9.[34] The quantity h is equal to Gauß’ quantity d, and the Gregorian exceptions are hidden in m. Thus, it is a Gaußian-type algorithm. The method

[32] R. W. Mallen, “Easter Dating Method,” Website of the Astronomical Society of South Australia, 2000.

[33] A. de Morgan, “A Budget of Paradoxes,” Volume 1, Chicago and London 1915, 356-357. The first edition appeared in 1872.

[34] Anonymous, “To find Easter ‘for ever’ ”, *Nature*, April 20, 1876, 487.

is of great beauty and has been adopted by the British people almost immediately; I mention here S. Butcher's Ecclesiastical Calendar and H. Spencer Jones' textbook.[35,36] Some decades ago, J. Meeus brought this algorithm to Continental Europe.[37]

Conclusion

"In considering the fact, that the date of Easter Sunday is available for sufficiently long periods of time [...], a discussion of the various ways which have been proposed for determinig Easter dates, is of interest neither to historians nor to chronologists."[38]

I have not followed this advice of F. K. Ginzel's although he is right in some sense. I would rather think, it is not very useful to construct new Easter algorithms, which necessarily must be variants of known algorithms. One might get the impression that some scholars are ignoring the work of others, and reinvent the wheel. In my personal opinion, Oudin's paper, a work of unique *clarté*, is the final keystone. Perhaps, I stand alone with my opinion. Being not a psychologist, I am aware, however, of the fascination of the lunar cycle, a cycle that "hath been already old of time, which was before us (Ec. 1 v. 10)." So, the race for the ultimate solution may go on.

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[35] S. Butcher, "Ecclesiastical Calendar," Dublin/London 1876, 226.
[36] H. Spencer Jones, "General Astronomy," 3rd edition, London 1951, 66.
[37] J. Meeus, "Astronomical Formulae for Calculators," 2nd edition, Richmond 1982, 31–33.
[38] F. K. Ginzel, "Handbuch der mathematischen und technischen Chronologie," Volume 3, Leipzip 1914, 265–266.