إفراد المقال في أمر الظلال

تصنيف الشيخ أبي الريحان محمد بن أحمد البيروني رحمه الله

THE EXHAUSTIVE TREATISE ON SHADOWS

by

Abu al-Rayḥān Muḥammad b. Aḥmad al-Bīrūnī

Translation & Commentary

by

E. S. KENNEDY

Volume II

COMMENTARY

INSTITUTE for the HISTORY of ARABIC SCIENCE

University of Aleppo

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RESEARCH PUBLICATIONS

of the

INSTITUTE FOR THE HISTORY OF ARABIC SCIENCE

UNIVERSITY OF ALEPPO

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PREFACE TO THE COMMENTARY

As remarked in the preface to the translation, the commentary has been organized in chapters corresponding to those of the original text. The chapters are subdivided into titled sections numbered serially independent of the chapters. The sections treat of successive portions of the text, references being made by page and line numbers of the printed Arabic text. The same pages and lines are indicated along the margins of the translation, so that the reader may have quick access to cognate passages in text, translation, and commentary. References to the commentary in the indices are given by section numbers in italics, following any page and line references to the printed text.

Each entry in the bibliography (beginning on page 181 of this volume) commences with an abbreviated title, the full title following it. In the body of the commentary, references to the bibliography cite the short title, in italics.

Paragraphs enclosed within parentheses and having the initials D.P. appended have been contributed by Professor David Pingree.

CONTENTS

SYMBOLS AND CONVENTIONS

For easy reference, symbols used consistently in the commentary are displayed below, arranged more or less alphabetically. Where applicable and convenient they are the standard modern astronomical symbols.

The medieval trigonometrical functions are, as is customary, distinguished from their modern counterparts by capital initals, thus

$$\text{Sin}\ x = R \sin x,$$

where R is the radius of the defining circle, usually $R = 60$. Where two or more such parameters are present in the same discussion, one may be shown as a subscript to avoid ambiguity, thus

$$\text{Sin}_{\rho}\ x = \rho \sin x.$$

As is usual, sexagesimals are transcribed in ordinary numerals with sexagesimal digits separated by commas. The semicolon is used as a "sexagesimal point".

All rules and expressions in the original Arabic (and in the English translation) are written out in words. In putting them into modern symbols we have frequently found it convenient to set up parenthetical equations within equations, say,

$$A = (\text{Sin}(B=C)).$$

This usage may appear strange, but it is an accurate reflection of the text.

a_i rising time in *sphera recta* of the i-th zodiacal sign (in Section 103).

az azimuth.

α right ascension.

α_ϕ	oblique ascension for a locality of latitude ϕ.
$\Delta\alpha_i$	rising time of the i-th zodiacal sign.
β	celestial latitude.
—	bar (vinculum) usually denotes a complement, e.g. $\bar{\theta} = 90° - \theta$, but it is sometimes used with the squares of line segments, say $\overline{AB}^2$. Occasionally, as in Section 135, the bar denotes a mean value.
d	half the arc of daylight; however, in Section 71 it denotes length of daylight; as a subscript it stands for "desired"; in Sections 62 and 63 it is used for the diameter of a sphere.
δ	solar declination.
Δ	the difference operator.
$\cdot$	a dot over a variable (as $\dot{\delta}$ in Section 84) indicates its derivative with respect to time.
e	equation of (half) daylight.
ε	inclination of the ecliptic.
ϕ	terrestrial latitude.
g	gnomon length; as a subscript, stands for "given".
h	altitude, the horizon coordinate.
L	length of daylight in day-minutes.
Λ	terrestrial longitude.
λ	celestial longitude.
n	(as a subscript) noon.
p	parallax.
( )	parentheses in the translation indicate material not in the text but added for clarity.
'	the prime sign, used with a decimal integer means sixtieths, e.g. 150' = 150/60 = 2;30.
R	the radius of the defining circle for the medieval trigonometric functions. Usually $R = 60$.
r	hour angle.
ρ	radius of the day circle.
s	shadow length; when used as a subscript it stands for *sun*.
s_n	noon shadow.
s_ϕ	equinoctial noon shadow.
t	time.
vers	the versed sine, vers $x = 1 - \cos x$, whence Vers $x = R$ vers x.
w	rising amplitude; horizon distance from the east point to the sun's rising point. Sometimes called ortive amplitude.

THE INTRODUCTION

1. Al-Bīrūnī's Dedication and Invocation (3:1 - 4:11)

The book begins with an extraordinarily long and involved sentence in which the author states that he will not undertake a philosophical discussion of the nature of light. He mentions the intromission versus emission controversy, already ancient in his day, but still then being argued. (See, e.g., *Ronchi*, p.13, and *Lindberg*, 2, p.478.) But he does not take sides in the matter. He will confine himself, he says, to a negative aspect of light — shadows, and to their mathematical implications.

Bīrūnī's great contemporary, Ibn al-Haytham, also wrote a treatise on shadows, and a remark in it (*Sabra* p.195) implies that there existed a distinct category of works on this subject. The only additional example known to us is the book by Ibrāhīm b. Sinān mentioned in Section 11 below. Bīrūnī and Ibn al-Haytham seem to have operated in ignorance of each other.

The Shaykh Musāfir, to whom the *Shadows* is dedicated, was a leading citizen of Nīshāpūr in Khurāsān. His locality was fixed by Professor Iḥsān ʿAbbās, who points out that another prominent Nīshāpūrī, the prolific scholar al-*Thaʿālibī* (961-1038), dedicated to Musāfir one of his books, the "*Khāṣṣ al-Khāṣṣ*" (pp.2, 178, 186, 191). Bīrūnī dedicated to the Shaykh two other works, a commentary on the astronomy of al-Farghānī (*Boilot* RG 14), and a discussion on an astrological technique (RG 16). We note that in the dedicatory passage here our author connects Musāfir with the determination of the times of the Muslim ritual prayers, and with astronomical instruments for finding them. Since some of these times are defined in terms of shadows, a good part of the book which follows is devoted to these topics.

The passage closes by invoking divine blessing upon the author and his patron. There follows in 4:12 - s35:13 a list of the chapter headings.

2. Al-Bīrūnī's Preface (s35:14 - s38:12)

Having announced the subject of the treatise, the author next states that the understanding reader will need to be conversant with cosmology and with mathematics. The rest of the section refutes the idea that the material is contrary to religion, showing that, on the contrary, its cultivation is necessary for the proper observance of the precepts of Islam.

Of the four poets who receive disparaging mention in s36:13, one, Abū Nuwās, (f.800) is well-known, indeed he is by some considered the greatest of the Arab poets. But both his life and his writings were highly licentious, which is doubtless the reason for his being mentioned by Bīrūnī. (See *Wagner*, p.10; *Nicholson*, p.203.)

The same can be said of the minor poet Ibn Ḥajjāj, who died in 1001 in Baghdad. An anthology of his poems, although partially expurgated, was regarded by the police of Baghdad as too obscene to be read in the presence of minors (*GAL*, vol.1, p.81, suppl. vol., p.130).

ʿAbd al-Salām ibn Raghbān, known as Dīk al-Jinn, "The Demon's Rooster", was born in Homs in 849. He championed the Syrians against the Arabs. Perhaps Bīrūnī held it against him that he was a Shīʿa, as was Ibn Ḥajjāj (*GAL*, vol.1, p.85; suppl. vol., p.137).

The fourth Abbasid poet, Abū Ḥukayma, Rāshid b. Isḥāq, is mentioned by *Yāqūt* (vol.11, p.122), who cites a poem of his composed during the last days of the Caliph al-Ma'mūn, hence c.833.

Five ritual prayers per day are incumbent upon every Muslim. The times during which they may be prayed are determined by astronomical considerations. These matters are discussed in great detail in Chapters 25 and 26 of this book.

Furthermore, the worshipper is enjoined to face toward Mecca when praying. The determination of this direction of prayer, the *qibla*, for an arbitrary point on the earth's surface is a problem in mathematical geography which is solved by Bīrūnī in various ways in the *Tahdīd* and the *Canon*, and by many other Muslim scientists.

Daylight fasting during one month of the year is also required of Muslims. The calendar is lunar, and the beginning of each month is defined as the first evening when the new crescent moon is actually sighted above the western horizon by some competent observer. The problem of determining whether or not (assuming clear weather) the new moon will be visible from a given place and on a given evening, and if so where, is a matter of great complexity. It also was worked over very extensively by the astronomers of medieval Islam. Bīrūnī's point is that, although the matter can be settled canonically only by an actual sighting, not computations, nevertheless prior calculations can assist the observer to the extent of saying beforehand whether or not the sighting is possible, and if it is, precisely where in the heavens the crescent will appear. Some of the authors prescribe the use of a sighting tube mounted like a telescope to indicate a celestial locality of predetermined horizon coordinates. This may be the *abṣār* (from *baṣara*, to see) named in the text at s38:3. (Cf. *Battāni*, zīj, vol.3, p.137, and *Sanjarī Zīj*, f. 90v.) The term is unfamiliar to us.

For the subjects mentioned above it is easy to cast science and mathematics in the role of hand-maidens to the faith. The rest is rather farfetched. Almsgiving is indeed one of the five pillars of Islam, and presumably the almsgiver is in need of arithmetic to calculate properly his income and the tithe thereof. Carrying the notion a step farther, he may have no income unless he engages in commerce, hence the need for metrology. The rules governing inheritance can be regarded as a part of Muslim law, and it is true that there are problems involving bequests in the algebra of al-*Khwārizmī* (pp.86-174). Finally, the notion that the scientist be supported in order that he create engines to be used in the holy war is a proposition which has become genuinely dangerous only in modern times.

CHAPTER I

ON THE PRIME MOTION

3. Importance of the Daily Rotation (s38:13 - s41:1)

This short chapter simply points out three consequences of (in modern terms) the earth's rotation, (in the medieval view) the turning of the celestial sphere about the north pole from east to west above the horizon.

For one thing, this prime motion supplies a method for determining the cardinal directions at any point on the earth's surface. To fix the meridian, simply bisect the arcs on the horizon bounded by the rising point and the setting point of any fixed star (s49:2).

Secondly, the motion about the celestial pole is useful for establishing the positions of localities on the terrestrial sphere (39:8). This subject is exhaustively discussed by Bīrūnī in another work, the *Taḥdīd*.

And thirdly, the prime motion is the source of units of time. Here Bīrūnī draws an analogy between space and time, pointing out that uniform motion gives the measure of magnitudes in both (s40:7). Although velocity is in principle unbounded (s40:12), the fastest of the celestial motions is the prime motion. Indeed the daily rotation is much faster than the proper motion in the celestial sphere of any of the planets. In the Aristotelian view these lesser displacements have been communicated successively to the various celestial objects from the prime motion.

CHAPTER 2

ON LIGHT AND DARKNESS

4. Definitions and Nomenclature (s41:2 - s45:7)

This chapter discusses the nature of shade and shadow, the Arabic words used to denote them, and related optical matters. The author states that any opaque object intercepts the rectilinear rays of the sun. If the object is a plane mirror, it will not itself be perceived, but a reflected image. If the object's surface is not highly polished it will be seen, illuminated by the sunlight. In either case the regions behind it, shielded from the rays, constitute the shadow. A distinction is made between this and pitch darkness (s42:1).

There follows a discussion of two Arabic words commonly used, *ẓill* and *fay'*, the former being the word for shadow consistently employed in the book itself. Various opinions are cited, one to the effect that the term *fay'* is restricted to shadows cast in the afternoon, when the sun is declining and the shadow lengthening, *ẓill* being the forenoon shadow. But Bīrūnī then cites from the literature four examples to the contrary. Two of them, at s42:12 and s42:14, have *ẓill* for the afternoon shadow; another, at s44:1, calls the morning shadow a *fay'*. A fourth example (s44:7) uses *ẓill* for both morning and afternoon.

The couplet of al-Shāmī (s44:11) has a shadow attaining its maximum length, then decreasing. But the shadow cast in the sunlight by a vertical gnomon on a horizontal plane does the opposite. Its maximum is at sunrise, whence its length decreases to a minimum at noon, thereafter increasing. Bīrūnī suggests that the shadow on a meridian-oriented wall and the earth's shadow in a lunar eclipse perform as described in the poem. But he hardly presents these as serious explanations of what the poet had in mind.

Of the individuals mentioned in the text, Khuwaylid b. Khālid al-Qatīl, Abu Dhu'ayb (43:11) fl.650, was the best known poet of the Hudhayl, an Arab tribe related to the Quraysh, the tribe of the Prophet. The poetry of the several tribes was collected in separate anthologies. Bīrūnī may have obtained the couplet quoted in s42:7 from the Hudhayl anthology. (See *Nicholson*, p.xix, *GAL*, vol.1, p.41; suppl. vol.1, p.42.)

The Abū Laylā named at s42:8 was evidently one of the early commentators on pre-Islamic and early Islamic Arabic poetry. He is mentioned once in the commentary on Labīd's poetry. (*Sharḥ dīwān Labīd*, Kuwait, 1962, p.133; supplied by I. Abbas.)

The poet al-Khalīᶜ al-Shāmi (s44:10-17) is mentioned in *Thaᶜālibī*, *Yatīma* (vol.1, p.287) as being a contemporary of the poet al-Buḥturī (fl.860), and as being in Aleppo as an old man.

Ru'ba (s42:18), d.c. 760, was a merchant who travelled between Khurāsān and Baṣra, but he was of Arab origin. His poetry was in the ancient *rajaz* meter (*GAL*, vol.1, p.60).

Ghaylān b. ᶜUqba, known as Dhū al-Rumma (s43:13), d.c. 730, was the last of the Arab poets to write in the archaic style of the pre-Islamic bedouin (*GAL*, vol.1, p.58).

Al-Faḍl b. Qudāma al-ᶜAjalī, Abū al-Najm (s43:15) d.c. 750, was another *rajaz* poet (*GAL*, vol.1, p.60).

The *Dīwān al-adab* (s44:6) was written by the Khurāsānī grammarian Abū Ibrāhīm, Isḥāq ibn Ibrāhīm al-Fārābī, d. 961 (*GAL*, vol.1, p.128).

5. Shadows in the Hereafter and in the Sky (s45:8 - s51:4)

The author next pursues the question of shadows into the afterlife. Since there time has no meaning, there is no need for the sun, which is simultaneously the best indicator of time and the prime source of shadows. Thus heaven, presumably, is comfortably shady throughout.

There remains hell, whose shade is smoke, a torment added to the fire. We can only conjecture why this material has been introduced into a scientific work. The passage is buttressed with quotations from the Qur'ān. Perhaps this literary excursus is to sustain the note of piety introduced in the preface.

At s47:9 the author moves on to consider celestial matters which are not articles of faith, the conical shadows cast by the earth and the moon in the light of the sun. Here also there is a certain amount of Qur'ānic exegesis brought on by a discussion of why night is not called a shadow.

Abū Muslim Muḥammad b. Baḥr al-Isfahānī (s46:19) was a theologian; he is listed in the *Fihrist* (p. 137).

Manṣūr b. Ṭalḥa was the last ruler of a semi-independent dynasty founded by an Abbasid governor of Khurāsān. In the *Taḥdīd* (cf. *Comm.*, p.41) Bīrūnī names a book on astronomy by Manṣūr, and cites a passage from it. We are unable to make anything of the remark (at s49:14) about the size of terrestrial mountains.

The remarks at s50:6 concerning the relative sizes and distances of the moon, sun, and earth are consonant with the accepted opinions of Ptolemy and the medieval astronomers. In *Almagest* V, 16 the ratio between the volumes of the earth and the moon is $39\frac{1}{4}:1 \approx 40$. That between the volumes of the sun and the earth is $6644\frac{1}{2} : 39\frac{1}{4} \approx 169$, also slightly off from Bīrūnī's 166. The ratio of the earth-sun distance to the maximum earth-moon distance is $1210 : 64;10 = 18.9 \approx 19$.

It is curious that Bīrūnī should mention (s50:17) a shadow cast by the earth in the moonlight. The earth being larger than the moon, the vertex of the shadow cone is indeed between the moon and the earth, but this vertex is not part of the shadow.

Of interest also is the statement (s51:1, s55:19) that there were different opinions as to whether or not the planets and the fixed stars are self-luminous.

6. Refraction and Reflection, al-Sarakhsī and Aristotle (s51:5 - s55:18)

In this passage a number of topics are touched upon briefly and vaguely. The author alludes again (in s51:15) to the emission controversy, naming Galen and Aristotle as protagonists. He next mentions (in s52:1) optical refraction, the bending of the rectilinear ray when it passes from one medium into a denser one. Reverting to the subject of reflection, for a concave spherical mirror he locates the approximate focus, the burning point, half-way between the center of the sphere and its surface (s53:17).

There is a discussion of why objects in shade are partially lighted (s54:3), which leads to a criticism of one Aḥmad b. al-Ṭayyib al-Sarakhsī. This individual was born in Khurāsān, c. 835, became a disciple of the philosopher al-Kindī, tutor of the prince al-Muʿtaḍid, and a boon companion of the latter when he became caliph. However, in 899 he was executed by al-Muʿtaḍid, probably for heresy. He wrote voluminously on religion, philosophy, politics, science, geography, history, and astrology, but only fragments of his works are extant. (See *Rosenthal* and *Moosa*.) This being the case, it is difficult to know what to make of his blackening of the air (s54:16) at high altitudes. We find nothing of this sort in Aristotles' *De Sensu*. Mount Demavand is indeed a very high peak, clearly visible from Ray (near modern Tehrān), where Bīrūnī spent some time (*DSB*, vol.2, p.148). In the *Meteorologica* 1, 13 (p.97) Aristotle mentions the Caucasus, its great height, and the illumination of its peak before sunrise (s55:17). However there is nothing about the stillness of the air there, or its blackness, or the ashes of sacrifices. Perhaps the repeated "he" in our passage is frequently a reference to al-Sarakhsī rather than Aristotle.

The Arabic version of the *Meteorologica* is by no means identical with the received Greek text, but the passage concerning the Caucasus is essentially the same (*Petraitis*, p.45). The Arabic version of *De Sensu* is not extant (*Peters*, p.45).

7. The Freezing of Hot Water (s55:19 - s57:19)

Of several subjects touched on this section, the greatest interest attaches to that which illustrates Bīrūnī's readiness to experiment. The notion that hot water freezes more quickly than cold water is expressed

by Aristotle in the *Meteorologica* 1, 12 (p.87). Abū Rayḥān demonstrated the contrary by two trials of a pair of samples each, in the second experiment making the temperature of the hot water higher than in the first.

With regard to habitable regions on the terrestrial globe (s56:11), Aristotle was indeed in error in claiming (in *Meteorologica* 11,5) that the tropical regions are uninhabitable, although he admitted the south temperate zone to be habitable.

The treatise on burning mentioned in s56:14 is to be added to the long list of Bīrūnī's lost works. We find no other mention of it in the literature.

CHAPTER 3

VARIATIONS IN SHADOWS

8. Altitude and Azimuth (s58:3 - s60:19)

The scientific content of this chapter is very slight. It commences with definitions; these merge into matters of usage and nomenclature, thence into fanciful religious and literary allusions.

The two horizon coordinates, altitude and azimuth, are first defined. It is remarked that when actual celestial objects move in the sky both coordinates vary simultaneously (s58:11), but that conceptually the two are independent. Variation of the one without the other, and conversely, produces the intersecting families of coordinate circles on the celestial sphere (s58:16). As with us, altitude is normally measured up from the horizon, the term depression (*inḥiṭāṭ*) being reserved for vertical angular distances below the horizon (s59:8).

Azimuths are horizontal angles measured from any one of the four cardinal directions. If a person stands facing the north his right side is to the east and his left to the west (s59:18). Aristotle's association of right and left with the cardinal directions is found in *De Caelo* II, 2. The peculiar reference to the animal, doubtless stems from the same passage in Aristotle. By contrast, an eastward-facing person is implicit in the two Arabic words, *yamīn* for both south and right(hand), and *shamāl* for both north and left (s59:19).

It is true that in medieval Islamic astronomy the term *khaṭṭ al-istiwā'* is generally reserved for the terrestrial equator (s60:14).

9. Literary Allusions - Shadows and Prostrations (s61:1 - 8:19)

As the sun crosses the meridian it attains its maximum altitude and thereafter declines, whence the term "line of declining" (s60:18). The time of decline is the accepted period for the Muslim noon prayer (s61:5). This topic is discussed in detail later, in Chapter 25. At the instant of culmination the sun stops rising. The notion of stopping seems to have been extended to the claim that at noon the sun actually desists from motion, thence to the idea of fasting (s61:6). Poetic quotations are adduced bv way of illustration (s61:8). Concerning Dhū al-Rumma, see Section 4 above. Next Bīrūnī remarks that although at noon the rate of change of the altitude vanishes, the sun's azimuth continues to change at this time (s61:16-s62:1). He also reverts to a difference already noted (in Section 4) between the meaning of *ẓill* and *fay'* (s62:4). Then ensues a discussion citing various terms used for shadows cast in moonlight (s62:9 - 17).

Much is made of the analogy whereby the object casting a shadow is likened to a worshiper prostrating himself in abasement (s62:18 - 8:10).

The name of the poet quoted in 7:5 is transcribed in the *GAL* (vol.1, p.240; suppl. vol.1, p.425) as Abū al-Faraj ʿAlī b. al-Ḥusayn b. Hindū (d. 1019). He lived in Nīshāpūr and was primarily a philosopher.

The writer Aḥmad b. Muḥammad b. Thawāba (7:7, fl. 200) was of an originally Christian family. The quotation is from his lampoon of the Abbasid *wazīr* Ṣāʿid b. Makhlad, also a converted Christian (*Fihrist*, p.143; *Yaqūt*, vol.4, pp.144-174.)

The Abū al-Fatḥ quoted in 7:10 was an eminent scholar from Bust (modern Qalʿa Bīst in Afghanistan) who was for a long time chief secretary in the Ghaznavid bureaucracy (*Bosworth*, p.42; *Thaʿālibī*, pp.viii, 10, 134). He is mentioned elsewhere by Bīrūnī (e.g. *India*, transl., vol.2, p.270). Abū Rayḥān also composed a poem in praise of al-Bustī, translated in *Aufsätze*, vol.2 p.480.

As to the curious legend about the Christian *qibla* (direction of prayer) retailed in 7:16 - 8:6, there is no mention of a shadow in connection with Mary

Magdalene's visit to the tomb in the accounts given by the four Gospels. John 20:14 has, "When she had thus said, she turned herself back, and beholdeth Jesus standing,..." All the versions agree that the time was very early in the morning. Of course the shadow at dawn extends due west on a morning near the equinox.

Whatever the source of the story, Abū Rayḥān makes use of it to indulge in a little anti-Trinitarian polemic. The inference seems to be that since the shadow did not prostrate itself to Jesus, he himself had a master, i.e., he is not God. The eastward *qibla* of the Christians is alluded to by Bīrūnī in *Taḥdīd* 210:16.

The theme of monotheism is the reason for the mention of Abraham at 8:6. In Qur'ān vi, 74-82 the patriarch is persuaded of God's oneness by watching the risings and settings of the sun, moon, and planets.

Abū al-Dardā' (8:7) was one of the Companions of the Prophet during his lifetime (*Nicholson*, p.225).

The title "The Shadow of God upon the Earth" was a common honorific of Muslim rulers. It is to this that Bīrūnī refers in 8:11. Abū Bakr the Veracious (*al-Ṣiḍḍīq*, 8:16) was the first of the four orthodox caliphs.

CHAPTER 4

CURVES DRAWN BY SHADOW ENDS

10. Conics Traced by the Sun's Shadow (9:3 - 10:11)

If the sun is assumed to be fixed in the celestial sphere, then the daily rotation of the latter will cause it to trace out a circle in the course of a day. But if the sun's proper motion is taken into consideration, then because of its variation in declination, its trace will be a slowly rising or falling spiral (9:8). Neglecting the proper motion, the line joining the sun to a gnomon end point will sweep out the lateral surface of a cone of revolution each day. The lower nappe of the cone is the shadow of the gnomon's end, and the vertex angle of the cone is $\bar{\delta}$. Any intersection of the shadow cone with a horizon will be a conic section.

At an equinox, $\bar{\delta}=90^\circ$, $\delta=0^\circ$, and the cone flattens out into a plane. On any such day the trace of the shadow end point will be a straight line, regardless of the value of ϕ. In particular it is not necessary, as Abū Rayḥān claims, that $\phi=0^\circ$ (9:15). He is right, however, in saying that then the axis of the cone is parallel to the horizon.

The conic will be a parabola if the angle between the cutting plane and the axis of the cone equals the cone's vertex angle. This requires that $\bar{\delta}=\phi$, or $\delta=\bar{\phi}$. Since max δ = ε, the least latitude at which this can occur will be when $\phi=\bar{\varepsilon}$ (cf.10:15).

The trace will be an ellipse when $\phi>\bar{\delta}$, and this will happen in any locality having a twenty-four hour day, the arctic regions (10:8). Whenever $\phi<\bar{\delta}$ the section is an hyperbola.

11. A Criticism of Thābit (10:12 - 11:12)

Thābit b. Qurra (d. 901) originated in Ḥarrān in Syria, and was a member of the sect known as the Sabians centered there (cf. Section 128 below). An intermediary between the Hellenistic and the Islamic worlds, he not only made translations from Greek into Arabic, but made solid original scientific contributions of his own. The list of his known books is extensive but the one named in 10:12 is not among them. (see *GAL*, vol.1, p.217; suppl. vol.1, p.384).

His grandson Ibrahīm b. Sinān (d. 946) was also a scientist. A book of his, Aya Sofya (Istanbul) MS 4832(15) is about shadows and may be the one mentioned in 10:14, although the titles are not identical (*GAL*, vol.1, p.218; suppl. vol.1, p.381).

The apparent diameter of the sun subtends an angle of about half a degree. Hence, regardless of the size of an aperture through which the sun is shining, the beam will have a conical rather than a cylindrical shape, as Abū Rayḥān argues. Perhaps Thābit disregarded the fact that all points on the solar disk emit rays in many directions. In particular, the point on the sun's limb at *A* (Figure 1) has a ray passing through the opposite edge of the hole at *D*, hence the conical shape. Or perhaps Thābit thought the vertex angle was sufficiently small to be disregarded.

CHAPTER 5

DISTORTIONS OF SHADOWS

12. Shadows with Indistinct Edges (12:3 - 13:7)

It is easily verifiable that shadows cast in, say bright sunlight, are sharply defined only when the shadow-casting object is close to the field upon which the shadow falls. As the distance between the two increases, the edges of the shadow become blurred and indistinct. There is no need, however to assume that this phenomenon is caused by reflections from airborne dust particles (12:5). It is due to the fact that most sources of light subtend an appreciable angle from stations at which shadows are cast.

Consider a thin opaque body with a pair of intersecting rectilinear edges *ABC* as shown in Figure C1. Assume that it is above the plane of the paper at a convenient distance, and parallel to it. If we further assume a point source of light at an infinite distance above the plane of the paper, the shadow of the object will have a sharp boundary, congruent with *ABC*. Let, however, the point source be replaced by the sun, a spherical object at a great distance, but so large that its disk subtends an angle of about a half a degree. The rays of light passing through any point, say *A*, on the shadow-casting object, will make up a cone with vertex at *A*, the elements passing through the periphery of the solar disk. The intersection of the cone with the plane of the paper will be a circle with center at *A* as shown, its radius being proportional to the distance the object *ABC* is above the paper. Now

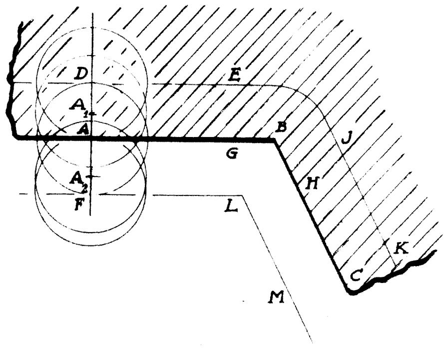


Figure C1

the shadow cast by *ABC* will no longer be sharply defined. The point on the paper immediately below *A*, for instance, will not be completely dark, but it will receive only half the illumination received by, say, *F*, because it is exposed to only half of the solar disk. The point on the paper below *A*, will receive even less than *A*, an amount proportional to the part of the disk which peers, as it were, around *AB*. It is the area of the circular segment having center at A_1. In like manner, A_2 will be more than half illuminated; *F* will have full lighting, and *D* no direct light at all.

Thus the curves of equal illumination on the paper, so long as they are not in the vicinity of *B*, will be families of lines, one parallel to *AB* and the other to *BC*.

The situation is altered in the vicinity of the corner, *B*. The curve of half illumination will not pass through the point on the paper below *B*, for that point is illuminated by the sector *BGH*, which is less than a semicircle. Thus the curves of equal illumination will bend smoothly about the corner, making up a single family enveloped by *DEJK* and *FLM*.

This is the explanation of the smoothing out of the shadow cast by an object with serrated edges (13:2).

Abū Yūsuf b. Isḥāq al-Kindī (d.c. 870), known as "the philosopher of the Arabs", wrote on numerous subjects. His work on optics was influential in medieval Europe, but we are unable to spot the individual passages in his writings referred to here (12:10) and later. (Cf. *Lindberg*, 2, also *GAL*, vol.1, p.209; suppl. vol.1, p.372).

13. Circular Image from a Small Aperture (13:8-17)

Let the sun shine through a minute hole in some opaque substance onto a plane surface normal to the central ray. Then, roughly speaking, from each point on the solar disk one ray only will pass through the hole, illuminating a corresponding small spot on the screen. The totality of such dots will make up a lighted circle, an (inverted) image of the solar disk. This is the phenomenon of the pinhole camera, evidently unknown to Bīrūnī. The argument above says nothing about the shape of the hole, only that it be small. Without being able to follow Bīrūnī's train of thought, we feel that it utilizes two facts: (1) that the sun is not a point source, and (2) that the aperture is of appreciable size. His realization of these is clear from his Figure 1 used in criticizing Thābit. Refer now to it, and regard *A* and *B* as endpoints of a diameter of the solar disk. If the aperture *DG* is a polygon, the light from *A* will illuminate a similar polygon on the screen bounded by *K* and *S*. Light from *B* will define a second polygon on the screen, *LM*, similar to *KS* (and to *GD*), and similarly placed. The composite of the two images will make up a sort of polygon of double the sides of *GD*. The same reasoning applies to any one of the infinite number of diameters of the solar disk, whence it follows that the illuminated region is circular. The successive doubling at 13:14 is not needed for the demonstration, and we do not see how it is supposed to arise. It is instructive to compare this passage with approaches to the same problem made by students of optics in medieval Europe (*Lindberg*, *1*).

14. Distortion of a Gnomon's Shadow (13:18 - 14:5)

We regard Figure 2 as a schematic illustration of a special case of the type of effect discussed in Section 12 above. Let *THMK* be the shadow which would be cast by a certain gnomon if the sun were a point source, and if no other shadow intervened. In fact, however, half of the field *AEZD* is in shade, namely *ABGD*. Then, by virtue of the blurring of edges and rounding of angles explained above, the angles between *TH* and *BL*, and between *KM* and *GS* will be filled in somewhat as indicated in the figure. Moreover, if the gnomon is sufficiently narrow, the shadow of its top will disappear altogether.

15. Images from Nearby Pinholes (14:6 - 15:5)

Consider a pinhole camera of the ordinary type, except that it has two small holes instead of the customary one. Then in sunlight each will produce an image of the sun. If the apertures are sufficiently close and the intercepting screen is sufficiently far from the holes, the two circular images will intersect, and the part in common between them, *AESGOZ* on Figure 3 will be twice as brightly illuminated as the remainder of the images.

Now introduce a long rectangular object between the sun and the holes, outside the camera, so that its axis is along the line of centers of holes. Assuming proper dimensions and distances so that the long object does not cut off the sunlight from the holes completely, there will then appear on the screen two joined images of the object, *THMK*. In the figure this terminates inside the circles; in our experiments object and image continued on beyond. What is striking, however, is that the shadow between *ZE* and *OS* is invariably attenuated, and for properly chosen dimensions and distances it is completely annulled in this region.

Abū Rayḥān notes this curious effect, but he attempts no explanation. We submit that the reasons for it are somewhat as follows. In the theoretical case where a single line penetrates each hole for each point on the solar disk the images would be perfectly sharp, and the region *ZESO* would be just as dark as

TZOK and *EHMS*. But this would imply apertures of zero diameter, whence no light at all would enter. For an actual device the holes must have a finite diameter. This being the case, a point such as P_1, on *OS* or *ZE*, will not be completely dark, for the images are formed by finite pencils of light which penetrate partially into the shadowed region. The same is true of such points as P_2, but the effect is half of that of P_1 because it is affected by light from only one hole, whereas P_1 adjoins a region of double illumination. Hence the shadow between *OS* and *ZE* is attenuated.

16. Double Shadows (15:6 - 16:6)

The little that is known about the philosopher and scientist al-Irānshahrī (fl. 870) has been collected in the *Taḥdīd*, *Comm.*, p.5. See also *Pines*, p.346.

The situation he reported is illustrated in Figure 4, slightly modified from the corresponding figure in the text. The man at *TG* has his shadow cast on the cliff behind him, the upper extremity of this shadow being at *Z*, the point at which the ray from the sun across the top of his head strikes the mountain. But, because of the mirror effect of the river, there is a virtual sun, say *S'*, located along *KE* produced in the direction of *E*, a distance equal to *ES*. This *S'* also casts a shadow of the man, its extremity being *K*, above *Z*.

17. Shadow as a Flow (16:7 - 18:7)

After harking back briefly to the polygons (16:7 and 13:12) with which he attempted to explain the circular image produced by a pinhole camera, Bīrūnī describes the following simple experiment. The subject places himself near a wall in bright sunlight so that the shadow of his face appears in profile on the wall. Then let him extend one hand, and let the forefinger slowly approach and touch the nose. When the two are in proximity the corresponding shadows seem to flow out to each other and merge before the actual objects touch (17:3).

What happens is that the blurred shadow edges reinforce each other as they merge, producing the seemingly plastic effect. This apparently reminds the author of a notion which he attributes to Plato (17:5), that shadows are liquids which freeze. (We find no reference resembling this in the Timaeus or in Proclus' commentary thereon. D.P.). Be that as it may, Bīrūnī ridicules the idea by citing superstitions and by referring to his own experiments with freezing (s56:17).

The ʿAbdullāh b. Muḥammad al-Nāshī al-Akbar mentioned in 17:16 was a well-known poet and theologian of the Muʿtazilite persuasion. He lived in Baghdad; moved to Cairo, and died there, c.905 (Ibn *Khallikān*, vol.2, p.57).

18. A Sphere-topped Gnomon (18:8 - 19:5)

The blurring at the edges of shadows cast in sunlight makes it difficult to observe the shadow of the end point of a gnomon. This is particularly the case if the gnomon is thin and has a sharp point. This in turn hampers precision in accurately measuring the lengths of gnomon shadows, a matter of great practical importance. To lessen the difficulty Bīrūnī suggests that a small spherical object be mounted at the end of the gnomon, and by implication that the gnomon height be reckoned from the base to the center of the sphere, the shadow lengths likewise. He does not claim that the idea originated with him. In fact the same thing was done in antiquity, in particular with an Egyptian obelisk transported to Rome and utilized as a gnomon by Augustus (*Sayili*, p.346; *Pliny*, bk. 36, 15).

19. Solar Parallax (19:6 - 20:14)

Figure 5, in both the text and the MS, has suffered considerable distortion at the hands of successive copyists. The version presented in the translation has been satisfactorily restored, except that in the text two distinct points have been marked with the letter *sīn*, now transcribed as *S* and *S'*. Since both are referred to in the text, the error may well go back to Bīrūnī himself. There seems to be no need for the point marked *H*, and the only time it is

used (at 23:5) it is used wrongly. But it is not very misleading.

The general idea is that celestial positions are computed as though the observer were at the center of the earth (see ***Parallax***). Since he is in fact on the earth's surface, the object's "apparent" position will be somewhat lower than its "true" altitude. The difference, $\angle TBE = \angle KEY$ on Figure 5, is called parallax. There will be a corresponding difference in the gnomon's shadow length, *YK*.

In the case of the sun, its distance is so immensely greater than the earth's radius that for many purposes, certainly measurements taken with gnomon shadows, the parallax can be neglected. Bīrūnī says the ratio between the earth's radius and the sun's distance is less than (20:3)

$$\frac{1}{2}\cdot\frac{1}{10}\cdot\frac{1}{6}\cdot\frac{1}{10}=\frac{1}{1200}=0;0,3.$$

which says that *ET* is about 0;3 if *EB* is taken as 1,0;0 = 60. This is consonant with Ptolemy's calculation of 1210 earth radii for the earth-sun distance (*Almagest* 5, 15).

Note that *BS* and *BD* can both be regarded as sines as stated at 20:5, but the radius of the defining circle differs, being *BT* for the first and *BE* for the second.

20. Lunar Parallax (20:15 - 22:2)

There was good practical reason for showing the situation of the moon on the same drawing as the sun, because the main application of parallax theory was for eclipse computations where, for a solar eclipse, the body of the moon interposes between the sun and the observer. When the moon is at *M* (in Figure 5) it has the same true altitude as the sun. But it is only when it is at *C*, or near it, having the same apparent altitude as the sun, that an eclipse can take place. At either position the lunar parallax exceeds that of the sun. Bīrūnī fully accepts Ptolemy's erroneous lunar model whereby the moon approaches the earth as close as 33:33 earth-radii (*Almagest* 5, 17). This is the origin of the text's 1/30 = 0;2 (21:13), which implies that lunar parallax is indeed appreciable.

21. Al-Kindī on Parallax (22:3 - 23:8)

Noted are two remarks by al-Kindī (concerning whom, see Section 12 above). They are both true, but of trivial consequence. The first (22:8) uses the fact that the solar orbit about the earth is eccentric. Hence, say, when the sun is in apogee, thus farthest from the earth, its parallax is less than when it is closest, in perigee. But the sun is at all times so far away that minor variations in the distance are to be disregarded. Abū Rayḥān cannot refrain from pointing out that if al-Kindī is going to be so fine about it he should also state that the apogee and perigee themselves are in motion.

The second observation (22:19) is that since the sun has an appreciable apparent diameter its light will reach farther around the gnomon tip, as it were, than if it were a point source located at the same position. Hence its shadow will be shorter.

CHAPTER 6

DEFINITIONS OF THE SHADOW FUNCTIONS

22. Quotations Concerning Shadows (23:11 - 25:11)

This short chapter resembles Chapter 3 in that it is padded out with largely irrelevant literary material on shadows.

The author of the long quotation at the beginning, Abū Zayd Aḥmad b. Sahl al-Balkhī (d. 934), a sometime student of al-Kindī, was best known as a geographer. He was born near Balkh, a great and ancient city, the site of which is in northern Afghanistan (*GAL*, vol.1, p.229; *EI*, vol.1, p.624).

Abū ʿUthmān ʿAmr b. Baḥr al-Jāḥiẓ (fl. 850), quoted at 24:19, was a famous prose-writer of Baṣra at the head of the Persian Gulf (*EI*, vol.1, pp.1000-1).

23. Definition of the Cotangent and Tangent Functions (25:12 - 28:18)

The author now defines two standard varieties of shadows cast on planes, introducing incidentally (at 25:19) two technical terms for the gnomon. There is no loss of generality in calling the plane a horizon, since any plane whatever is parallel to some plane tangent to the earth, hence a horizon.

When the gnomon is normal to the horizon the shadow it casts is called *al-ẓill al-mustawī*, the direct shadow, *GE* in Figure 6. The same term is used interchangeably for the number expressing the length of *GE* in units determined by the length of the gnomon. It depends upon the altitude, angle *E*, and in this sense is the trigonometric function called in English the *cotangent.* Nowadays the parameter, the length of the gnomon, is invariably taken as unity. As will be seen below, this was never the case with the medieval function.

In like fashion, the length *BE* measured in the same units is the cosecant function, in Arabic *quṭr al-ẓill (al-mustawī)*, the hypotenuse of the (direct) shadow.

24. Definition of the Tangent and Secant Functions (21:1 - 28:18)

The second kind of shadow is that cast on a vertical plane by a gnomon perpendicular to it, the gnomon lying in the vertical plane which contains the celestial source of light. It is *al-ẓill al-maᶜkūs* (the reversed shadow), *EG* in Figure 7. Thought of as a number, its length measured in the same units as the gnomon *BG*, it also is a function of the altitude, now angle *B*. It is the *tangent* function. The length of *BE* in the same units is *quṭr al-ẓill al-maᶜkūs* (hypotenuse of the reversed shadow) the *secant* function.

Al-Ḥallāj, whose statement in 27:9 Bīrūnī criticizes, was a famous mystic (*ṣūfī*) who was executed for heresy in 922. The "red sulfur" in the name of his book was supposed to be a very rare variety of sulfur, probably mythical, which played a role in alchemical theory. (See *EIne*, vol. 3, pp.99-104; *EI*, vol.2, p.989.)

The chapter closes with a reminder to the reader (28:8) of the results obtained at the end of the preceding chapter, that solar parallax may be neglected insofar as shadows are concerned, but that in the case of the moon and other nearby objects it is considerable.

CHAPTER 7

LENGTHS OF GNOMONS

25. Gnomon Units - Introduction (29:2 - 32:10)

In this chapter also the prosaic subject matter is interlarded with semi-philosophical and literary material. The author asserts that the existence of a shadow in a transparent medium is to be perceived only by interposing in it an object upon which the light and shade may fall. As an example he adduces the entry of the moon into the earth's shadow to produce a lunar eclipse (29:9). The resulting circular shadow is used as evidence for the sphericity of the earth. It is curious that here (29:15) he gives the relative smallness of mountains as a reason for their non-appearance in the shadow, whereas in 13:2 he has a different explanation.

Next it is stated that the angle between the horizontal and the line from the end of the shadow to the top of the gnomon equals the altitude of the light source (29:18). When the source is on the horizon the shadow is infinite; as it rises the shadow decreases in length, disappearing under the gnomon tip if the light source reaches the zenith.

This phenomenon of the disappearing noon shadow provides the occasion for the display of four poetic excerpts in which the same idea is conveyed by various figures of speech. In the first (30:13), perhaps the poet sees the animal and rider as the leg of a sock, the shadow constituting its foot. In the second (30:15) the advancing animal treads upon (the shadow of) its own neck. In the third (at 31:1) the familiar theme of prostration recurs (cf. Section 9 and s62:18).

In the fourth (at 31:3) the text is garbled. It has been restored by Professor Iḥsān ʿAbbās, who finds the correct version in the *dīwān* composed by the Abbasid poet and caliph, Ibn al-Muᶜtazz.

Concerning Abū al-Najm (30:16), see Section 4 above.

These remarks are based on shadows cast by a vertical gnomon. Bīrūnī next points out (31:7) that the situation is reversed for the "reversed shadow" defined in Chapter 6 (hence probably the name). When the gnomon is horizontal and pointed along the azimuth of the source, zero altitude of the latter corresponds to a zero shadow length. As the altitude of the source increases, the shadow length also increases, becoming infinite if the source reaches the zenith.

Precise relations between altitudes, zenith distances, shadows, and the other trigonometric functions (31:12) are developed in the chapters which follow. Preliminary to these are definitions of the standard units into which the gnomon length g is subdivided for measuring shadow (32:1), the subject of the present chapter. The units are

1. Sexagesimal parts, used by the Occidentals and the "moderns". $g = 60 = 1,0$,
2. Digits, used by the Orientals. $g = 12$,
3. Feet, used by the Muslims. $g = 7$, or $6\frac{1}{2}$

26. The Sixty Part Gnomon (32:11 - 33:8)

It is true that sexagesimal subdivisions were taken over by Islamic astronomers from Ptolemy and other Hellenistic scientists. It is also correct that the use of the shadow functions can simplify many trigonometric computations. But, of course the sine function was unknown to Ptolemy, who used a table of chords.

The basic reason for using this unit, however, is that sexagesimal (rather than decimal) place-value computation was standard for astronomical work. Then,

the length of the gnomon being 60 = 1,0, division by it involved only, as we would say, moving the sexagesimal point one (sexagesimal) place to the left, the individual digits being unchanged. Multiplication by the gnomon length was carried out by a one place shift in the other direction, or up (*rafᶜ*, "elevating", 33:3) in the medieval parlance. Division by sixty was called *ḥaṭṭ*, depressing.

Bīrūnī and his great contemporary Abū al-Wafā' (Section 34) took the intelligent ultimate step of putting $R = 1$ (33:6), thus eliminating even the trivial operations described above and making their trigonometric functions identical with those used today.

27. The Twelve Digit Gnomon (33:9 - 35:15)

This type of subdivision was indeed used by the Indians. Their application of it to fix zones of terrestrial latitude (33:10) is described in Chapter 22 of our text.

The Sanskrit word for a digit (of a gnomon) is *aṅgula* (cf. *Pañca*. IV, 48; pt. 1, pp.67-68), which means also finger, as does digit in Latin and English. A minute (sixtieth) of a digit would be *vyaṅgula*. Bīrūnī gives (in 33:13-14) the Sanskrit words transliterated into Arabic characters. The two renderings of *aṅgula* differ by a letter because standard Arabic has no character to render the value of g in the original. His own transliteration uses a *kāf*. The second he has obtained from the Arabic Arkand where the translator apparently chose to use a *jīm*.

The word *arkand* is apparently the Arabic transliteration of Sanskrit *ahargaṇa*, meaning the number of days which have elapsed since epoch. The word was attached to more than one Sanskrit astronomical document after it was translated into Arabic. The one Bīrūnī refers to was based on the *Khaṇḍ*, and was translated in 735 (see *al-Hāshimī*, comm. Section 4).

The reason given for the twelvefold division is that both the (hand-)span and the digit (finger breadth) are units always available, and the larger contains twelve of the smaller. If the gnomon used by a particular person equals his own span, it is convenient to measure its shadow with both hands.

The variant method Bīrūnī describes (34:8) was even more crude. It assumes that the bent middle finger projects three digits beyond the palm. The latter being held horizontal, the finger's shadow is measured with the digits of the other hand. If the shadow length is S, then $4S = 12S/3$ will be the shadow cast by a twelve digit gnomon.

It is true that eclipse magnitudes were measured in digits, twelfths of the diameter (or area) of the eclipsed disk (*Almagest* VI,7). However, to claim (in 35:5) that this is because the lunar or solar disk subtends a hand-span seems a farfetched explanation. At an arm's length the hand subtends far more than a half a degree, which is the order of magnitude involved.

At 35:8 is another reference, to the Arkand Zīj, also in connection with eclipse digits, 1 *māshah* = 4 *kākī*. Bīrūnī knows the *māshah* as a weight: 1 *tūlah* = 3 gold dirhems = 12 *māshah*, 1 *māshah* = 4 *wandī* = 16 *jawa*.

(1 *māṣa* = 4 *kākiṇī*, but these terms apply only to weights. Usually it is stated that 1 *tulā* = 100 *pala* = 64,000 *māṣa*. However, some authorities mention a smaller *tulā* (sometimes called a *talaka*) which is equal to $\frac{1}{4}$ or $\frac{1}{8}$ of a *pala*. This is close, but not equivalent to Bīrūnī's *tūlah*. I cannot explain *wandī* from Sanskrit. *Jawa* (barley corn) is a unit, but of length. D.P.)

28. The Seven Foot Gnomon (35:16 - 37:11)

The rationale for the use of this unit resembles that for the digit – the observer uses himself as a gnomon, and steps off the length of his own shadow with his own feet as a measure. It is assumed that each individual's height is seven foot-lengths.

The flavor of the quaint anecdote (36:19) about the bigotted muezzin may be enhanced, for non-Muslim readers, by some exegesis. The Muslim calendar is strictly lunar, hence its months move forward with respect to the solar year, slipping around through all the seasons in turn. This was insisted on by the Prophet in order to make it distinct from the calendars of other faiths. In particular, he strictly forbade

the intercalation of a month from time to time (as in the Jewish calendar) which would serve to stabilize the months with respect to the seasons. The Christian calendar, on the other hand, is based on the solar year, so that the same month always falls in the same season. Now the Muslim times of prayer, as will be seen in Chapter 25, vary with the seasons. Hence on an instrument for finding the daily times of prayer throughout the year, a scale graduated with the Christian month names is useful, and the Muslim months cannot be substituted for them.

29. The Six-and-a-half Foot Gnomon (37:12 - 38:10)

At a time when the solar altitude is 45° and a man is facing his shadow, its endpoint will not be the shadow of the top of his head, but of a point on his forehead where the sun's ray is tangent to it. The vertical dropped from this point is supposed to bisect his foot, so half a foot is to be taken from his seven foot shadow, counting from the heel, and $7 - \frac{1}{2} = 6\frac{1}{2}$. It is not much of an argument, and Bīrūnī does not seem to take it very seriously. But it gives him an excuse for the excursus which follows.

30. Peoples Who Alter their Head-shapes (38:11 - 39:-2)

The Khwarazmian practise of head flattening is remarked by other sources (*Thaᶜālabī*, p.143), reportedly to disfigure children so that they would not be taken as slaves. The Chinese pilgrim Hsüa Tsiang (603-668) reports the custom at Kucha in Sinkiang province, far to the east of Khwārazm (*Beal*, p.89).

(Bīrūnī refers to Hippocrates' On Airs, Waters, and Places (περὶ ἀερων ὑδάτων τόπων) 14, wherein are discussed the peculiarities of the Macrocephali (Long-head ones, not Wide-headed) who lived near the Sea of Azov. Hippocrates does say that the length of their heads was a mark of nobility, but neither "taking pride in bravery" nor "and it appears to him an impossibility" are justified by the Greek. Hippocrates does also state that long-headedness becomes an inherited characteristic, but I do not find Galen's criticism in the edition of his Greek works. D.P.)

(On the use of a man as gnomon, the earliest Sanskrit word for the shadow of a gnomon was *paurușī chāyā*, or man-shadow; see Kauṭilya, Arthaśāstra 2, 20, 38; Sūryaprajñapti 9, 22; and Śārdūlakarṇāvadāna, pp.54-55 Mukhopadhyaya. The term was in use, then, from the fourth century B.C. in India, and the practice may have existed contemporaneously with the use of the actual gnomon. D.P.)

31. The Usages of Various Scientists (39:13 - 40:18)

This passage is useful to the extent that it supplies tidbits of information about the contents of a number of documents which no longer exist.

Abū Maᶜshar (39:13), Jaᶜfar b. Muḥammad al-Balkhī (d.886) was the famous astrologer known in the Latin West as Albumasar, a man upon whom Abū Rayḥān delighted in pouring out his sarcasm. Abū Maᶜshar's zīj is not extant (See *DSB*, vol.1, pp.32-39).

Abū al-ᶜAbbās, al-Faḍl b. Ḥātim al-Nayrīzī (39:15) wrote extensively in astronomy, but neither of his zījes is extant. He died c.922 (see *Suter*, p.45).

Muḥammad b. ᶜAbd al-ᶜAzīz al-Hāshimī (39:16, fl. 930) wrote, among other things, a zīj called al-Kāmil which is not extant (see *Suter*, p.79; *Taḥdīd*, comm., p.125).

Al-Ḥasan b. al-Sabbāḥ (39:18) was one of three brothers who were astronomers. His Mukhtariᶜ Zīj is named nowhere else in the literature, to our knowledge (*Suter*, p.19; *Taḥdīd*, comm., p.82).

By the Bāṭini in (40:2) Bīrūnī probably meant the adherents of the Ismaᶜīliya sect, Shīᶜa Muslims who hold that the imamate terminates with Ismaᶜīl. They maintain that there is an inner (*bāṭin*) significance which transcends the literal meaning of sacred writings. (*EIne*, vol.1, p.1098).

The treatises of the Ikhwān al-Ṣafā' (40:8) are early Ismaᶜīlī tracts (*EIne*, vol.3, p.1071).

In Bīrūnī *Canon* (vol.1, pp.325, 343) both the sine and the tangent functions are tabulated with R=1, thus verifying his statements in 40:18 and 33:6.

32. Parameters for the Sine Function (40:19 - 41:10)

$R = 2\frac{1}{2} = 150'$ (40:19) is indeed the value used by Brahmagupta in *Khaṇḍ.* 1, 30.

$R = 54\frac{1}{2} = 54;30 = 3270'$ (41:1) is Brahmagupta's in the Brahmasphuṭasiddhānta (*Brahmasp.* in the bibliography), which Bīrūnī calls the Brahmasiddhānta. The restoration of the word for four, missing in the text, seems justified, since in at least one other place (*K & M*, p.119) he gives the same peculiar parameter correctly.

$R = 57 + 1/5 + 1/10 = 57;18 = 3438'$ (41:2) is used by Āryabhaṭa in *Āryabhaṭīya* 1, 10. Concerning the mass of material connected with the words Pulisa, Paulus, and Pauliśasiddhānta, see *Pauliśasidd.* It suffices here to state that Bīrūnī's attempt to connect this Pulisa (or Pauliśa) with Alexandria was erroneous, as were later identifications of Pulisa with the astrologer Paulus Alexandrinus.

The two other Indian astronomers mentioned in this passage and later are:

Brahmagupta (b.598), son of Jiṣṇugupta, lived in Rajasthan. The two books named above are his only extant works (see *DSB*, vol.2, p.416).

Āryabhaṭa I (b. 476) worked in the city now called Patna. He also wrote two books, but only the *Āryabhaṭīya* is extant (see *DSB*, vol.1, p.308).

CHAPTER 8

CHANGES OF UNITS

33. Transformation of Shadow Lengths - The General Rule (41:12 - 42:14)

In the preceding chapter four standard units for subdividing gnomons are described. They are named below, together with the number of each contained in the gnomon:

1.	(sexagesimal) parts	60
2.	digits	12
3.	feet	7
4.	fractional feet	$6\frac{1}{2}$

The subject matter of this chapter consists of a set of rules whereby if a shadow length expressed in one of these units is given, it may be transformed into its equivalent in any one of the other units. Let S denote a shadow length and u the unit, while the subscripts g and d stand for "given" and "desired" respectively. Then the relation given verbally at 42:9 may be expressed symbolically as

$$S_g/u_g = S_d/u_d \quad , \quad \text{whence}$$

(42:11) $$S_d = S_g \cdot u_d/u_d \ .$$

That is all there is to it, but the author is not satisfied with this general approach. The length may be given in any one of four ways. It may then be transformed into any one of three other units. Hence there are 4 × 3 = 12 special cases, and he systematically gives a rule and a worked example, frequently with variants, for each of the twelve.

34. From Sexagesimal Parts into Other Units (42:15 - 45:1)

For the first of the twelve cases the rule becomes

(43:12) $$S_{12} = S_{60}\cdot\left(\frac{12}{60}\right) = S_{60}\cdot\left(\frac{1}{5}\right) = S \cdot 0;12,$$

where now the numerical subscripts indicate the units used.

The worked example puts $S_{60} = 10$, whereupon

(43:5) $$S_{12} = 10 \cdot \frac{12}{60} = \frac{120}{60} = 2 \text{ digits}$$

(43:10) $$= 10 \cdot \frac{1}{5} = 2 \text{ digits}$$

Several manuscript versions of zījes are extant written by Kūshyār b. Labbān al-Jīlī (fl. 1000 in Baghdad, see ***DSB***, vol.7, pp.531-533, ***Survey***, pp.125, 156-7). His version of the rule is

(42:15) $$S_d = S_{60}\cdot u_d \div 1{,}0,$$

where division by 1,0 = 60 is carried out by "depressing" the result one sexagesimal place (see Section 26 above).

Abū al-Wafā' al-Buzjānī, (fl. 980, see ***Suter***, p.71, ***Carra***, and the ***Survey***, p.134) was a very able scientist who, like Kushyār, worked in Baghdad. Since he took the gnomon length as unity, for him the transformation is simply

(43:1) $$S_d = S_1 \cdot u_d \ .$$

For the second case the rule is

(44:1) $$S_7 = S_{60} \cdot \left(\frac{7}{60}\right),$$

the worked example being

$$10 \cdot \frac{7}{60} = \frac{70}{60} = 1\frac{1}{6} \text{ feet.}$$

The third case gives

(44:9) $$S_{6\frac{1}{2}} = S_{60} \cdot \frac{6\frac{1}{2}}{60} = S_{60} \cdot \frac{13}{120} = S_{60} \cdot \frac{6;30}{1,0;0} \quad ,$$

(44:9) $= S_{60} \cdot \frac{390'}{3600'}$,

the worked example resulting in

(44:19) $10 \cdot \frac{390'}{3600'} = 1;5$ fractional feet.

Note that the sentence at 43:1 is the prime authority for ascribing to Abū al-Wafā' a zīj distinct from his Almagest.

35. From Digits into Other Units (45:2 - 47:14)

To convert from digits into sexagesimal parts the rule is

(45:5) $S_{60} = S_{12} \cdot \frac{60}{12} = 5\ S_{12}$.

For a two digit shadow the conversion is 5×2 = 10 parts.

To convert into feet, use

(45:15) $S_7 = S_{12} \cdot \frac{7}{12} = S_{12} \cdot (\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{6})$,

or

(46:4) $= (\frac{S_{12}}{2}) \cdot 7 \cdot (0;10)$,

or

(46:11) $= (S_{12} \cdot 5 \cdot 7)/(5 \cdot 12) = S_{12} \cdot (0;35)$,

the first line being a reversion to the ancient unit fractions. The two digit shadow becomes $2 \times 7/12 = 1\frac{1}{6} =$ 1;10 feet.

For the other kind of feet,

(46:18) $S_{6\frac{1}{2}} = S_{12} \cdot \frac{6\frac{1}{2}}{12} = S_{12} \cdot \frac{13}{24}$,

or

(47:5) $= S_{12} \cdot \frac{1}{2} (1 + \frac{1}{2} \cdot \frac{1}{6})$.

The two digit shadow of the example becomes $1 + \frac{1}{2} \cdot \frac{1}{6}$ fractional feet.

At 47:10 the author reverts to the blunder of Abū Maʿshar already cited in 39:13. In the explanatory

text of Abū Maʿshar's zīj he gives the rule of 46:18. But in his tables $S(45^\circ) = 6\frac{2}{5}$ (=g), which betrays the difference of parameters.

36. From Feet into the Other Units (47:15 - 49:6)

The conversion from (sevenfold) feet to parts is

(47:15) $S_{60} = S_7 \cdot \frac{60}{7}$.

For the conversion to digits,

(47:17) $S_{12} = S_7 \cdot \frac{12}{7}$,

which is correct, but the author then says

(47:18) $S_{12} = \frac{S_7}{2} \cdot \frac{7}{6} = \frac{S_7}{2} \cdot 7 \cdot (0;10)$,

which is wrong. It is the inverse operation, already given in 46:4. But the next variant is all right:

(48:2) $S_{12} = 2S_7 \cdot \frac{6}{7} = S_7 \cdot 2(1 - \frac{1}{7}) = S_7 \cdot \frac{60}{35}$

$= S_7 \cdot (\frac{1}{0;35})$,

the last expression being, as he says, valid but useless.

Now, starting from fractional feet,

(48:14) $S_{60} = S_{6\frac{1}{2}} \cdot (\frac{120}{13})$,

(48:19) $S_{12} = S_{6\frac{1}{2}} \cdot (\frac{24}{13})$,

and

$S_7 = S_{6\frac{1}{2}} \cdot (\frac{14}{13})$.

CHAPTER 9

ALTITUDE FROM SHADOW, AND CONVERSELY

37. The Identity Connecting Cosecant and Sine (49:9 - 50:2)

The chapter opens by stating the identity equivalent to the modern

$$\csc \theta = 1/\sin \theta \ .$$

We translate here as "cosecant" the Arabic *quṭr al-ẓill*, literally the "hypotenuse of the (direct) shadow", which is what the trigonometric cosecant is.

Bīrūnī gives it as

(49:9) $$\text{gnomon } (=g) \ / \ \mathrm{Csc}_g\, h \ = \mathrm{Sin}_R\ h \ / \ R,$$

which is easily proved by using Figure 8. On it,

(49:14) $$EL = g,$$

$$HT = \mathrm{Sin}_R\, h,$$

$$ET = \mathrm{Cos}_R\, h,$$

$$LK = \mathrm{Cot}_g\, h,$$

and $$KE = \mathrm{Csc}_g\, h.$$

By using pairs of sides of the similar triangles *HTE* and *ELK*,

(50:1) $$EL\,/\,KE = HT\,/\,EH,$$

which is equivalent to 49:9 above.

(This rule is given by Brahmagupta in *Khand.* 3, 13(9), and in *Brahmasp.* 3, 29; he probably derived it - as many other things - from the *Mahāb.* (3,5) of the first Bhāskara. D.P.)

38. Altitude from Shadow Length (50:3 - 51:8)

Next is a rule whereby, given $s = \mathrm{Cot}_g\, h$, the altitude h may be determined. First calculate

(50:4) $$\sqrt{\mathrm{Cot}^2 h + g^2} = \mathrm{Csc}_g\, h,$$

then

$$g \cdot R \ / \ \mathrm{Csc}_g\, h \ = \mathrm{Sin}_R\, h,$$

which is equivalent to 49:9. Now find h by operating inversely in a sine table. The author tacitly assumes that no tangent or cotangent table is at hand, although he himself gives one in Chapter 12.

In applying this rule the value of the numerator $g \cdot R$ will, of course, depend upon the units being used. Bīrūnī displays the possibilities:

For Ptolemy

(50:12) $$g \cdot R = 60 \times 60 = 3600.$$

If the sexagesimal gnomon is used with the Indian R it will be

$$g \cdot R = 60 \times 2\tfrac{1}{2} = 60 \times 150' = 150.$$

With the seven-foot gnomon and Ptolemy's R it is

(51:2) $$g \cdot R = 7 \times 60 = 420.$$

The same gnomon and the Indian R give

$$g \cdot R = 7 \times 2\tfrac{1}{2} = 17\tfrac{1}{2} = 35/2.$$

With the fractional gnomon and the sexagesimal R, it is

(51:4) $$g \cdot R = 6\tfrac{1}{2} \times 60 = 390.$$

The same gnomon with the Indian R is

(51:5) $$g \cdot R = 6\tfrac{1}{2} \times 2\tfrac{1}{2} = 16\tfrac{1}{4} = 65/4.$$

Use of the Indian g for the gnomon with the sexagesimal R gives

(51:7) $$g \cdot R = 2\tfrac{1}{2} \times 60 = 150.$$

Finally, the Indian R for both yields

(51:8) $$g \cdot R = 2\tfrac{1}{2} \times 2\tfrac{1}{2} = 6\tfrac{1}{4} = \frac{25}{4}.$$

39. The Authors of Various Zījes (51:9-13)

Four of the authors or documents mentioned in this passage have appeared previously in the text. They are listed below, each name followed by the number of the commentary section in which they are identified:

Al-Nayrīzī, 31
Abū Maᶜshar, 31
Abū al-Wafā', 34
Kūshyār, 34

The other six are mentioned for the first time. They are:

Muḥammad b. Ibrāhīm al-Fazārī (d.c. 770) an astronomer who worked for the Abbasid dynasty, his writings exhibiting strong Indian influence. No zīj written by him is extant, but fragments of his work have been assembled in ***Fazārī***. See also ***Suter***, pp.3-4, and ***Taḥdīd***, comm., p.91.

Yaᶜqūb b. Ṭāriq (fl. 760) is frequently associated with al-Fazārī, and indeed the remarks made about the latter apply to him also. See ***Yaᶜqūb***, and ***Suter***, p.4.

Al-Khwārizmī (fl. 820) was the author of the famous arithmetic, algebra, zīj, and geography, all of which have survived in one form or another. See ***al-Khwārizmī***, ***biogr.***, and ***zīj***.

Ḥabash al-Ḥāsib (fl. 840) carried out extensive astronomical observations and wrote zījes. Two such sets

of tables are extant, numbers 15 and 16 in the ***Survey***. See also ***Ḥabash***, ***biogr.***

Al-Battānī (d. 929), an able astronomer, was of Sabian origin. His zīj is extant and has been published (***al-Battānī***, ***zīj***, see also ***al-Battānī***, ***biogr.***).

Al-Zīj al-Shāh (51:19) is the Arabic designation for any one of a series of astronomical canons compiled at the behest of the Sasanian dynasty of Iran. Written in Middle Persian, these documents in their original form have disappeared. Reconstructing their contents, and the relations between them is a matter of piecing together the numerous references scattered through the later literature. (See ***Shāh***, also ***Hāshimī*** and ***Māshā'allāh***.)

40. Their Application of the Rule (51:14 - 53:5)

All these zījes use the rule of 50:4 above to convert from S to $\mathrm{Csc}_g\, h$, there being four possibilities for g^2, depending upon the gnomon units:

	g	g^2
	12	144
(51:18)	$6\frac{1}{2}$	$42\frac{1}{4}$
	7	49
	60	3600

Note that for the Shāh Zīj, g = 12 (51:19).

(In this connection, see also the ***Brahmasp.*** 3, 32 and the ***Khaṇḍ.*** 3, 14(10); cf. ***Aryabhaṭīya***, Gaṇitapāda 14. For the method of 52:3-5 see ***Mahāb.*** 3,5. D.P.)

Instead of using 50:5 to obtain h, al-Khwārizmī, al-Nayrīzī, Kūshyār (in the Jāmiᶜ Zīj), and al-Battānī convert the cosecant into a cosine by the relation

(52:4) $$(\mathrm{Cot}_g\, h) \cdot R \,/\, \mathrm{Csc}_g\, h = \mathrm{Cos}_R\, h,$$

which Bīrūnī proves by invoking from the similar triangles of Figure 8 the proportion

(52:6) $$LK(=S)\,/\,KE(=\mathrm{Csc}_g\, h) = TE(=\mathrm{Cos}_R\, h)\,/\,EH(=R).$$

Of those who convert into a sine, for al-Fazārī, al-Khwārizmī, Yaᶜqūb, Abū Maᶜshar, and the Shāh, rule

50:5 takes the form

(52:18) $R \cdot g / \mathrm{Csc}_g\, h = 1800'/\mathrm{Csc}_g\, h = 150' \times 12 / \mathrm{Csc}_g\, h$,

so that for all these zījes $R = 150' = 2\frac{1}{2}$, $g = 12$.

In the extant version of the ***Khwārizmī zīj*** (ed., p.169) $R = 60$ for the sine table, demonstrating a departure from the original form.

For Ḥabash and al-Battānī it is

(53:2) $720 / \mathrm{Csc}_g\, h = 60 \times 12 / \mathrm{Csc}_g\, h$,

so $R = 60$ and $g = 12$, which is easily verified for ***Battānī*** (*zīj*, vol.2, pp.56, 60). However, in both of the extant versions of ***Ḥabash***, $R = g = 60$.

41. The Inverse — Shadow from Altitude (53:6 - 55:11)

To obtain a rule for calculating S in terms of h, use the similar triangles in Figure 8 to write

(53:7) $HT(=\mathrm{Sin}_R\, h) / TE(=\mathrm{Cos}_R\, h) = EL\,(=g) / LK(=g = \mathrm{Cot}_g\, h)$.

Solving for the shadow,

(53:9) $g \cdot \mathrm{Cos}_R\, h / \mathrm{Sin}_R\, h = S$.

(See ***Mahāb.*** 3, 55 and ***Brahmasp.*** 3, 28. D.P.)

The only new information concerning zījes which comes out of this passage is that for al-Nayrīzī $R = g = 60$ (53:15). It is confirmed that for Abū al-Wafā' $g = 1 = 60'$ (cf.43:1)

ʿAbd al-Jalīl al-Sijzī was a friend of Bīrūnī's who wrote extensively in mathematics, astronomy, and astrology. His book on the astrolabe is extant (*Suter* p.80; *GAL*, vol.1, p.219).

His method reverses the steps presented in Section 38 above. He uses the proportion given in 50:1 to put

(54:11) $g \cdot R / \mathrm{Sin}_R\, h = \mathrm{Csc}_g\, h$,

and then applies the Pythagorean theorem to triangle ELK in Figure 8 to obtain

(54:15) $\sqrt{\mathrm{Csc}_g^2\, h - g^2} = S$.

A clause seems to have slipped out of the sentence beginning in 54:17. As restored in the translation it makes sense. Al-Fazārī and Ḥabash are said to have used the same rule worked out by al-Sijzī. In the case of the former, expression 54:11 became

(54:18) $1800' / \mathrm{Sin}_R\, h = 12 \times 150' / \mathrm{Sin}_R\, h$,

corresponding to $g = 12$, and $R = 60$. These are the parameters noted above (in Section 40) for these individuals.

In both cases the second step, 54:15, would be

(55:1) $\sqrt{\mathrm{Csc}_g^2\, h - 12^2} = S$.

Since for Abū al-Wafā' $R = g = 1$ we have for him the completely modern expressions

(55:5) $1 / \sin h = \csc h$, and $\sqrt{\csc^2 h - 1} = \cot h$.

Another method is obtained by forming from the familiar triangles of Figure 8 the proportion

(55:9) $EK(=\mathrm{Csc}_g\, h) / KL\ (=S) = EH(=R) / ET(=\mathrm{Cos}_R\, h)$,

which gives $\mathrm{Csc}_g\, h \cdot \mathrm{Cos}_R\, h / R = S$.

42. Alternative Methods - The Calculus of Chords (55:12 - 57:1)

Some anonymous works have the rule

(55:12) $\sqrt{(975 / \mathrm{Sin}_R\, h)^2 - 42\frac{1}{4}} = S$.

This is explained by noting that the radical is

$$\sqrt{(6\tfrac{1}{2} \times 150 / \mathrm{Sin}_R h)^2 - (6\tfrac{1}{2})^2} = \sqrt{(g \cdot R / \mathrm{Sin}_R h) - g^2} ,$$

which makes it clear that the method is an amalgamation of expressions 54:11 and 54:15, where $g = 6\frac{1}{2}$ and $R = 150'$.

A suitable modification would take care of the case $g = 7$ (56:1).

The concluding passage in this chapter is a faithful description of how Ptolemy in *Almagest* 2,5 calculated S in terms of h. His sole "trigonometric" function was a table of the lengths of chords inscribed in a circle of radius $R = 60$, the independent variable being the arc of the chord.

In Figure 8 put $EK = 2R = 120$. Then EL and LK will be chords in the circle with diameter EK. And since the arc intercepted by an inscribed angle is twice the angle, in the units just stated, $EL = \mathrm{Crd}_R\, 2h$ and $LK = \mathrm{Crd}_R\, 2\bar{h}$. Then in units such that the gnomon EL is g,

(56:13) $EL\ (= g) / LK (= S) = \mathrm{Crd}_R 2h / \mathrm{Crd}_R 2\bar{h}$.

So $S = g \cdot \mathrm{Crd}\ 2\bar{h} / \mathrm{Crd}\ 2h$.

Since for all θ, $\mathrm{Crd}\ 2\theta = 2\ \mathrm{Sin}\ \theta$, the last expression above can be written

$$S = g \cdot 2\ \mathrm{Sin}_R \bar{h} / 2\ \mathrm{Sin}_R h = g \cdot \mathrm{Cos}_R h / \mathrm{Sin}_R h,$$

which, as Bīrūnī remarks, has appeared before, at 53:9.

The Abū al-Ḥasan al-Ahwāzī (57:1) may be the same as the Aḥmad al-Ḥusayn al-Ahwāzī al-Kātib credited by Bīrūnī with a book on the sciences of the Greeks (*Chron.*, transl. pp.284, 288; *India* transl., vol.2, p.19; *Suter*, p.57).

CHAPTER 10

FROM TANGENT TO ANGLE AND CONVERSELY

43. From the Tangent into Other Functions (57:4 - 58:8)

Now the author works out relations between the tangent and the other trigonometric functions. From the similar triangles of Figure 9 he puts down the proportions

(57:6) $LK(=S=\mathrm{Tan}_g h) / KE(=\mathrm{Sec}_g h) = HT(=\mathrm{Sin}_R h) / HE(=R)$,

and

(57:7) $HT(=\mathrm{Sin}_R h) / TE(=\mathrm{Cos}_R h) = LK(=S) / LE(=g) = \mathrm{Tan}_g h / g$.

The first problem attacked is, given $\mathrm{Tan}_g h$ find h, again under the tacit assumption that no tangent tables are at hand. The rule is, calculate

(57:9) $\sqrt{\mathrm{Tan}_g^2 h + g^2} = \mathrm{Sec}_g h$,

an application of the Pythagorean theorem, then put

(57:10) $(\mathrm{Tan}_g h) \cdot R / \mathrm{Sec}_g h = \mathrm{Sin}_R h$,

which follows from 57:6 above. Now, presumably, use a sine table to obtain h.

Kushyār's rule (57:13) is equivalent, since to depress (see Section 26) the denominator is equivalent to multiplying the numerator by $R = 60$.

Conversely, given h to find the tangent, calculate

(57:16) $(\mathrm{Sin}_R h) \cdot g / \mathrm{Cos}_R h = \mathrm{Tan}_g h$, an application of expression 57:7.

Kūshyār's variant (57:18) simply puts $g = R = 60$ into the rule.

For Abū al-Wafā', since $R = g = 1$, the same triangles give

(58:3) $HE(=1) / ET(= \cos h) = KE(= \sec h) / LE(= 1)$,

so $1 / \cos h = \sec h$.

Should the cosecant be available (58:7), the cotangent may be found as described in the preceding chapter.

44. When to Use the Direct and Reversed Shadows 58:9 - 59:2)

Bīrūnī suggests that the direct shadow is best used for observing the winter solstice, while the reversed shadow is to be preferred for the summer solstice. His attitude was probably prompted by some such consideration as the following. For his latitudes (at Ghazna $\phi \approx 33\frac{1}{2}°$), the solar noon altitude at the winter and summer solstices would be about 33° and 80° respectively. The first value is rather small, and the second near a quadrant. When h is small, cot h (the direct shadow) varies drastically with small change in the argument, whereas it is much more stable when h is in the vicinity of 90°. On the other hand, tan h (the reversed shadow) varies ever more violently as it approaches 90°, hence small changes in h are more readily detected if it is used for the summer solstice.

Of the two instruments mentioned by the author at 58:10, the *mukḥula* is a portable hand-held sundial with a horizontal gnomon (see *Livingston*). We have no information concerning the *saut̤*.

CHAPTER 11

RELATIONS BETWEEN THE SHADOW FUNCTIONS

45. Identities Relating Cotangent to Tangent (59:6 - 61:2)

The objective of this short chapter is to demonstrate the identities

$\cot \theta = \tan \bar{\theta}$, and $\cot \theta = 1/\tan \theta$,

as we would put them. The discussion and results are somewhat complicated by Bīrūnī's not putting g, the gnomon length, equal to unity, in contrast to Abū al-Wafā'. In the *Canon* (vol.1, p.343), presumably written after the *Shadows*, Bīrūnī in fact puts $g = 1$. As usual, the discussion is geometric.

To prove the first identity, note that in Figure 10,

(59:9) $LK = \mathrm{Cot}_g \widehat{AH} = \mathrm{Tan}_g \widehat{BH}$.

or $\mathrm{Cot}_g h = \mathrm{Tan}_g \bar{h}$,

which is the first relation.

Bīrūnī's form of the second is

(59:12) $(\mathrm{Cot}_g h) / g = g / (\mathrm{Tan}_g h)$.

To prove this, note that

(59:17) $LM = \mathrm{Tan}_g \widehat{BT}$, and since $AH = BT$,

$$KL = \mathrm{Cot}_g\, \widehat{BT}\ .$$

In the right triangle MEK with altitude EL,

(60:2) $$\frac{KL\ (= \mathrm{Cot}\ h)}{LE\,(=g)} = \frac{LE\,(=g)}{LM\,(= \mathrm{Tan}\ h)}\ ,$$

the second relation, 59:12. Or

(60:3) $$g^2 = \mathrm{Cot}\ h \cdot \mathrm{Tan}\ h$$

Alternatively, on the right-hand part of Figure 10, put $EZ = EL = g$.

Then $$ZO = \mathrm{Tan}_g\, \widehat{AH} = LM = \mathrm{Tan}_g\, \widehat{BT}\ ,$$

and in the right triangle SEO with altitude EZ

$$SZ\,/\,EZ = EZ\,/\,ZO\ ,$$

which also leads to 60:3.
Hence, in general,

(61:1) $\mathrm{Cot}_R\theta = R^2\,/\,\mathrm{Tan}_R\theta$, and $\mathrm{Tan}_R\ \theta = R^2\,/\,\mathrm{Cot}_R\theta$.

CHAPTER 12

TABLES AND IDENTITIES

46. Use and Precision of the Tables (61:5 - 62:1)

The table given in this chapter has been left out of the printed edition. In the translation it has been transcribed from the MS. Its entries give, to one fractional sexagesimal place, $\mathrm{Cot}_g\,\theta$ (and $\mathrm{Tan}_g\,\theta$) for each of the four standard gnomons. Bīrūnī states that not all such tables have separate arguments for the two functions. If only one argument appears, then one uses the identity

$$\mathrm{Cot}\ \theta = \mathrm{Tan}\ \bar{\theta}$$

to obtain the other function.

The table was recalculated with an electronic computer and the results compared with the text. In our transcription, numerals inside square brackets are correct values restoring erroneous entries in the text. In general, the original entries are precise to the one fractional sexagesimal digit given. The total number of errors or illegible places involves just about a third of the 360 entries, and the great bulk of these are accumulated scribal miscopyings, not mistakes in the original computation. Many of the errors are results of the very easily made slip of omitting (or inserting) a dot to denote a *nūn*(=50), the absence of which indicates a *yā'*(=10).

As medieval Islamic tables go, this one is not very impressive. For instance, Bīrūnī's tangent table in the *Canon* (vol.1, pp.341-5) is calculated to a far higher degree of precision, four significant sexagesimal digits instead of the two here.

47. Character of the Tangent Function (62:2 - 64:7)

This section examines the conditions under which linear interpolation is useful, with particular reference to the shadow functions. Bīrūnī first notes that if the first differences are constant, then linear interpolation is completely accurate. The shadow functions are not like this, and successive differences increase ever more rapidly as they approach their respective singularities.

For this reason Kūshyār and others tabulate the tangent function only up to 45°. Indeed, in the Leiden copy of Kūshyār's Jāmiʿ Zīj (Cod. 523(1) Warn.) on f.32r there is a table of $\mathrm{Tan}_{60}\theta$, calculated to three sexagesimal places, with tabular differences, for $\theta = 1°$, $2°, 3°, \ldots, 45°$. However, on the same folio is a table of $\mathrm{Cot}_7\theta$, to two places, with $\theta = 1°, 2°, 3°, \ldots, 90°$. On f.34v is another table of $\mathrm{Tan}_{60}\theta$ with the same format as that on f.32r, but going up to 60°. On f.35r are tables of $\mathrm{Cot}_{12}\theta$ and $\mathrm{Cot}_7\theta$ to two places for $\theta = 1°$, $2°, 3°, \ldots, 90°$.

The author notes that for some functions the accuracy of linear interpolation may be improved by adopting a smaller interval for the argument in regions where the second differences are large (62:18). This expedient is used by Ptolemy in many tables of the ***Almagest***.

Bīrūnī proceeds to argue that, as we would put it, if θ_n , $n = 1,2,3,\ldots$, are successive equally-spaced values of the argument, then

$$\Delta\,\mathrm{Tan}\,\theta_n < \Delta\,\mathrm{Tan}\,\theta_{n+1}$$

or the function is concave upwards.

For this he uses Figure 11, establishing in a straightforward manner that

(63:13) $$ZB < ZH ,$$

which is a special case of the theorem for $n = 1$.

Next he notes that

$$\text{area of } AZH\ (=ASH) \,/\!/\, \text{area of } AHT = ZH / HT ,$$

from which it follows that

$$ZH < HT ,$$

the case for $n = 2$, at which stage he leaves off.

It may have been his reflections on the inadequacy of linear interpolation which led Bīrūnī to the erroneous second order interpolation scheme in the *Canon* (vol.1, p.327; *MR*, vol.23(1962), p.583).

48. Reciprocal Relation Between Tangent and Cotangent (64:8 - 65:4)

To obviate the need for using arguments for the tangent function in excess of 45°, the author says if it is necessary to multiply a number, say T, by such a functional value, multiply instead by the tangent of the complement, i.e.,

(64:10) $$T \cdot \mathrm{Tan}\,\theta = T / \mathrm{Tan}\,\bar{\theta} = T / \mathrm{Cot}\,\theta ,$$

and likewise for division

(64:11) $$T / \mathrm{Tan}\,\theta = T \cdot \mathrm{Tan}\,\bar{\theta} = T \cdot \mathrm{Cot}\,\theta .$$

However, this is the case only if $g = 1$, that is if Abū al-Wafā's or the modern functions are used. The rule will not hold for the functions tabulated in the text. The correct expression, or its equivalent, has already appeared in 61:1 above.

Apparently Bīrūnī has led himself astray in the course of the demonstration which follows, done in the manner of the Greek geometrical algebra. In it magnitudes are represented by lines to admit the possibility of incommensurables.

Figure 12 shows segments A and B as the tangent and cotangent of the given arc respectively. G represents the gnomon, which Bīrūnī says can be regarded as unity. Insofar as choosing a scale for the drawing is concerned, this is all right. But numerically the ensuing argument holds only if A Tan $\theta = \tan\theta$, and $B = \mathrm{Cot}\,\theta$, that is, $g = 1$ throughout. Segment T is the number to be multiplied or divided by the given shadow function.

Carry out geometrically the division of T by A, and let the result be the segment E. Now

(64:16) $$A / G = G / B ,$$

as was proved in 59:12 above.

The author then says

(64:19) $$A \cdot B \neq G ,$$

but this holds only if $G = 1$. Since by construction $T / A = E$, hence

(64:19) $$A \cdot E = T ,$$

and

(65:1) $$B / E = G / T .$$

So

(65:2) $$B \cdot T = E \cdot G = E ,$$

again using the fact that $G = 1$. That is

$$T \cdot B = E = T / A ,$$

which is expression 64:11 above.

49. Functions of Two Arcs (65:5 - 69:3)

It is now shown that the tangents of two arcs are in inverse proportion to their cotangents. In Figure 13,

(65:8) $KL = \text{Cot}\ \widehat{AH} = \text{Cot}\ \theta_1$, and $KC = \text{Cot}\ \widehat{AZ} = \text{Cot}\ \theta_2$,

(65:12) $MS = \text{Tan}\ \widehat{AH} = \text{Tan}\ \theta_1$, and $MO = \text{Tan}\ \widehat{AZ} = \text{Tan}\ \theta_2$.

(66:1) $$LK / CK = MO / MS ,$$

or $$\text{Cot}\ \theta_1 / \text{Cot}\ \theta_2 = \text{Tan}\ \theta_2 / \text{Tan}\ \theta_1 .$$

Consider now Figure 14, in which CS has been drawn parallel to EO. It is stated that

(67:3) $$\frac{SC}{SE} = \frac{\text{Sin}(SEM = HEA = \theta_1)}{\text{Sin}\ ECS\ (=\text{Sin}(SCM = ZEA = \theta_2))} = \frac{\text{Sin}\ \theta_1}{\text{Sin}\ \theta_2} .$$

This holds, by the law of sines for plane triangles applied to the oblique triangle ECS. This theorem was known to Bīrūnī, but he does not explicitly invoke it here.

Next

(67:5) $$SE / EO = FE / EZ (=EH) ,$$

by use of the similar triangles SEO and FEZ. In turn,

(67:6) $$FE / HE = EK / ET ,$$

from the similar triangles EFK and ETH. Combining these,

$$\frac{SC}{SE} \cdot \frac{SE}{EO} = \frac{\text{Sin}\ \theta_1}{\text{Sin}\ \theta_2} \cdot \frac{EK}{ET} .$$

But

(67:9) $$SC / OE = MS(=\text{Tan}\ \theta_1) / MO(=\text{Tan}\ \theta_2) ,$$

from the similar triangles SCM and OEM.

So

(67:10) $$\frac{\text{Tan}\ \theta_1}{\text{Tan}\ \theta_2} = \frac{\text{Sin}\ \theta_1}{\text{Sin}\ \theta_2} \cdot \frac{\text{Cos}\ \theta_2}{\text{Cos}\ \theta_1} = \frac{\text{Sin}\ \theta_1}{\text{Cos}\ \theta_1} \cdot \frac{\text{Cos}\ \theta_2}{\text{Sin}\ \theta_2} .$$

Further,

(68:1) $$SE / EO = FE / EZ = EK / ET ,$$

whence

(68:3) $$\text{Sec}\ \theta_1 / \text{Sec}\ \theta_2 = \text{Cos}\ \theta_2 / \text{Cos}\ \theta_1 .$$

Without proof the author states that

(68:7) $$\frac{\text{Cot}\ \theta_1}{\text{Cot}\ \theta_2} = \frac{\text{Sin}\ \theta_2}{\text{Sin}\ \theta_1} \cdot \frac{\text{Cos}\ \theta_1}{\text{Cos}\ \theta_2} ,$$

and

(68:11) $$= \frac{\text{Cos}\ \theta_1}{\text{Sin}\ \theta_1} \cdot \frac{\text{Sin}\ \theta_2}{\text{Cos}\ \theta_2} .$$

CHAPTER 13

SHADOW FUNCTIONS ON THE ASTROLABE

50. General Description of the Instrument

Since this chapter, as well as the following, and Chapter 26, are involved primarily with the astrolabe, it seems well to sketch here the leading characteristics of this commonest of all ancient and medieval astronomical instruments. The reader who is interested in a detailed description may commence with, e.g., ***Hartner***. There is an enormous literature on the subject.

The device consists of a flat, circular brass ***mater***, hollowed out on one side to receive a number of thin ***plates***, one for each of the terrestrial latitudes for which the astrolabe was to be used. On top of the plates is a filigree ***rete***, and on the back of the mater an ***alidade*** (from Arabic ***al-ʿaḍāda***) with a pair of peep sights. Alidade, mater, plates, and rete are held together by a central pin passing through all.

The astrolabe has two functions:

1. ***It serves as an observational instrument*** for taking celestial altitudes. The mater has a ring linked to its edges so that when the whole assemblage is suspended from the ring the instrument hangs in a vertical plane: To take the altitude of the sun, say, rotate the astrolabe until its plane contains the sun, then rotate the alidade about the pin until a ray of the sun shines through both sights. Now read the altitude off the protractor scale engraved along the rim of the mater.

2. ***It serves as an analogue computer***. Each plate bears a pair of orthogonal families of circles engraved on it. These make up the stereographic map of

the coordinate curves for the horizon system of a particular terrestrial latitude, ϕ. The center of projection is the south celestial pole, and the map of the north celestial pole is the plate center. The rete, in turn, is the stereographic map of the celestial sphere, including the ecliptic circle. The latter has graduations so that the map of the sun for any given celestial longitude, λ, can be marked upon it. The maps of prominent fixed stars are indicated by tongues on the rete. This being the case, rotation of the rete about the plate fixed underneath reproduces quantitatively the motions in the sky of these celestial objects in the course of the daily rotation. For any particular time their horizon coordinates, altitude and azimuth, may be read directly off the rete.

51. Etymology of Astrolabe (69:6-13)

Ḥamza al-Iṣfahānī (fl. 940, see ***GAL***, vol.S1. p.221, and ***EIne***, vol.3, p.156) was an historian and philologist who worked under the patronage of the Būyid dynasty of central Iran. The book of his referred to by Abū Rayḥān (full title ***Kitāb khaṣā'iṣ al-muwāzina bayn al-ʿarabiyya w'al-fārisiyya)*** is extant only in fragments, but apparently had many improbable etymologies in defense of Persian versus Arabic culture. Here Ḥamza obtains astrolabe from Persian ***sitāra-yāb***, "star-finder" (***yāb*** from ***yāftan***, "to find"). Of course, Bīrūnī is right in obtaining it from Greek ἀστρολάβος (star-taker), involving ἄστρον, whence also ἀστρονομία and ἀστρολογία. In fact the Persian, Greek, and English words for star are all from the same Indo-European root, but the same does not hold for the last syllable of astrolabe.

52. Astrolabes Among the Indians (69:14 - 70:4)

Bīrūnī is also right in stating that (in his time) the Indians had no knowledge of the astrolabe.

(The first Sanskrit treatise on the astrolabe was the Yantrarāja of Mahendra Sūri (composed in 1370), which has been edited with the commentary of his pupil, Malayendu Sūri, and with the Yantraśiromaṇi of Śriviśrāma (1615) by Kṛṣṇaśaṅkara Keśavarāma Raikva, Nirṇayasāgara Press, Bombay 1936. Mahendra was the pupil of Madana

Sūri, an astronomer of Bhṛgupura (the same as Bhṛgukaccha, modern Broach?). He was patronized by Fīrūz Shāh (1351-1388), the monarch who also encouraged the algebraist Nārāyaṇa and sponsored the Persian translation of Varāhamihira's Bṛhatsaṃhitā. Mahendra freely admits his indebtedness to the "Yavanas" (Muslims).

As mentioned above, Indians did not construct astrolabes of any material before the end of the fourteenth century; but they did commonly use wood for armillary spheres and the like (cf., e.g., *Āryabhaṭīya*, Golāpada 22 and Sūryasiddhānta 13, 3). D.P.)

53. Shadow Graduations on the Rim (70:5 - 73:7)

This section describes the calibrations which appear on backs of many astrolabes, so arranged that if the instrument is used for taking an altitude, the length of the shadow cast at that time by a standard gnomon may be read directly off the calibrations by reading the number engraved on the rim at the point where the alidade edge intersects it.

In order to make the marks, the instrument is placed with its back upward in such fashion that the line *GL* (in Figure 15) may be drawn tangent to the rim at its bottom point *G*. *EG* = *GK* may then be taken as the gnomon's length, and *GK* is subdivided according to the units chosen for the gnomon. If digits are chosen, *K* will correspond to twelve, *Y* to seven, say, and so on, the subdivisions extending beyond *K* in the direction of *L*. Each of the points of subdivision is projected onto the quadrant *GD* from the center *E*. Thus *Y* projects onto *Z*. At *Z* a mark is made and the number 7 written, and so on.

As the shadow gets longer, the distance between successive marks on the quadrant becomes less. Eventually the marks approach each other so closely that the numbers can no longer be written, and the graduations leave off. The photograph of an actual astrolabe with such markings is Plate 1 in *Mayer*.

An alternative method is to use the legs *EC* and *CH* of the isosceles right triangle *ECH* as the gnomon length, subdivide *CH* extended, and project the subdivisions from *E* onto *GD*.

The above assumes that the direct shadow (associated with the cotangent function) is to be marked. If the reversed shadow (tangent function) is preferred, the procedure is the same except that the tangent *DO* is subdivided, and the projected marks will then accumulate in the region of *GZ*.

The last sentence in the chapter (at 73:7) may be a reference to two types of alidade. In one (as in *Mayer*, Plate IX) the alidade is so mounted that one edge is always a diameter of the instrument. In the other (as in *Mayer*, Plates I and III) the alidade is so constructed that on, say, its lower limb the left hand edge is a diameter; on the upper half it shifts to the right hand edge.

CHAPTER 14

THE ASTROLABE LADDER SHADOW

54. Layout (73:10 - 76:3)

In order to read off directly a shadow length corresponding to a particular altitude setting of the alidade, it is possible to graduate two sides of a square laid out in the interior of the back. This square, illustrated in Figure 17, is an alternative to the graduations on the rim described in the preceding chapter. It has the advantage over rim graduations that the differences between successive graduations are constant; the lines do not cluster, as on the rim. The method of marking is clearly described in the text, and need not be duplicated here. Note that for $90° > h > 45°$ the base of the square is used, hence the direct shadow (cotangent) is read; whereas for the rest of the quadrant, $45° > h > 0°$, the reading is taken from the right hand side of the square, and the reversed shadow (tangent) is obtained. This is exactly analogous to the custom, noted in Section 47, of tabulating the tangent function only up to 45°.

Should a user wish to obtain a shadow length not directly marked on the instrument, say Cot h for $h > 45°$, he is to make use of the relation $g^2 = \text{Cot } h \cdot \text{Tan } h$ derived in 60:3, and further discussed below. Lest he forget the value of g^2, this is to be inscribed in the interior of the square, its amount depending on the gnomon divisions used:

	digits	12^2	144
(75:6)	feet	7^2	49
	fractional feet	$(6\frac{1}{2})^2$	$42\frac{1}{4} = \frac{169}{4}$

This "ladder shadow" is a standard feature, appearing on all astrolabes known to us. Its ascription to al-Khwārizmī (see Section 39) is of interest. In al-Khwārizmī's own treatise on the astrolabe the construction and use of the ladder shadow are described (*Frank*, p.10), but with no indication that the author regards himself as the inventor.

We find Abū al-Qāsim al-Aḥwalī mentioned nowhere else in the literature.

55. Why "Ladder" Shadow? (76:4 - 77:9)

An obvious explanation for the name is that the right hand side of the scale, with its pair of long, parallel members joined by a series of short horizontal cross hatchings, like rungs, resembles a ladder. But this apparently did not occur to Abū Rayḥān, who goes through an involved and far-fetched discussion in order to drag in a ladder or so. He refers to books on algebra and *ḥisābāt al-muṭāraḥa* ("conversational computations"?, an unknown term to us) as having a problem dealing with a ladder leaning against a wall. Dr. Jacques Sesiano informs us that in a Paris MS (Bibl. Nat., Lat. 7377A, f. 198r), an anonymous translation from Arabic, just such a ladder problem is discussed. This is all well enough, but no shadows are involved. The author therefore shifts his ground and presents us with a pair of parallel walls of equal height seen in profile in Figure 18. Solar rays coming from *D*, *A*, and *M* cast shadows as shown. By using corresponding sides of the similar triangles *ETG* and *EZB* we have

(77:3) $$ET / TG = BZ / ZE ,$$

whence $$TG = ET \cdot EZ / BZ .$$

To complete the analogy with the astrolabe, let the distance between the walls equal their common height, so $ET = EZ$, and

$$ET^2 = TG \cdot BZ .$$

Now we have shadows but no ladder. Abū Rayḥān easily gets around this by the remark (76:19) that if the walls are high we need a ladder to measure the shadows on them.

56. The Ladder Shadow in the Canon

In Bīrūnī's zīj (*Canon*, vol.1, p.337) is a related discussion. If at some time the shadow cast by a gnomon (say *ET* in Figure 18) exceeds the gnomon length (i.e., $h < 45°$), its length may be determined as follows. Set up a second gnomon, *ZH*, of length equal to the first, in the plane determined by the first gnomon and the sun, so that the distance between the two equals their common length. Then part of the second gnomon will be shaded by the first. This shaded part, *BH*, is called the ladder shadow (*ẓill al-sillum*). Then the direct shadow is given by either of the expressions

$$TG = g + \frac{BH \cdot g}{g - BH} ,$$

or

$$TG = g^2 / (g - BH).$$

Proof is immediate by use of similar triangles.

57. A Derivation Repeated (77:10 - 79:13)

The passage commences with a reminder that the alidade edge used to identify the proper graduation must be a diameter of the astrolabe back (see Section 53). There follow specific directions for the use of the device. If the direct shadow (cotangent) is sought and the alidade edge falls somewhere along *TH* (in Figure 19), simply read the result from the numbered graduations. But if the alidade edge intersects *HZ* somewhere ($h < 45°$), the reading will be the reversed shadow (the tangent); transform it by the rule

(78:2) $\text{Cot } h = g^2 / \text{Tan } h.$

If, on the other hand, the tangent is desired, read it directly from *HZ* if $h < 45°$. If the alidade edge enters *TH*, read the cotangent and transform by using

(78:6) $\text{Tan } h = g^2 / \text{Cot } h.$

This has already been proved in Chapter 11. But Bīrūnī restates that

(78:8) $\text{Tan } h / g = g / \text{Cot } h$.

He now says, working from Figure 19,

(78:9) $KZ / TO = (KZ/ZE) \cdot (KZ/ZE)$,

which is true, but which seems unnecessarily complicated. From this he infers the desired 78:6 and its equivalent

$$\text{Cot } h = g^2/\text{Tan } h .$$

We miss the point of this redundant demonstration. The author further belabors the issue by putting down the proportions

(74:1) $KZ(= \text{Tan } h) / ZE(=g) = ET(=g) / TO(= \text{Cot } h)$,

and

(79:4) $TN(= \text{Cot } h) / TE(=g) = EZ(=g) / ZM(= \text{Tan } h)$.

Suppose (79:6) either shadow function is given and we wish to find *h*. If the shadow is less than the gnomon, rotate the alidade until its edge crosses the proper ladder shadow scale at the point corresponding to the given shadow. Then read *h* off the protractor scale engraved around the periphery of the back. If the shadow exceeds the gnomon, transform it into the other variety by use of 78:6, and proceed as described previously.

58. Al-Sijzī's Derivation (79:14 - 80:1?)

The astrolabe book by Abū Saʿīd has been mentioned before (in 54:2 and Section 41). We paraphrase below the passage from it quoted at 79:17.

In Figure 19 the (direct) shadow cast by a gnomon *ES* will be *SK*. In our notation $\text{Cot}_{ES} h = SK$. But we seek $\text{Cot}_{12} h$, hence a change of scale is required, proportional to the ratio between the two gnomons, 12(= *ET* = *KS*) and *ES*. So

$$\text{Cot}_{12} h / \text{Cot}_{ES} h = 12 / ES ,$$

and $\text{Cot}_{12} h = 12(\text{Cot}_{ES} h = SK = 12) / (ES = ZK)$

(80:9) $= 144 / ZK$.

This, of course is the familiar $\text{Cot } h = g^2/\text{Tan } h$ for the case of the 12-digit gnomon.

The chapter closes with the remark that the ladder shadow graduations may be replaced by graduations on the rim obtained by central projection of the former. Then there will be no need to have one alidade edge a diameter, since both angles and shadows will be read along the rim. This would be the system described in Chapter 13, except that the lower half of the graduated quadrant would have cotangents and the upper half tangents.

CHAPTER 15

SHADOWS ON SLOPES AND SPHERES

59. Altitude from Shadow and Conversely - Graphical Determinations (81:3 - 83:8)

The methods described in this section are independent of the rest of the chapter. They are simply scale constructions from which the desired quantities are read off from a protractor or linear scale. If the direct shadow (cotangent) is given it is laid off to scale as the base, *DE* in Figure 20, of a right triangle whose altitude is the gnomon length. Then angle *E* is the corresponding altitude. If a circle of radius *R* is drawn with center *E*, and a perpendicular dropped from *G* to *DE*, *GT* = Sin *h* and *TE* = Cos *h*.

If the given shadow is the reversed one (the tangent), then in the same configuration as before, now Figure 21, with *DE* the shadow and *WD* the gnomon, the desired argument is the angle at *W* or the arc *KM*, and Tan *W* = *DE*.

Conversely, given the arc, to construct its cotangent (direct shadow), lay off the arc on some suitable circle, say *BG* in Figure 22, draw its central angle, and draw a segment of length equal to the gnomon and perpendicular to one side. Through the end of this perpendicular, draw a line *AW* parallel to this side, *DE*. Extend the second side until it intersects the parallel at *W*. From *W* drop a perpendicular to *DE*. Then *DE* = Cot *E* is the required shadow. The analogous construction for the reversed shadow is not given.

Concerning Ḥabash al-Ḥāsib, see Section 39.

60. Adjusting a Shadow Cast on a Slope (84:1 - 86:3)

If a gnomon is set up on a plane which is not level, in general the shadow it casts will be inclined and should be corrected accordingly. The method for doing so is taken from a book *Fī'l-ᶜIlal* by Yaᶜqūb b. Ṭāriq (Section 39), a work known to us only from the three places where Bīrūnī mentions it in the *Shadows*.

It seems tacitly assumed that the gnomon is emplaced vertically, which should not be hard to do by using a plumb line. A further tacit assumption is that the observer is able to measure the difference in level (ET in Figure 24) between the end point of the shadow and the foot of the gnomon. This also should not be difficult, with the aid of the try-square, or some such instrument. Then

(84:14) $$AH = g \mp BH = AB \mp ET \ ,$$

where if plus and minus signs occur together the upper symbol refers to the "first" part of Figure 24 (shadow uphill) and the lower to the "second" (shadow downhill).

Now compute the "mean shadow"

(85:4) $$EH = BT = \sqrt{\overline{BE}^2 - \overline{BH}^2 \ (= \overline{ET}^2)} \ .$$

By similar triangles the "equation"

(85:5) $$GT = BT \cdot TE / AH \ .$$

Finally, the corrected shadow,

(85:8) $$BG = EH \mp GT \ .$$

Alternatively,

(85:9) $$BG = BT \cdot AB / AH \ .$$

Or, if AE has been measured, the mean shadow may be calculated as

(85:12) $$EH = \sqrt{\overline{AE}^2 - \overline{AH}^2} \ ,$$

and the computation completed as above.

The shadow having been computed, the solar elevation h can also be calculated, if desired, by the various methods explained previously.

61. Inclination of the Shadow (86:4 - 19)

The idea now is to find the shadow's inclination, angle EBG, by using the expression

(86:6) $$EB / BG = \text{Sin}(EGB = h) / \text{Sin}\ EBG \ ,$$

where everything is known save the desired angle at B. But this equation is invalid. Application of the sine law to triangle EBG would give, e.g.,

$$EB / BG = \text{Sin}\ EGB / \text{Sin}\ BEG \ ,$$

and, the angle at E once at hand, the angle at B can be found very readily. The invalid expression cannot be blamed on a copyist, for the text now has

(86:12) $$(\text{Sin}_R h)(BG{=}S) / BE = ET = \text{Sin}_{EB} EGB \ ,$$

which is equivalent to the invalid 86:6.

The ET allegedly found is said to be in units such that Sin 90° = EB (86:15). Hence a change of scale is accomplished by calculating

(86:16) $$(\text{Sin}_{EB} EGB) \cdot EB / R = ET$$

in the units of the gnomon, R being the parameter for Sin h. But ET was previously treated as known. All told, the passage is thoroughly unsatisfactory. We have no way of telling whether it is the work of Yaᶜqūb, or a contribution by our author, so there is no hope of spotting an early application of the law of sines, even if it had been valid.

62. Gnomon on a Sphere — Ibn al-Farrukhān's Rule (87:1 - 15)

The individual mentioned in this passage and later at 108:11, Muḥammad b. ᶜUmar b. al-Farrukhān al-Ṭabarī, fl.c.780, was one of the group of Abbasid scientists who worked at Baghdad. His family was from the Caspian provinces of Iran, and his father, also an astronomer, made translations from Pahlavi into Arabic. In the *Shadows* we have two fragments from Ibn ᶜUmar's non-extant zīj. (See *Suter*, p.17).

Bīrūnī distrusts the transmission of the rule, and does not undertake to analyze it. If, however, we emend the text (as has been done in the translation) at 87:5 and 87:7 to read "half the diameter" instead of the "the diameter", it is not hard to interpret the rule in terms of the first part of Figure 23. There a gnomon of length g is depicted, erected at the top of a sphere of diameter d. A solar ray at altitude h casts the gnomon's shadow, arc AD, on the sphere.

The emended rule says, calculate

(87:5) $(g+d/2)\ \text{Sin}\ h/R\ (=(g+d/2)\ \sin h = BT)$.

That BT is indeed Ibn al-Farrukhān's "retained quantity" is easily verifiable from the figure. Next compute

(87:8) $(g+d/2)\ \text{Cos}\ h/R\ (=(g+d/2)\ \cos h = ET)$

and $\sqrt{(d/2)^2 - \overline{ET}^2}\ (= \sqrt{\overline{ED}^2 - \overline{ET}^2} = DT)$.

(87:10) $BT - DT = BD$,

and

(87:11) $\widehat{AD} = \text{arc Sin}_{d/2}\left(\frac{BD \cdot \text{Cos}_{d/2} h}{d/2}\right)$.

It is implicit in the rule, though not stated, that we must have $R = d/2$ for the trigonometric functions last used. To convert the result from degrees into the arc length in digits, put

(87:14) $\frac{3\frac{1}{7} d}{360^\circ}\ \widehat{AD}^\circ\ (\approx \pi d\ \widehat{AD}^\circ / 360^\circ)$.

63. Bīrūnī's Solution (87:16 - 89:10)

Abū Rayḥān first works out his procedure by using arcs and lines on the figures, and then reformulates it in general terms without reference to the figures. We will carry along both arguments simultaneously.

First

(88:6, 88:18) $EB = d/2 \pm g = r_1$,

where r_1 is the "first retained", r_2 the second, and so on, and the plus sign applies to the first part of Figure 23, the minus to the second.

Now since

$$EB / BH = \text{Sin}\ h / R\ (= \sin h)\ ,$$

hence

(88:6, 89:1) $R(g \pm d/2) / \text{Sin}\ h = r_2 (=(g \pm d/2) / \sin h) = BH$.

Moreover, since

(88:10) $BH / EH = R / \text{Cos}\ h$,

(the text has Sin BH, an error), hence

(89:1) $BH\ \text{Cos}\ h / R (= BH \cos h) = EH = r_3$.

By similar triangles,

(88:12) $\overline{EH}^2 = BH \cdot HT$,

so $\overline{EH}^2 / BH = r_3^2 / r_2 = HT = r_4$.

By the Pythagorean theorem,

(88:14) $\overline{HE}^2 - \overline{HT}^2 = \overline{ET}^2\ (= r_3^2 - r_4^2)$,

and $\sqrt{\overline{ED}^2 - \overline{ET}^2} = \sqrt{(d/2)^2 - (\overline{HE}^2 - \overline{HT}^2)} = DT$.

By similar triangles

(88:15) $HD / DZ = HB / BE$,

so

(89:6) $HD \cdot EB / BH = (DT + r_4) \cdot r_1 / r_2 = DZ$,

the result being $\text{Cos}_{d/2} AD$. The phrase "we transform" at 89:7 must imply

$$\widehat{AD} = \text{arc Cos}_R \left(\frac{R \cdot DZ}{d/2}\right) ,$$

or some words may have fallen from the text. The conversion from degrees to digits of arc is as in the section above.

64. An Alternative Procedure (89:11 - 90:8)

Instead of calculating the cosine of the desired arc, its sine may be determined. Again Bīrūnī has a pair of parallel arguments, the first based on the figure. As before we will consolidate the two.

The "difference between the two previous amounts" is

(90:1) $r_3^2 - r_4^2 = \overline{ET}^2$

calculated in 88:14 above. So ET can be found. Now

(89:11, 90:3) $\sqrt{(d/2+ET)(d/2-ET)\cdot 4} = \sqrt{4(\overline{ED}^2-\overline{ET}^2)} = 2DT=DK=(r_5)$.

This is the "retained root" of 90:4. Now form

(90:4) $(\sqrt{(g+d)d+(r_5/2)^2} - r_5/2)\; r_3/r =$ Sin $\widehat{AD}$ $(=MD)$.

To prove this, note that it can be written as

(1) $(\sqrt{OB\cdot AB + \overline{TD}^2} - TD)\; HE\,/\,BH = (\sqrt{KB\cdot BD + \overline{TD}^2} - TD)\cdot HE\,/\,BH$,

(89:13) $OB\cdot BA = KB\cdot BD$,

for if secants are drawn to a circle from a point, the product of the two segments formed at the intersections with the circle is a constant. In the range of points B, D, T, K, the midpoint of DK is T, hence

(89:17) $KB\cdot BD + \overline{TD}^2 = \overline{TB}^2$,

a standard identity of the ancient "gnomonic algebra". Bīrūnī also puts down the valid expression

(89:15) $KD\cdot DB + \overline{DB}^2 = KB\cdot BD$,

but does not seem to use it. Substituting in equation (1) the right-hand side of 89:17 for the left-hand side gives for (1),

(2) $(\sqrt{\overline{TB}^2} - TD)\cdot HE\,/\,BH = (TB - TD)\cdot HE\,/\,BH = DB\cdot HE\,/\,BH$,

and since

(89:19) $DB\,/\,DM = BH\,/\,HE$,

equation (2) equals DM = Sin $\widehat{AG}$ as claimed in 90:4.

We regard all of this as a mathematical exercise of no practical import. A gnomon on top of a sphere would have to be quite small, else the end of the shadow would escape the sphere completely while the sun was still high in the sky. And no matter how small the gnomon, this escape of the shadow would always occur before sunset. It would be more practical to set up a segment of a hollow sphere with the gnomon inside its top, flush with the top of the segment. Actual sundials exist having a hollow hemispherical surface with the gnomon point at the center

of the sphere. But then there is no need for computation — the celestial sphere maps onto the inner spherical surface with no changes of arc lengths.

CHAPTER 16

DETERMINATION OF THE NOON SHADOW

65. The Problem in General Terms (90:11 - 92:8)

The chapter commences with a straightforward presentation of how to calculate the solar altitude at culmination. As can be seen from Figure C2,

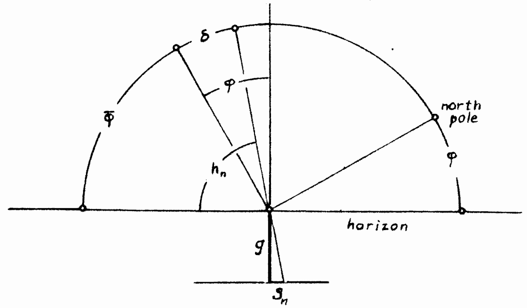


Figure C2

$$h_n = \bar{\phi} + \delta .$$

Bīrūnī's three cases are

(1) $\delta < \phi$ (91:7),

which implies that $h_n < 90°$.

If

(2) $\delta > \phi$ (91:8),

which implies $\phi < \varepsilon$, i.e., the locality is in the tropics, the culmination will be north of the zenith, and

(91:9) $h_n = 180° - (\bar{\phi}+\delta)$

measured from the north.

The intermediate case is

(3) $\delta = \phi$ (91:10)

for which

(91:11) $h_n = \bar{\phi} + \phi = 90°$

and the sun culminates in the zenith.

At the equinoctial points

(91:12) $\delta = 0$ and $h_n = \bar{\phi}$.

The tropical zone is characterized as the region "having two shadows", since part of the year s_n will be south and the rest of the time north.

Although Bīrūnī says nothing about it in this passage, the transformation from noon altitude to noon shadow is immediate and will be needed later on. It is

(4) $s_n = g \cot h_n = \mathrm{Cot}_g(\bar{\phi} + \delta)$.

Note that since δ depends on ε, the latter is an implicit parameter of h_n and s_n.

66. The First Indian Scheme (92:9 - 17)

Of the approximate Indian rules, the first cited can be expressed as

(92:15) $$s_n = \frac{(B_\phi - L)(60-L)}{L} ,$$

in digits, where the number B is the "base" in "day-minutes", and L is the length of the particular day in the same units. B depends on the latitude of the locality, hence the subscript in the expression above. Presumably ***astarkī*** (?) is a transliterated Indian word, but we find no Sanskrit word from which it may have come.

A day-minute is a sixtieth part of a day, so 60 day-minutes = 24 hours = 360 time-degrees, the last unit being time measured in degrees of daily rotation.

The text indicates later (in 92:19) some reason for connecting *B* with the maximum length of daylight, hence we set about calculating the latter as a function of ϕ. The "equation of (half) the daylight", arc $EC(=e)$ in Figure 44, is half the difference between the actual length of daylight and twelve hours (or thirty day-minutes, or 180 time-degrees). By solving the spherical right triangle *ECF* in the same figure, it can be shown that

$$e = \text{arc sin } (\tan\delta \tan\phi) .$$

So

$$\max e = \text{arc sin } (\tan\varepsilon \tan\phi) ,$$

since

$$\max \delta = \varepsilon .$$

Hence

$$\max L = 30 + 2 \max e .$$

Displayed below are the results of calculating this quantity for the two latitudes named by Bīrūnī in this passage:

ε \ ϕ	30° (Sind)	32° (Lahore)
23.5°	34.9	35.3
24°	35.0	35.4
B	36	38

The juxtaposition of the related *B*'s justifies our author's remark that they are max *L*'s, or "perhaps a minute more".

67. Examination of the Rule

We are now in a position to examine expression 92:15. The length of a civil day being 60 day-minutes, 60-*L* is the duration of the night, the second factor in the numerator.

Table 1 displays on its second line the results of using the rule to calculate S_n for the summer solstice

(max h_n), the equinoxes (δ=0), and the winter solstice (min h_n). For purposes of comparison the first line of the same table shows the results of accurate calculations of S_n in digits (g=12), for the same times and for the latitude indicated. There is a reasonable correspondence between the accurate values and those given by the rule, the latter being consistently below the former.

Table 1. Rules for Calculating Noon Shadows

	Found in the text at	The Rule	Parameters	Remarks	S_n in digits: Summer solstice	equinoxes	winter solstice
1		$12 \cot(\phi + \delta)$	$\phi=30°$, $\epsilon=24°$	This is the accurate computation.	1.2	6.9	16.5
2	92:11	$\frac{(B_\phi-L)(60-L)}{L}$	$B=36$, max $L=35$		0.8	6.0	15.4
3			$B=\max L=36$		0.0	6.0	18.0
4	92:18	via al-Sijzī $(B_\phi-L)(5/4)$	$B=\max L=36$	linear	0.0	7.5	15.0
5	93:8	$57-2\delta/15$	max $\delta=\epsilon=24°$	linear, manifestly absurd	53.8	57	60.2
6		emended to $(57-2\delta)/15$	max $\delta=\epsilon=24°$	linear, still very poor	0.6	3.8	7.0
7	93:13	Abū ʿĀṣim $\text{Tan}_{12}\,\phi \begin{cases} 13\frac{1}{3}\delta, & \delta\geq 0 \\ 25\delta, & \delta<0 \end{cases}$	$\phi=30°$	piecewise linear	1.4	6.9	16.9
8	93:17	$\frac{216-d}{5\frac{1}{2}} + \left\lvert\frac{180-d}{18}\right\rvert$	max $d=216$ time-deg.	piecewise linear	2.0	6.9	15.7

We cannot surmise why the unknown inventor of rule 92:15 chose the form he did. We do note that the element (60-*L*)/*L* becomes unity at the equinoxes, so that the *B* in the remaining factor, then (*B*-30), can be set so as to yield any desired equinoctial S_n. This having been done, the inventor has expended the only degree of freedom left him.

Since, as will be seen below, there is some reason for associating the *B* with a max *L* of 36, we have also calculated S_n for these parameters. The results appear in line 3 of the table. This has the summer solstitial sun passing through the zenith (since S_n = 0) in the latitude of Sind. Any actual observer should realize that this contradicts the facts.

68. Al-Sijzī's Report (92:18 - 93:3)

Abū Saʿīd has appeared previously; for background on him, see Section 41 above. The rule he passes on is

(93:1) $s_n = (B_\phi - L)(5/4) = (36-L)5/4$.

The results given by it appear in line 4 of the table. Comparison of them with the first line gives the impression that the rule is a fair linear approximation to a piece of the curve implied by the first line.

Here 36 is categorically said to be the max L. The corresponding min L=24, and 36/24 = 3/2. This ratio of maximum to minimum daylight was a standard ancient Babylonian approximation. Its use among the Indians is otherwise attested, and is an undoubted instance of a borrowed parameter (see *Pingree*, 1, p.4).

69. A Rule in Terms of Declinations (93:4-10)

This is also linear, but the independent variable is δ. Expressed symbolically it is

(93:8) $s_n = 57 - 2\delta/15$,

since the text prescribes addition when the declination is southerly. The results, shown in line 5 of the table, are so absurd that an attempt has been made to emend the text to say "add the doubled elongation to fifty-seven, and then divide by fifteen", i.e.,

$$s_n = (57 - 2\delta)/15.$$

The results are much better, but hardly impressive.

70. From the Zīj of Abū ʿĀṣim (93:11-16)

Save for this mention, we have no information at all about Abū ʿĀṣim ʿIṣām. Because of his connection with the East Iranian Barmecid family, powerful under the ʿAbbāsids, he also may well have come from Khurāsān.

His rule may be expressed as

(93:13) $$s_n = \mathrm{Tan}_{12}\phi - \begin{cases} 13\tfrac{2}{3}{}'\,\delta, & \delta \geq 0 \\ 25'\,\delta, & \delta < 0 . \end{cases}$$

Comparison of the results it gives, shown in line 7, with the accurate computations makes it clear that the two constant coefficients were so chosen as to insure

that this piecewise linear function would give good results at the end points as well as in the middle of the range.

71. A Rule Involving Time-Degrees (93:17 - 94:2)

The last approximate scheme puts

(93:17) $$s_n = \frac{216-d}{5\frac{1}{4}} + \left|\frac{180-d}{18}\right| ,$$

the absolute value signs being clearly implied by the verbal formulation. Here the argument, d, is again length of daylight, but now measured in time-degrees.

This function is also piecewise linear, and its results, tabulated in line 8 of the table, are seen to be almost as good as those of the rule just before.

The motivation behind the rule can be grasped if we recall that

6 time-degrees = 1 day-minute

whence 216 time-degrees = 36 day-minutes,

the same ancient Babylonian max L used before. The first term is suppressed at the summer solstice, when d = max d = 216. The second term is suppressed at the equinoxes when d takes its mean value of 180. The constant coefficients, $5\frac{1}{4}$ and 18, have been chosen to give favorable results.

72. The Valid Rule (94:3-12)

The last set of directions, giving the noon shadow in feet, may be expressed in symbols as

(94:3) $$s_n = \sqrt{\left(\frac{975}{\mathrm{Sin}\, h_n}\right)^2 - 42\tfrac{1}{4}} .$$

If we put R = 150 and $g = 6\frac{1}{2}$ it becomes

$$s_n = \sqrt{\left(\frac{150 \times 6\frac{1}{2}}{\mathrm{Sin}_{150} h_n}\right)^2 - (6\tfrac{1}{2})^2} = \sqrt{\frac{g^2}{\sin^2 h_n} - g^2}$$

$$= g\sqrt{\csc^2 h_n - 1} = g \cot h_n ,$$

which is our expression (4) above.

73. Comparison with Indian Methods

Although the clear implication of the text is that all these methods are Indian, none of them appear in the extant Sanskrit literature. Our schemes, however, exhibit certain resemblances. In one of the oldest gnomon-texts, the Arthaśātra, the summer solsticial shadow is taken as zero, that of the winter solstice as a gnomon length, twelve, and linear variation is assumed in between. (See *Pingree*, 1, p.5). This implies that the equinoctial S_n shall be six, a situation found in lines 2 and 3 of our table. The $S_n = 0$ at the summer solstice appears in lines 3 and 4, and a strictly linear relation in 4, 5, and 6.

In the later Indian works, *Khaṇḍ*. 3, 13-4, and *Brahmasp.* 3, 49, the method of 94:3 is given, but without the use of the curious "fractional feet" $g = 6\frac{1}{2}$, which seems to be associated with Muslim contexts. (See Section 29).

CHAPTER 17

THE EQUINOCTIAL MERIDIAN SHADOW

74. Definition and Discussion (94:15 - 95:7)

The subject of this short chapter is a special case of the problem discussed in the previous chapter. It is S_n when $\delta = 0$. Under these circumstances expression (4) of that chapter degenerates into

(94:18) $$S_\phi = \mathrm{Cot}_g \bar{\phi} = \mathrm{Tan}_g \phi ,$$

where S_ϕ is the "equinoctial shadow".

This is trivially equivalent to the rule of Nayrīzī (see Section 31) and Ya*q*ūb b. Ṭāriq (see Section 39).

(95:1) $$S_\phi = g \cdot \mathrm{Sin}\ \phi \,/\, \mathrm{Cos}\ \phi .$$

Bīrūnī is right in remarking that al-Battānī (see Section 39) does not use the standard Arabic term for "sine", *jayb* (from Sanskrit *jyā*) already well established in his day. Instead he calls them "half-chords" (cf. *Battānī*, *zīj*, vol.2, p.55; vol.3, p.15), utilizing the fundamental identity connecting sine and chord,

$$\sin \theta = \tfrac{1}{2}\,\mathrm{crd}\ 2\theta .$$

Yaʿqūb calls the sine function *watar mustaqīm*, "straight chord", instead of *jayb*, a term which had become standard by Bīrūnī's time. The archaic usage recurs at 127:19 in another rule of Yaʿqūb (see Section 99).

Since S_ϕ is a function of ϕ it can be used to characterize latitudes. In Chapter 22 Indian rules are given which involve S_ϕ, and it is probably this which

Bīrūnī has in mind at 95:6. But this is hardly a matter of delimiting terrestrial zones as with the Greek *klimata*.

75. Al-Kindi's Parallax Again (95:8 - 96:18)

Here our author alludes again to a remark of al-Kindī, noted in 22:7 and Section 21, to the effect that at the two equinoxes the sun is at different distances from the earth, hence its parallaxes, apparent altitudes, and noon shadows will all differ at these two times. Al-Kindī correctly gives the Ptolemaic solar apogee as having a longitude of Gemini 5;30° (*Almagest* 3,4). Hence, since the sun is nearer the apogee at Aries 0° than at Libra 0°, its distance at the former time (*EK* in Figure 26) will exceed its distance at the latter. And since the head of the gnomon is not in the equatorial plane, there will be a parallactic difference.

In principle this is correct. But Bīrūnī is also right in stating that the gnomon length is so small in comparison to the earth-sun distance that the effect is wholly imperceptible. The number he gives at 96:10 should be 1210 (*Almagest* 5, 15), not 1110 earth-radii. The slip, whether made by the author or by a scribe, does not affect the argument. By Bīrūnī's time precession had carried the apogee quite close to a condition of symmetry with respect to the equinoxes. In the *Canon* (vol.3, p.693) he makes it Gemini 25:10° as of 400 Yazdigird (= 1031 A.D.). If the apogee were midway between the equinoxes the two distances would be the same. As in the previous discussion (at 22:14), he reminds the reader that the great variation in the moon's distance from the earth as well as its relative nearness makes the situation quite different for it.

76. An Erroneous Manichean Rule (97:1 - 98:19)

The sect founded by Mānī retained many followers in Bīrūnī's time, and some of our knowledge about their beliefs comes from him. The notion reported in this passage is apparently connected with their idea that the north pole is the highest, hence the best place, to which the worshipper should turn in prayer (*Chron.*, transl., p.329).

It is correct that if an individual in the northern hemisphere travels north the equinoctial shadow, S_ϕ = Tan ϕ, increases. The Manichean rule, however, went beyond this, implying a linear relation between S_ϕ and ϕ so that the distance travelled north is alleged to be $120 \cdot \Delta S_\phi$ in *farsakhs* (1 farsakh = 3 miles). To demolish this, it suffices to show that the tangent function is not linear. This was already done by Bīrūnī in 63:2 - 64:7 (Section 47). Here he gives another demonstration, using Figure 25. He shows that equal increments in ϕ give increasing increments of S_ϕ.

Of the two stars, or groups of stars, mentioned in the rule, the first, the Great Bear, is unequivocally the nearest to the North Pole. As for the second, the text says *Suhayl al-suflā*. Now Suhayl is Canopus (α Carinae), a bright star far south of the equator. Various modifiers are attached to Suhayl, but we have not encountered *suflā* (inferior, or under) among them. However, some sources mention a star under (*taḥt*) Suhayl called *Suhayl balqīn*(?), conjectured to be α Pictoris (cf. *Kunitzsch*, p.106). Perhaps this was the star intended. In any case, the general idea is clear. The rule uses stars as far as possible to the north and to the south.

CHAPTER 18

DETERMINING THE MERIDIAN LINE BY TWO EQUAL AZIMUTHS OR SHADOWS

77. The North-South Line Bisects the Angle of Equal Shadows (99:3 - 101:11)

In much of what follows, observations are made and constructions carried out graphically upon a horizontal surface. The precision of the results depends upon the accuracy with which this plane is laid out, and the opening remarks of the chapter are directed toward this end. In *Schmidt*, p.43 will be found a description of related ancient levelling techniques.

The elaborate discussion which follows has as its object to demonstrate that the angle made by two equal shadows, one cast before noon, the other after, is bisected by the north-south line. Furthermore, the time interval between the instants at which the two shadows are cast is bisected by the local noon. This is under the assumption that there is no change in solar declination during the daylight hours. The consequences of taking the actual variation of declination into consideration are discussed in the next chapter.

It is useful to define at this stage the technical terms introduced in the passage. The *day circle*, not referred to explicitly here, is the parallel of declination traced out by the sun in the course of the day, *THZ* in Figure 27. Its radius is $\rho = \cos \delta$. The arc of (half) daylight, or the *day arc* (*qaws al-nahar*), *d*, is half of that portion of the day circle which is above the horizon. It is a measure of the length of

daylight. The *day sine* (*sahm al-nahār*), more properly the "versed sine of daylight" is Vers ρ; on Figure 27 it is *BH*.

The *day triangle*, *HBD* in the same figure, is the right triangle having the day sine as hypotenuse and an acute angle equal to ϕ. The *time triangle*, *TKG'* is similar to the day triangle, but with altitude equal to Sin *h* where *h* is the solar altitude at the time.

Figure 27 once being at hand, considerations of symmetry would seem to insure the desired conclusions, that $KT = LZ$ implies $\angle CEG = \angle YEG$ and $\widehat{TH} = \widehat{HZ}$, without the elaborate discussion of the text.

78. Dependence of Azimuth upon Altitude (101:12-103:15)

This passage, based on Figure 28, demonstrates that at a single place and on a given day the azimuth depends upon the altitude. Let the sun be at *O*, the day circle being *LO*, and the altitude (which determines the shadow) *OT*. If the same altitude is to be preserved the sun must be somewhere on the almucantar *OZK*. But no other point (on the forenoon half of the almucantar) will do, for on that day the sun must move along *LO* – to shift the day circle would be to shift the season. Hence, since the intersection *O* is unique, the azimuth *BT* is also unique, since there can be only one great circle through *O* normal to the horizon.

In general, a shift in the locality will entail a change in both altitude and azimuth. This also is illustrated on Figure 28, although the drawing is faulty. It is assumed that the local zenith moves from *S* to *C* along the original meridian. Then the east point on the new horizon should remain *B* as before, but the text has the two horizons intersecting in a distinct point, *N*.

Be that as it may, Bīrūnī compares the old and new altitudes, *OT* and *OH* respectively. Since $OH > OF > OT$ (103:9) they differ, and the same goes for the azimuths.

79. One Horoscope and Altitude at Two Times (103:16 - 105:4)

The relation between azimuth and altitude apparently suggests to Bīrūnī a somewhat analogous situation as between altitude and the horoscope. The latter, also called the ascendant, is the ecliptic point rising across the eastern horizon at any particular time, *E* in Figure 29. For a given horoscope and a given locality there are in general two times at which the sun will have a particular altitude, although this, the author indicates, seems improbable at first blush.

To demonstrate it, let *Z* in the same figure be on the ecliptic and at a quandrant's distance from both *E* and *H*. The arc *ZS* is the peculiarly named "latitude of visible climate" (*ʿarḍ iqlīm al-ruʾya*, 104:6). Then any pair of ecliptic points such as *Y* and *O*, or *K* and *M*, which are symmetrically disposed with respect to *Z*, satisfy the requirements set. For *Y* and *O* are at the same altitude; yet the events the two configurations represent must take place at different times, for if the sun is at *O*, some time must elapse before it arrives at *Y*, travelling slowly along the ecliptic at about a degree per day. The corresponding solar azimuths, arcs *AC* and *GF* in this case, will be different. They may both be easterly (like *N* and *B*) or one east and one west (like *C* and *F*), or both westerly.

This particular peculiarity seems to have intrigued Abū Rayḥān, for in the *Taḥdīd* (comm., p.98) he gives a numerical example of precisely this phenomenon.

80. The Indian Circle (105:5 - 107:8)

The text now reverts to the main topic, noting that, as has been established, in principle the meridian may be found by time measurements, say by splitting the time from sunrise to sunset and noting the direction of the shadow at the half-time, noon. This is impractical for many reasons, the lack of reliability of clepsydras being among them. Alternatively one may bisect the angle between equal forenoon and afternoon shadows. For this the simple device pictured in Figure 30 was common. Although in Muslim parlance it was called "Indian", there is no reason for thinking it originated in India.

A gnomon *AB* is erected on a level surface, and on the latter a circle with center *A* and suitable radius drawn. In the forenoon the observer marks *H*, the point

at which the end of the shortening shadow crosses the circle. In like manner, *D* is where the shadow end passes outside the circle in the afternoon. The meridian is then the bisector of angle *DAH*.

Concerning the Arkand, see Section 27; for Pulisa, Section 32.

Vijayanandin (or Vijāyananda) of Benares (fl. 966) the author of the *Karaṇatilaka*, is not to be confused with the earlier astronomer having the same name referred to by Varāhamihira in *Pañca.*, XVII, 62 (see *Vijayanandin, biogr.*, also Section 108 below).

The "fish-shaped figure" is the narrow, pointed shape bounded by two circular arcs which results from the standard construction for bisecting an angle. There is none in Figure 30, but two appear in Figure 33.

(The Indian circle was described first in the Pauliśasiddhānta (*Pañca.*, IV, 19), then in Mahābhāskariya 3,2, and in the Uttarakhaṇḍa of the *Khaṇḍ.* D.P.)

81. Suitable Radii for the Indian Circle (107:9 - 109:3)

It was considered well to make the circle sufficiently large that the longest shadow encountered at any season in any of the inhabited portions of the globe would still cut its periphery. The longest noon shadow at any locality occurs at the winter solstice, when $\delta = -\varepsilon$. So the condition to be satisfied is (see Figure C2, for example) that

$$\text{the radius} > s_n = \text{Cot}_g(\min h_n) = \text{Cot}_g(\bar{\phi} - \varepsilon).$$

Bīrūnī correctly reports Ptolemy's $\phi = 63°$ for Thule (107:13), see *Almagest* 2,6, vol.1, p.78). He says that

$$\min h_n = 27° - \varepsilon = 3 + \frac{1}{4} + \frac{1}{6} = 3;25°,$$

whence

$$\varepsilon = 27° - 3;25° = 23;35°,$$

which is indeed the value he accepts (*Taḥdīd*, comm., p.53).

Now

(107:16) $\text{Cot}_{12}\ 3;25° = 201\frac{1}{4} = 3,21;15 = 12 \times 16;46,15 \approx 12 \times 16\frac{3}{4}$

the accurate value being

$$3,20;59,43.$$

Apparently this is too big to be practical in more normal latitudes, and Bīrūnī tries a maximum $\phi=48°$. Then min $h_n = \phi - \varepsilon = 42° - 23;35° = 18;25° = 18 + \frac{1}{4} + \frac{1}{6}$ as in 108:4, and

$$\text{Cot}_{12}\ 18;25° = 36\frac{3}{10} = 36;18 \approx 3 \times 12.$$

whereas the accurate value is 36;2,18.

He concludes that a circle with a radius of four gnomon lengths will be ample (108:7).

The author's information about the location of the Volga Bulgars seems to have come from an ambassador from these people who presented himself at the court of Sulṭān Maḥmūd (*Taḥdīd*, comm., p.81). If $\phi = 45° = \bar{\phi}$ (108:10), min $h_n = \phi - \varepsilon = 45° - 23;35° = 21,25°$, and cot $21;25° = 2;32,58 \approx 2\frac{1}{2}$, so a radius three times the gnomon would be ample.

The reference to Abū Bakr b. al-Farrukhān (see Section 62) is obscure. The only other time Bīrūnī mentioned him (at 108:11) was in connection with a gnomon erected on a sphere , and there was nothing about gnomon lengths associated with particular localities.

82. Two Techniques for Finding the Meridian (109:4 - 111:12)

The chapter closes with descriptions of two methods of applying the principles previously expounded. They can be grasped most easily by use of our Figures 31 and 32, which are drastic adaptations of the extremely crude diagrams bearing the same numbers in the text.

The first consists essentially of a Heronic diopter (see *Heron*, *biogr.*) mounted so that it is free to rotate in a horizontal plane and having a plumb line attached to one end. The sights are used to fix the rising and setting directions of the sun, *AK* and *AT*. Bisection of the resulting angle gives the meridian.

The reading of the second description, which commences at 110:1, offers several difficulties, but Figure 32 offers a reasonable interpretation. The use of balances in related operations is attested (see, e.g., *Schmidt*, p.68). We assume the tongue to be horizontal and normal to the balance arm, although this is certainly

not the case with ordinary balances. Choosing the two times when the shadow of the tongue bisects the upright insures equal solar altitudes at the two times. The bisector of angle *TFO* or *TAK* gives the meridian. Here the sides of the angle bisected are respectively normal to the sides in the preceding method, but the bisector is the same in both cases.

Should the solar azimuth be due east (111:11) when the shadow of the tongue bisects the upright, *TA* and *CO* will be parallel. But then both are along the meridian.

CHAPTER 19

CORRECTION OF THE INDIAN CIRCLE METHOD

83. Source of the Error - Its Approximate Maximum (112:3-18)

As Bīrūnī indicates, the method of the Indian circle involves measuring two azimuths at which the sun crosses a particular almucantar, once in the forenoon and once in the afternoon. If the almucantar chosen happens to be in the horizon, then the method consists simply of bisecting the arc on the horizon between the rising and setting points on a particular day.

In general this involves a small error, because in the course of one day the solar declination varies slightly, hence the day's path will diverge in a spiral from the parallel of declination implicit in the Indian circle technique.

Clearly the error is proportional to the rate of change of the declination, and this rate is maximum at the equinoxes and zero at the solstices. For a day and locality at which a solstice happens to occur at noon there will be no error at all. For whatever value the declination may have at the forenoon observation it will have returned to it by that of the afternoon.

The fifth of a degree max δ mentioned at 112:18 is readily explained as follows: At an equinox, for $\varepsilon = 23;35°$, and for $\Delta\lambda = 1°$, $\Delta\lambda = 0;24,0°$ (*Battānī, zīj*, vol.2, p.57). But in a civil day the sun does not travel a degree, but about 0;59,8°, so for this time

$$\Delta\lambda = 0;24,0 \times 0;59,8 = 0;23,39°.$$

At an equinox, sunrise and sunset are half a

day apart, so for this span

$$\Delta\delta = \frac{1}{2} \times 0;23,39° = 0;11,50° \approx 0;12° = \frac{1°}{5}.$$

84. Seasonal and Altitude Effects (112:9 - 113:11)

Just after the vernal equinox and just before the autumnal the change in declination ($|\Delta\delta| \approx |\dot{\delta}|$, see Figure C3) will still be near its maximum while the daylight length will be rapidly increasing beyond its mean of twelve hours. Hence in these regions the accumulated $\Delta\delta$ from sunrise to sunset will tend to exceed the 1/5° arrived at in the section above. As the summer solstice approaches the $\dot{\delta}$ will eventually decrease to the point where it damps out the effect of daylight increase.

Conversely, just before the vernal equinox and just after the autumnal the length of daylight will be less than twelve hours, so that in spite of the near maximum $|\dot{\delta}|$ the $\Delta\delta$ between sunrise and sunset will be less than 1/5°.

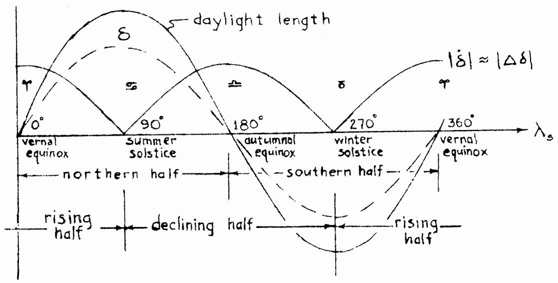


Figure C3

If an almucantar is used which is smaller than the horizon, the $\Delta\delta$ will be decreased, other things being equal, because there will be less time for the variation to occur.

85. Direction of the Error (113:12 - 114:3)

So long as the solar declination is increasing, from the summer to the winter solstice, the sun's setting point will shift to the south of where it would be if $\Delta\delta$ were zero. Hence the error in the south point as determined by the Indian circle will then be to the east. In the other half of the year, from winter to summer solstices, the declination will be increasing and the error in the south point will be to the west.

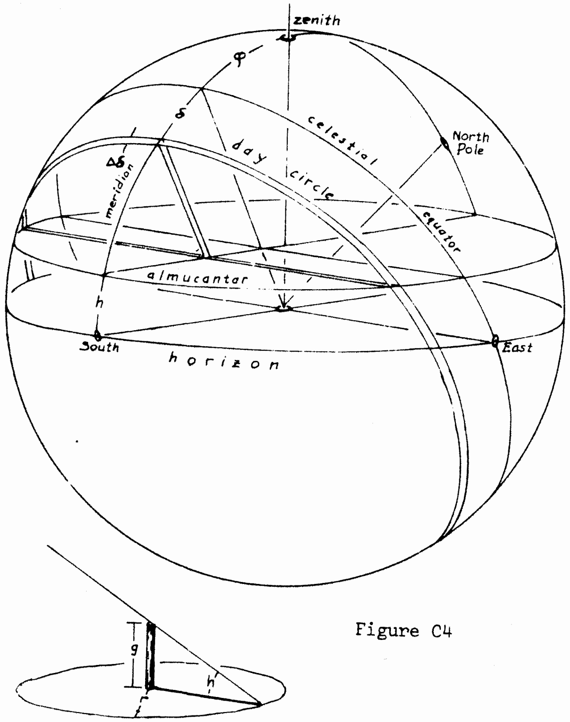


Figure C4

The author gives the dimension of the horizontal ring, fifteen cubits ($\approx$ 8.1 meters), used for his Khwārazm observations. This enables us to date the reference to about 995, since he describes the operation there in detail in the *Taḥdīd*(text, p.109; comm., p.49), where the same dimension is given.

86. Size of the Error

Preparatory to examining the rule from the non-extant Puliśasiddhānta, it is useful to set up and solve the problem involved, an estimate of the correction needed for an Indian circle meridian determination. The situation is shown in Figure C4, where h is the solar altitude at which the end of the gnomon shadow crosses the Indian circle, δ is the solar declination for the forenoon observation, and $\Delta\delta$ its change between the two observations. The required correction is either one of the two arcs on the almucantar of altitude h intercepted between the pair of nearby day circles which define $\Delta\delta$. Note that $\Delta\delta$ is not independent of h; as the latter increases the time between the two observations decreases, thus cutting down $\Delta\delta$ proportionately.

Precise relations between these quantities are obtained by means of the analemma (see Section 90) of Figure C5. If g is the gnomon length and r the radius of the Indian circle, then

$$\tan h = g/r .$$

For the analemma, choose the scale in such fashion that both $g < 1$ and $r < 1$. The entire construction then lies within the unit circle. On it *HH'*, *AA'*, *EE'*, and *DD'* are the projections on the meridian plane of the horizon, the almucantar, the equator, and the two daily circles respectively. Then the horizontal distance between the last named, marked "correction" on the analemma, is also the projection on the meridian plane of the desired corrective arc, T_3, on the Indian circle. To obtain an expression for it, we recall that $\Delta\delta$ is small, of the order of a fifth of a degree or less. Hence there will be very little error involved in identifying the chord of $\Delta\delta$ with its arc length. Then, treating the small triangles marked T_1, T_2, and T_3 (magnified below

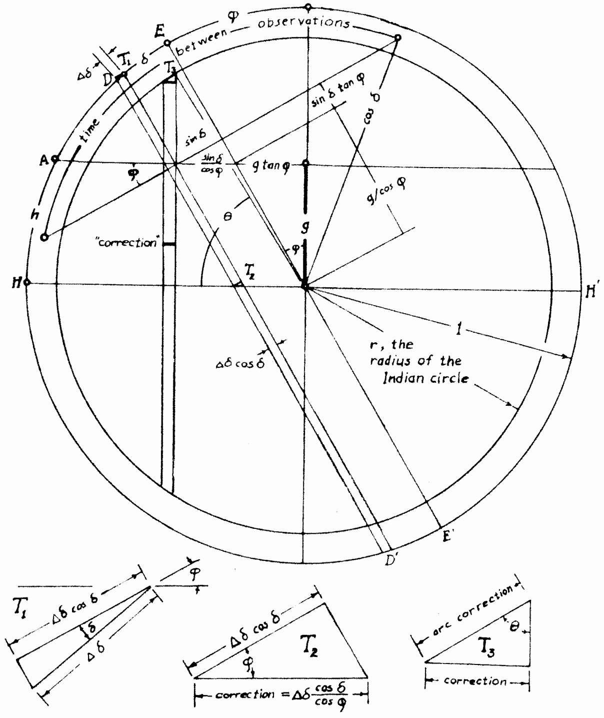


Figure C5

the main drawing) as plane figures, we have that the

$$\text{"correction"} \approx \Delta\delta \; \frac{\cos\delta}{\cos\phi} ,$$

provided that $\Delta\delta$ is in radians so that its arc on the unit circle is $\Delta\delta$ itself. The length of the arc T_3 is

$$\frac{\text{"correction"}}{\sin\theta} = \frac{\Delta\delta \cos\delta}{\cos\phi \sin\theta} ,$$

where, as seen from Figure C5,

$$\theta = \text{arc cos}\left(\frac{\frac{\sin\delta}{\cos\phi} + g\tan\phi}{r}\right)$$

$$= \text{arc cos}\left(\frac{\sin\delta + g\sin\phi}{r\cos\phi}\right)$$

In angular measure the correction arc T_3 on the Indian circle is

$$\frac{\Delta\delta \cos\delta}{r\cos\phi \sin\theta} ,$$

which is valid regardless of the units of angular measure radians, degrees, or otherwise.

However, if degrees are used, then the linear

$$\text{"correction"} = \Delta\delta \cdot \left(\frac{\pi}{180}\right) \cdot \frac{\cos\delta}{\cos\phi} .$$

87. Special Cases

These results are for the general situation. Two special cases are of interest:

In *sphera recta*, $\phi = 0$, so the

$$\text{"correction"} = \Delta\delta \cos\delta$$

$$= \text{arc cos}\left(\frac{\sin\delta}{r}\right)$$

and the angular correction is

$$\frac{\Delta\delta \cos\delta}{r\sin\theta}$$

in any units.

If the *horizon* is used for the almucantar, $h = 0 = g$, and $r = 1$, the "correction" is unchanged, but

$$\theta = \text{arc cos}\left(\frac{\sin\delta}{\cos\phi}\right) ,$$

and the angular correction will be

$$\frac{\Delta\delta \cos\delta}{\cos\phi \sin\theta} .$$

For the very special case of *sphera recta* and *horizon* used together, $\cos\theta = \sin\delta$, so $\theta = \bar{\delta}$ and the angular correction is simply

$$\frac{\Delta\delta \cos \delta}{\sin \delta} = \Delta\delta ,$$

the change in declination between the observations.

The time between observations, the arc shown as such on Figure C5, is

$$\Delta t = 2 \text{ arc cos} \left(\frac{\sin \delta \tan \phi + g / \cos \phi}{\cos \delta}\right)$$

$$= 2 \text{ arc cos} \left(\frac{\sin \delta \sin \phi + g}{\cos \delta \cos \phi}\right).$$

88. Pulisa's Rule (114:4 - 115:10)

Having built up this apparatus, the comment upon the passage is anticlimatic — essentially we endorse Bīrūnī's own comments. The rule says Pulisa's correction is

(114:6) $$\Delta\delta \cdot \frac{m}{60} \cdot \frac{d}{R} ,$$

where m is the minutes of days between the observations, and d is the radius of the Indian circle in digits. Since the gnomon length does not enter, use of the horizon as almucantar is implicit. The fraction $m/60$, the sixtieths of a day between the observations divided by sixty, is simply the fraction of a civil day between the observations. To multiply this by the change in declination during that time would make no sense. But to multiply it by the $\Delta\delta$ in an entire day, as Bīrūnī suggests, would be to interpolate linearly for the portion of the day involved, a sensible procedure. If $\phi \neq 0$ the rule should involve it, but it does not; presumably ϕ is regarded as sufficiently small to be neglected. There remains the factor d/R. If R is taken to be the length of the radius in the units used to measure arcs on the defining circle (such as the R = 3438' = 57;18 of the *Āryabhaṭīya*), then multiplication by d/R gives a change of scale whereby the small correction arc $\Delta\delta$ is measured in the "digits", whatever they are, which measure the radius of the Indian circle.

(No rule resembling this has thus far turned up in the extant Sanskrit literature. D.P.)

CHAPTER 20

DETERMINING THE MERIDIAN BY THREE SHADOWS

89. An Approximate Method by Brahmagupta (115:13 - 116:8)

Instead of obtaining a pair of equal shadows, the objective now is to find the meridian line by using any three shadows observed during a single day.

The solution first given is illustrated in Figure 33, where A, B, and G are the end points of the three shadows. K is the intersection of the perpendicular bisectors of segments AB and GB. E being the foot of the gnomon, KE is alleged to be the desired meridian.

The construction is at best approximate, for, as Bīrūnī implies, the shadow end sweeps out a hyperbola, the axis of which is the meridian. But in general, the perpendicular bisectors of chords of a hyperbola do not intersect on its axis. Brahmagupta gives the method in *Brahmasp.* 3,2, and it is found in other Indian works.

90. The Analemma of Diodorus (116:9 - 117:9)

Aside from its intrinsic interest as an elegant example of very early descriptive geometry, this passage is useful in that it is the only extant fragment of the work of Diodorus of Alexandria, a mathematician of the first century B.C. Until this fragment turned up, about all that was known about him was that he had written a book called "Analemma", and that he was a maker of gnomons (*Heath*, vol.2, p.287).

The construction is given on Figure 34, where A, B, and G are the three shadow end points; E is the foot of the gnomon, and the radius of the circle is *g*. Points *Z*, *H*, and *T* are laid out on the circle so that angles *ZEA*, *HEB*, and *TEG* are all right angles. Arcs *OZ* and *HO*, which intersect at *O*, have centers at *A* and *B*, and radii of *AZ* and *BH* respectively. Arc *CF* has center at *O* and radius equal to *TG*. Lines *CF* and *AB* intersect at *L*. Then *LG* is an east-west line, and hence *EK*, perpendicular to it is the meridian through the foot of the gnomon.

91. Proof that the Construction Is Valid (117:10 - 119:19)

Our Figure 35 bears the letters of the cognate figure in the text. The drawing itself, however, has been adapted from one by O. Neugebauer which shows the situation in space, as that of the text does not. As in the construction, *A*, *B*, and *G* mark three positions of the shadow's end, but now the whole gnomon is shown as *EZ*. In the course of the day the ray from the sun through *Z* sweeps out a right circular cone, portions of both nappes of which are shown. The axis of the cone is *ZN*. A plane through *G* normal to *ZN* cuts the horizon plane in an east-west line, it being the object of the construction to find a point other than *G* on this line, thus determining it.

The plane just referred to cuts the lateral surface of the cone in a circle, *GFC*. The line through *F* and *C* is in this plane, hence the line's intersection with the horizon at *L* is a suitable point to determine the east-west line. The objective is to consider the triangle in Figure 35 as rotated down into the horizon about side *AB*. The rotation will not shift the location of *G*, so that if *ABZ* can be constructed to proper size in the horizon plane, together with points *F* and *C*, the location of *L* can be discovered. To this end, right triangles *ZEA*, *HEB*, and *TEG* were constructed with one leg equal to *g*, and the other the respective distances from *E* to *A*, *B*, and *G*. This insures that *ZA*, *HB*, and *TG* are the respective distances in space from the gnomon tip *Z* (in Figure 35) to *A*, *B*, and *G* respectively. Further, since the construction makes *OA* = *ZA*, and *OB* = *HB* (= *ZB* in the space drawing), triangle *OAB* in the construction is congruent with *ZAB* in the space drawing, so the rotation has been effected. On the space drawing the element segments *ZG*, *ZF*, and *ZC* are all equal, and the construction has insured that on Figure 34

TG (= *ZG* in space) = *OC* = *OF*. Therefore *F* and *C* are properly located on the construction. On it the intersection of *CF* with *AB* is the desired point *L*.

Abū Rayḥān drops vertical lines *CS* and *FM* from *C* and *F* and draws *ST* in the horizon parallel to *AB*. He then adduces

(118:8) $AC > BF$,

(118:9) $AC / CZ = AS / SE > BF / FZ = BM / ME$,

(118:11) $AS / SE = BT / TE > BM / ME$,

so that

(118:13) $BT > BM$,

the objective being to insure that *MS* must intersect *AB*. Furthermore, since *F*, *C*, *S*, and *M* are coplanar (118:18), the intersection of *MS* with *AB* is *L*, the intersection of *FC* with *AB*.

92. The Version by al-Ḍarīr

The solution with its attribution to Diodorus was first described and published as *Merid.*, 1. Upon reading it, Dr. H. Hermelink kindly pointed out to us that essentially the same problem had been presented by *Schoy*. The earlier version is by a certain Abū Saᶜīd al-Ḍarīr (the Blind) al-Jurjānī (fl. 850), about whom practically nothing is known (*Suter*, p.27). Abū Saᶜīd does not associate the method with any particular person, saying only that it is from the book "Analemma". His treatment also has two figures, but the lettering is quite different, and the order of presentation is reversed. Abū Saᶜīd first discusses the situation in space, making use of a drawing much like our Figure 35. He then proceeds to a scale construction in one plane resembling Figure 34. So if either one of the two treatments is a literal translation of the lost Greek original, the other cannot be. Both may be free renditions of Diodorus.

CHAPTER 21

THE MERIDIAN FROM A SINGLE OBSERVATION

93. The Shadow as a Function of Azimuth (120:3 - 121:3)

The *rising amplitude* (w) is defined as the arc on the horizon between the east point and the sun's rising point for that day and latitude. Bīrūnī does not say how to calculate it, but it is not difficult to show that.

$$w = \text{arc sin } (\sin \delta \, / \cos \phi)$$

Once it has been ascertained, if one observes the sun's rising azimuth with a horizontal protractor, and then simply lays off an arc of w in proper direction from this azimuth, the east-west line will be that from the center of the protractor to the terminal point of the arc just constructed. The meridian is, of course, normal to the east-west line.

In this method $h = 0$. A generalization is to assume a convenient solar azimuth and to calculate the corresponding h. Again, the author fails to describe how this is done, but in the *Taḥdīd* (see the commentary, pp.62-64) he gives and proves a rule for the computation. Now, a gnomon having been set up at the center of the horizontal protractor, describe on its plane and with center at the foot of the gnomon, a circle of radius $\text{Cot}_g h$. Mark on this circle the point at which the end of the gnomon's shadow crosses it, either in the forenoon or the afternoon. From here on the procedure is that of the special case above, the assumed azimuth being treated as was w before.

94. A Construction for Zero Azimuth (121:4-14)

The author now describes a different special case, that at the instant when the sun has no azimuth, that is, its shadow is along the east-west line. This can happen only during the half of the year when it rises to the north of the east point, i.e., when $\delta > 0$. Let h_o be the solar altitude of zero azimuth. The method at hand is a graphical construction, in the manner of an analemma, for the length of the shadow cast at this instant. Whenever the shadow has this length it will be perpendicular to the meridian.

On the horizontal protractor, arcs $BD = \phi$ and $ZG = \delta$ are laid off as shown in Figure 36. H is the point at which the parallel to AG through Z intersects DE. T is marked on AG in such fashion that $ET = EH$. From T erect a perpendicular to AG intersecting ED at K. From K a parallel is drawn to AG intersecting the protractor at M. Then arc AM is asserted to be the required h_o. In the rectangle $MLES$ the angle LSE will then be h_o. So if $OE = g$, and OF is drawn parallel to SL, FE will be the required shadow length, $\text{Cot}_g h_o$.

To use this result, draw a circle with center E and radius FE. A gnomon having been erected at E normal to the plane of the circle, at the instant when the end of the gnomon's shadow passes through the circle, the shadow will be in the east-west line.

95. Proof that the Construction Is Valid (122:1 - 123:10)

Just as was done with the Diodorus construction in Section 91, the method here is to portray the situation in space on a separate drawing (here Figure 37), showing that the elements drawn to scale on the analemma have the magnitudes required by the space configuration. On Figure 37 the day and time triangles of Section 77 reappear. Now the time is so chosen that the sun is H on the prime vertical. We note that the base of the time triangle is Sin w, and that it is the hypotenuse of a right triangle having $\bar{\phi}$ as one acute angle. But in the analemma ET was constructed thus, hence it is indeed Sin w as marked. Next, HZ = Sin h_o is one leg of a right triangle in which the angle opposite HZ is

$\bar{\phi}$ and the other leg is Sin w. But this is the manner in which triangle *ETK* was constructed in Figure 36, hence *KT* = Sin h_o and arc *MA* = h_o. Finally, in the same figure, ∠*LSE* = ∠*OFE* = h_o, so *FE* = $\text{Cot}_g h_o$, the required shadow.

This special situation is also discussed in the *Taḥdīd* (comm., p.55).

96. An Analemma for the General Case (123:11 - 125:11)

All the methods described in this chapter thus far have been applicable at special times only. Now, however, a construction is given which can be applied at any time when the sun casts a finite shadow. With trivial modifications it appears in two other works of Bīrūnī, the *Taḥdīd* (comm., p.217) and the *Canon* (vol.1, pp.450-451, see also *Merid.*, 2).

The analemma is shown in Figure 38, where *EG* has the direction and length of the shadow at the instant of the observation. *ED* = g being perpendicular to *EG*, draw *ZG* = R (the radius of the graduated circle) through *D*. Through *Z* draw *HL* parallel to *EG*. Lay off *EK* = s_n, the noon shadow, and draw *KDL*. Drop *LM* and *HT* perpendicular to *AB*, and with center *E* and radius *EM* describe a semicircle. Draw *TO* tangent to the semicircle at *O*. Then *OEC* is the required meridian.

As will be seen below, the construction is faulty in the determination of *K*. It can be repaired without much difficulty, but we see no way of restoring the text, and believe the error goes back to Bīrūnī himself. To explain the solution we make use of Figure C6.

EG being the shadow and *ED* the gnomon, the angle at $G = h = \widehat{AH}$, whence *TH* = Sin h, the altitude of the "time triangle". In the same manner, since *EK* = s_n, angle *EKD* is h_n not $\bar{\phi}$ as Bīrūnī claims at 125:1. Because of this, *MLK* cannot be the time triangle, its acute angles being ϕ and $\bar{\phi}$. To generate the time triangle in proper position as shown on Figure C6, draw *XY* (not on the text figure) so that $\widehat{YB} = \bar{\phi}$. Then draw *LK*, not as prescribed in the text, but so that its distance from *HE* is Sin δ. In general it will not pass through the end of the gnomon. Now *MLK* will be the time triangle in proper position. Comparison with Figure C6 demonstrates that its base, *MK*, is the sum of Sin w, the sine of the rising amplitude,

and *ME*, the "argument of azimuth" (125:7). Further, we note from the figure that the angle between the lines of the shadow and the meridian is the acute angle of a right triangle in which the adjacent leg is the argument of azimuth and the hypotenuse is Cos h. But triangle *TOE* has been constructed so that it satisfies these requirements. Hence *EOC* is the line of the meridian.

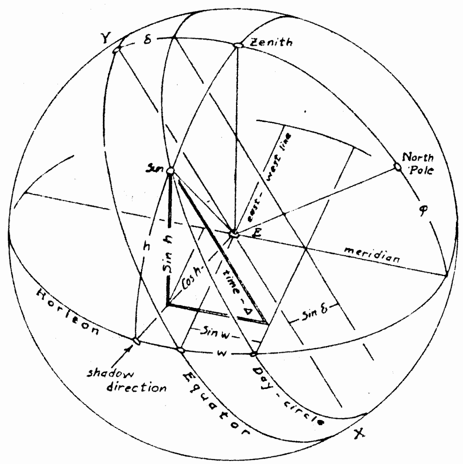


Figure C6

CHAPTER 22

DAYLIGHT LENGTH AND RISING TIMES

97. The Equation of Daylight, Indian Rules (125:14 - 126:19)

As its title indicates, this chapter deals with the length of daylight as affected by the seasons and by terrestrial latitude. The material was first studied by *Lesley*, but since the appearance of his paper additional background material has become available.

We will indicate by S_ϕ the "shadow of Aries" (126:6), the equinoctial S_n. By definition (cf. Section 65),

$$S_\phi = \text{Cot}_g \bar{\phi} = \text{Tan}_g \phi = g \tan \phi .$$

The radius of the day circle, *KS* in Figure C7, is Cos δ, and we will designate it as such, using no special notation. Usually the word *istiwā'* means "terrestrial equator"; here, however, *ẓill al-istiwā'* means S_ϕ and we translate it as "equatorial shadow".

For the equation of (half the) daylight, in Arabic *taʿdīl (niṣf) al-nahār*, the symbol *e* will be used. It is the amount by which half the arc of daylight exceeds a quadrant. For the situation shown in Figure C7 *e* is negative. An equivalent definition is to say that *e* is half the amount by which the time from sunrise to sunset exceeds twelve hours. It may be measured in hours, or in "time-degrees" (Arabic *azmān*) of daily rotation, $360° = 24^h$, or in analogous units.

Bīrūnī first quotes a rule of Brahmagupta (see Section 32). It amounts to calculating first

(126:10) $$\text{Cos}_R \delta = \sqrt{R^2 - \text{Sin}_R^2 \delta} ,$$

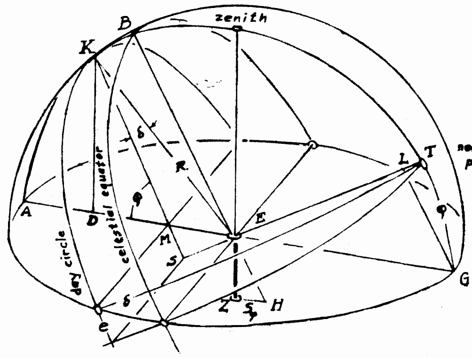


Figure C7

and then

(126:12) $$\text{Sin}_R e = (\text{Sin}_R \delta \cdot S_\phi / 12) \cdot R / \text{Cos}_R \delta .$$

This is equivalent to

$$R \cdot (\text{Sin}_R \delta / \text{Cos}_R \delta) \cdot (\text{Tan}_g \phi / g)$$

$$= R \cdot \tan \delta \cdot \tan \phi ,$$

or, as we would put it,

$$\sin e = \tan \delta \cdot \tan \phi .$$

As shown in the next section, this is correct.

To obtain *e*, it is necessary to find the arc sine of the expression on the right. Indian sine tables tended to give the argument in minutes. Hence the result would be in these units, a sixtieth of a three hundred and sixtieth of a civil day (twenty-four hours). In Sanskrit each such unit is a *prāṇa*, a respiration. Larger units are the *vināḍī* and the *ghaṭī*, defined as follows:

1 *vināḍī* = 6 *prāṇas* = 0;0,1 of a civil day, a "day second".

1 *ghaṭī* = 60 *vināḍīs* = 0;1 of a civil day, a "day minute".

A variant of Brahmagupta's rule by Vijayanandin (see Section 79) is

(126:17) $\text{Sin } e = [(S_\phi \cdot \text{Sin } \delta) / (\text{Cos } \delta \cdot g)] \cdot R$,

which involves only a shift in the order of the operations.

(The radius of the small circle of the star (sun) is known in Sanskrit as the *dyujyā* (day-sine), and is defined as by Bīrūnī in *Pañca.* 4, 23; *Āryabhaṭīya*, Golapāda 24; Mahābhāskarīya 3, 6; and *Brahmasp.* 2, 56 and 3, 16. The ratio mentioned by Bīrūnī — sine of latitude to cosine of latitude — appears in *Sūryasiddhānta*, 2, 6, but the more usual Indian practice is to employ the equivalent ratio of the equinoctial shadow to the gnomon; this ratio when multiplied by the sine of the sun's declination produces the *kṣitijyā* (earth-sine), as is stated in *Āryabhaṭīya*, Golapada 26; Mahābhāskarīya 3, 6; and *Brahmasp*, 2, 57. The product of the *kṣitijyā* and R divided by the *dyujyā* is the *carajyā* (sine of ascensional difference) in *prāṇas*; see *Pañca.* 4, 26; Mahābhāskarīya 3, 7; and *Brahmasp.* 2, 57-58 (the whole process bears some relation to a method of the Pauliśasiddhānta described in *Pañca.* 4, 45-47. D.P.)

98. Proof that the Rules Are Valid (127:1-15)

Bīrūnī illustrates his remarks with Figure 39, showing a configuration in the plane of the meridian only. To ease the task of the reader, we have embedded the same representation in the space drawing of Figure C7. Note, however, that whereas in Figure 39 the length of daylight is shown as exceeding twelve hours, in C7 it is less than twelve hours.

On either figure

(127:10) $MS / SE(= \text{Sin}_R \delta) = HZ(= S_\phi) / ZE(= g)$,

by similar triangles. Now MS is the sine of the desired e, provided the radius of the defining circle is that of the day circle. In order to change the scale, the author puts

(127:12) $SK(= \text{Cos}_R \delta) / SM(= \text{Sin}_{SK} e) = R(= BE) / \text{Sin}_R\, e$.

Solving expression 127:10 for SM we have

$$SM = S_\phi \cdot \text{Sin}_R \delta / g \ .$$

Substituting this in expression 127:12 and solving the latter for $\text{Sin}_R e$ there results

$$\text{Sin}_R e = R \cdot S_\phi \cdot \text{Sin}_R \delta / (g \cdot \text{Cos}_R \delta) ,$$

which is the rule. After introducing modern trigonometric functions and cancelling parameters, this reduces to

$$e = \text{arc sin } (\tan \phi \tan \delta)$$

as remarked above.

99. A Variant Rule by Yaʿqūb b. Ṭāriq (127:16 - 129:1)

This passage has the second mention of the book "On Causes" by Yaʿqūb (see Sections 39, 60, and 102), here with a quotation. The rule is

(127:18) $\text{Cos } \delta = 3438 - \text{Vers } \delta$,

and then

(127:19) $\text{Sin } e = (\text{Sin } \delta \cdot S_\phi / g) \cdot 3438 / \text{Cos } \delta$.

The truth of the first expression above is immediate if we recall that in the *Āryabhaṭiya* $R = 3438' = 57;18$, and the identity $\text{vers } \theta = 1 - \cos \theta$. By the same token the second expression is the same as the rule of Brahmagupta, equation 126:12 in Section 95 above, except for the change of the parameter R.

The Sanskrit units are also the same as in that section. However, a synonym for *vināḍī* is introduced, *caṣaka* (pronounced *chashaka*) transliterated in the text as *jashaha*. A Northwest Indian pronounciation of this is *cakha*, where *kh* is an aspirated gutteral. This is the text's *jakaha* (129:1).

Yaʿqūb also makes use of technical terms characteristic of the earliest Arabic mathematical writings. These words dropped out of standard medieval usage. Examples are "straight chord" for sine and "reversed chord" for versed sine. (These are literal translations of Sanskrit *kramajyā* and *utkramajyā* respectively. D.P.) Another example is Arabic *ṭawq*, here translated as "hoop". The same word appears in the *Taḥdīd* (comm., p.147) and in the *India* (transl., vol.1, p.312).

100. A Table of al-Khwārizmī (129:2-5, 130:10-18)

The equation of daylight function, e, is useful in other ways than the calculation of daylight lengths. The points of the ecliptic may be mapped upon the celestial equator in many ways. When the mapping is along great circle arcs orthogonal to the equator (ϕ=0), the resulting function is called ***right ascension*** (α). When the projecting arcs make a constant oblique angle $\bar{\phi} < 90°$, with the equator the mapping is called an ***oblique ascension*** (α_ϕ). Tables of the latter were widely used in medieval astronomy. If follows from the definitions above that for a given ϕ and a given ecliptic point (λ), hence a fixed solar declination (δ), the equation of daylight is the difference between the two varieties of ascensions:

$$\alpha_\phi(\lambda) = \alpha(\lambda) - e(\phi, \delta(\lambda)).$$

Recalling from expression 126:12 above that

$$\mathrm{Sin}_R e = S_\phi \cdot \frac{R}{g} \cdot \frac{\mathrm{Sin}_R \delta}{\mathrm{Cos}_R \delta} = S_\phi \cdot \frac{R}{g} \tan \delta,$$

we note that the first element on the right, S_ϕ, depends upon ϕ, the locality only, whereas the second depends upon δ only, i.e., the season. Suppose that an individual is in possession of a table of the function

$$c(\lambda) = \frac{R}{g} \tan \delta(\lambda) \; ,$$

called the "ascensional difference" (129:2), as well as a table of right ascensions. He can quickly calculate an oblique ascension for his locality by looking up the value of $c(\lambda)$, multiplying it by S_ϕ for his locality, and subtracting algebraically the arc sine of the result from the proper $\alpha(\lambda)$.

Bīrūnī here informs us that such a table was in al-Khwārizmī's zīj, although it has disappeared from the extant and published Latin translation of the

recension by al-Majrīṭī. Furthermore, this passage has led to the identification of the table itself with one found in a Newminster manuscript written in 1428. (Cf. Al-*Khwārizmī*, *Zīj*, transl., pp.50-55). Other tables of ascensional differences have been noted in the literature (*Lesley*, p.127; *King*, 4).

Of course the values of R and g will affect the values of the function c. For Khwārizmī R = 150' = 2;30 and g = 12 . So

$$R / g = 2\tfrac{1}{2} / 12 = 5/24 = 5 \cdot 0;2,30 = 5 \cdot 150'' = 1/4\tfrac{4}{5},$$

and this number must be multiplied by the $\tan\delta = \mathrm{Sin}\delta / \mathrm{Cos}\,\delta$. Bīrūnī rightly says at 130:16 that $\tan\delta$ is to be divided by $4\frac{4}{5}$, but in the next sentence he seems to have forgotten a factor of five. He should have said "Division by four and four fifths is the taking of five parts of a part of twenty-four parts". Likewise the factor five has been omitted in a preceding passage. The end of the sentence at 129:5 should read, e.g., "...its declination, multiplied by ***five times*** a hundred and fifty seconds".

For the units current in Bīrūnī's own time

(130:10) $$R / g = 60 / 12 = 5$$

to be multiplied by $\tan\delta$.

101. An Alternative Proof of the Rule (129:6 - 130:9)

This passage is essentially another demonstration of expression 126:12 (in Section 95) already proved. Now, however, the author works with entities on the surface of the sphere rather than inside it.

He first asserts that, in Figure 40,

(129:12) $$\mathrm{Sin}\, GZ / \mathrm{Sin}\, GD = \mathrm{Sin}\, EB / \mathrm{Sin}\, BT.$$

This does not seem obvious, but it can be proved by invoking the ***Rule of Four***. This theorem states that in any two right spherical triangles having a pair of acute angles equal (say $A = A'$),

$$\sin a / \sin c = \sin a' / \sin c'$$

and

$$\tan a / \sin b = \tan a' / \sin b'$$

where capital letters denote angles, the cognate small letters designate the sides opposite them, and $C = C' = 90°$ (see the *Overview*).

Two applications of this rule, to the triangles *AGZ* and *AGD*, give

$$\text{Sin } GZ / \text{Sin } EB = \text{Sin } AG / \text{Sin } AB = \text{Sin } GD / \text{Sin } BT ,$$

from which the above expression follows.

Next it is stated that

(129:13) $\text{Sin } ZG / \text{Sin } GD = g / S_\phi$.

Calling *ZG* "the altitude" is a reminder that it is the equinoctial h_n. So the expression says only that

$$\sin \bar{\phi} / \sin \phi = \cos \phi / \sin \phi = \frac{g}{g \tan \phi} = \frac{g}{\text{Tan}_g \phi} = \frac{g}{S_\phi} .$$

Now, by combining 129:12 and 129:13,

(129:14) $\text{Sin } EB / \text{Sin } BT = g / S_\phi$.

Application of the Rule of Four to triangle *DAE* gives

(129:16) $\text{Sin } BT / \text{Sin } BD = \text{Sin } AE / \text{Sin } (DE = 90°)$.

These two expressions may be written as

(130:1) $S_\phi / \text{Sin } BT = g / \text{Sin } BE$,

and

$$\text{Sin } BT / \text{Sin } AE = \text{Sin } DB / \text{Sin } (ED = 90°).$$

Multiplying these two proportions together gives, provided $g = R$,

$$S_\phi / \text{Sin } (AE = e) = \text{Sin } (DB = \delta) / \text{Sin } (BE = \delta),$$

or

(130:4) $\text{Sin } e = S_\phi \cdot \text{Sin } \delta / \text{Cos } \delta$,

the special case of 126:12 when the radius of the defining circle equals the gnomon length.

102. Yaʿqūb's Rule Using Right Ascensions (131:1 - 132:8)

Now comes another excerpt from the *Kitāb al-ʿIlal*. This is a rule for finding the equation of daylight without using declinations. It is stated in terms of the

specific signs Aries and Virgo, but the intent is general. Put into symbolic form it says

(131:2) $\text{Sin } e = S_\phi \cdot \text{Sin } \alpha / S_{\bar{\varepsilon}} = S_\phi \cdot \text{Sin } \alpha / 26;58$,

which is equivalent to

$$\text{Sin } e = \text{Tan}_{12}\phi \cdot \text{Sin } \alpha / \text{Tan}_{12}\bar{\varepsilon} = R \cdot \text{Sin } \alpha \cdot \tan \phi / \cot \varepsilon.$$

In point of fact the text says S_ε, not $S_{\bar{\varepsilon}}$, but the latter is correct, as will be seen below, and intended, for the number at 131:4 is $26;58 \approx \text{Cot}_{12} 24° = \text{Tan } \bar{\varepsilon}$, not Tan ε. A more precise value is 26;57,9.

Bīrūnī's alleged proof is incomplete. Using Figure 41 he asserts that

(132:2) $\text{Sin } AE / \text{Sin } AZ = \text{Tan } BE / \text{Tan } ZG$,

and $\text{Sin } AZ / \text{Sin } AM = \text{Tan } ZG / \text{Tan } LM$,

both of which are valid applications of the Rule of Four (see Section 101 above) to triangle *AZG*.

Multiplying together the above two expressions gives

(132:4) $\text{Sin}(AE{=}e) / \text{Sin}(AM{=}e_{\text{solst.}}) = \text{Tan}(EB{=}\delta) / \text{Tan}(LM{=}\varepsilon)$.

Another application of the Rule of Four, now to triangle *OKH* yields

(132:6) $\text{Tan } EB / \text{Tan}(HK{=}LM{=}\varepsilon) = \text{Sin}(OE{=}\alpha) / \text{Sin}(OK{=}90°)$,

which Bīrūnī seems to think settles the matter. This is by no means the case. It is equivalent to the equation

$$\tan \delta / \tan \varepsilon = \sin \alpha ,$$

or

$$\tan \delta = \sin \alpha \cdot \tan \varepsilon .$$

We employ it to complete the proof, noting that the first expression marked 132:2 above is, in modern terms,

$$\sin e = \tan \delta / \tan \bar{\phi} .$$

Now substitute in this the expression just obtained for $\tan \delta$, so that

$$\sin e = \sin \alpha \cdot \tan \varepsilon / \tan \bar{\phi} = \sin \alpha \cdot \cot \bar{\varepsilon} / \cot \phi$$
$$= \tan \phi \cdot \sin \alpha / \tan \bar{\varepsilon}.$$

This is equivalent to 131:2, Yaʿqūb's rule.

Note that in this as in other passages the text says only *ẓill*, "shadow", without indicating whether the tangent or cotangent function is intended. This must be inferred from the context.

103. Rising Times - Definitions and Notation (132:9 - 133:8)

We will denote by $\Delta\alpha_i$, $i = 1,2,3,\ldots,12$, the time required for the i-th zodiacal sign to cross the eastern horizon. This set of functions is what Bīrūnī refers to as the "differences of risings for the heads of the signs". It is a curious fact that if the sun is at the end point of any sign, the length of daylight on that day will be the sum of the rising times for that locality of the next six signs. For this and other reasons, rising time lore played a fundamental role in ancient astronomy.

Let a_i be the rising time of the i-th sign in *sphera recta* ($\phi = 0$), and let e_i be the equation of (half) daylight for the end point of the i-th sign. Put $\Delta e_i = e_{i-1}$. Then contemplation of a suitable diagram will convince one that the identity

(1) $$\Delta\alpha_i = a_i - \Delta e_i$$

holds for all ϕ and all i's. Symmetries on the sphere insure that there are only three independent Δe's (dependent upon ϕ) and the same number of a's (independent of ϕ), so that for any locality a complete set of twelve $\Delta\alpha_i$ can be built from six numbers. The schemes Bīrūnī reports on in the second part of this chapter are made up of approximate values for numbers. In order to have a solid base for comparison he first calculates by trigonometric methods an accurate set.

He puts $\phi = 24°$ (133:1), evidently using the latitude sometimes assumed for Ujjain, the Greenwich of ancient India. Hence $S_\phi = \text{Tan}_{12} 24° = 5;21$ (133:2), the entry for this argument in his own table on f. 205b of the manuscript. A more precise determination is 5;20,33,53.

His value leads to

$$e_1 = 5;13°$$
(133:3) $$e_2 = 9;28°$$
$$e_3 = 11;12° ,$$

in time degrees by use of, say, expression 131:2. He also uses

$$a_1 = 27;53°$$
$$a_2 = 29;54°$$
$$a_3 = 32;13°.$$

These right ascensional differences are not given explicitly in our text, but they are implicit, and may be verified, e.g., from al-*Battānī's zīj* (vol.2, p.62) where $\varepsilon = 23;35°$, Bīrūnī's value.

Use of definition (1) above gives successively

$$\Delta e_1 = 5;13°$$
(133:5) $$\Delta e_2 = 4;15°$$
$$\Delta e_3 = 1;44° ,$$

and

$$\Delta\alpha_1 = 22;40°$$
(133:7) $$\Delta\alpha_2 = 25;39°$$
$$\Delta\alpha_3 = 30;29° .$$

104. Rising Times in the Khaṇḍakhādyaka (133:9 - 134:7)

The first Indian scheme given is that of the *Khaṇḍ.*, which Bīrūnī seems to have in the original as well as in the Arkand translation (cf. Section 27). He says

$$\Delta e_1 = S_\phi \cdot (159/16)$$
(133:11) $$(\Delta e_2 = S_\phi \cdot (65/8))$$
$$\Delta e_3 = S_\phi \cdot (10/3) .$$

There is a lacuna in the text, so that the middle equation is missing, but the first and third are found in *Khaṇḍ.* 3, 1. The units are *palas*, a synonym for *vināḍis*, day-seconds (Section 95). Bīrūnī transliterates *pala* as *bala*, whereas the Arkand has it as *fala*, there being in Arabic no character for the sound of *p*. He notes that the *pala* is also a unit of weight of fifteen dirhams. It is indeed a weight, equal to four or eight *tulās*. Compare this information with the units in Section 27.

See also the *India*, transl., vol.1, p.165. We are unfamiliar with the Persian word for degree, *sas*(?) given at 134:1.

Substituting s_ϕ = 5;21 from 133:2 and multiplying by 0;6 to convert from day-seconds into time-degrees we obtain

$$\text{(133:15)}\qquad \Delta e_1 = 5;19^\circ \qquad \Delta e_2 = 4;21^\circ \qquad \Delta e_3 = 1;47^\circ\ ,$$

rounded off to two significant digits.

Now apply expression (1) in the section above, subtracting these Δe's from the respective right ascensional differences given in the same section to obtain

$$\text{(133:16)}\qquad \Delta\alpha_1 = 22;34^\circ \qquad \Delta\alpha_2 = 25;33^\circ \qquad \Delta\alpha_3 = 30;26^\circ\ .$$

In *Khaṇḍ*. 3, 4 itself a set of right ascensional differences is given, in *vināḍis*, in decimals, correctly reported by Bīrūnī in sexagesimals, then converted into time-degrees:

(134:4)	a_1	= 4,38 day-seconds	=	27;48°
	a_2	= 4,59 " "	=	29;54°
	a_3	= 5,23 " "	=	32;18°

105. A Persian Correction (134:8-15)

The rule given here seems to imply that the *Khaṇḍ* arrangements described above were intended for localities where s_ϕ = 5 digits. The equinoctial shadow is associated with Ujjain in the Indian literature (*Pingree*, 2, p.73). For a different s_ϕ the Δe_i are to be modified by putting

$$\Delta s = s_\phi - 5\ ,$$

then calculating

$$\text{(134:10)}\qquad c = \frac{8^h}{60}\,\Delta s = \left[\left(\tfrac{1}{5}\right)^h / 1\tfrac{1}{2}\right]\cdot\Delta s$$

and adding it to the Δe_i to be subtracted in turn from the a_i.

It looks as though the inventor of the rule felt that the linear variation of Δe_i with s_ϕ was insufficient and should be reinforced.

There is difficulty about the units intended. Bīrūnī attempts to apply the rule to his numerical example.

$$\text{(134:13)}\qquad \Delta s = 5;21-5 = 0;21 \text{ digits},$$

and

$$\text{(134:13)}\qquad c = 0;8 \times 0;21 = 0;2,48\ ,$$

presumably in hours. Now since

$$24^h = 360 \text{ time-degrees},$$

$$1^h = 15 \text{ time-degrees} = 15,0,0 \text{ time-seconds},$$

the result in time-seconds, the units of the *Khaṇḍ*., would be

$$0;2,48 \times 15,0,0 = 42,0,$$

which is impossibly large. In hours it is

$$0;2,48 \times 15 = 0;42,$$

whereas Bīrūnī says 0;0,42 at 134:14. In view of the paucity of information, it seems fruitless to attempt further investigation. The rule comes from Persian documents, probably versions of the *Khaṇḍ*.

106. The Shahriyārān Scheme (134:16 - 135:7)

Found "in some of the books" is the rule

$$\text{(134:16)}\qquad \Delta\alpha_1 = \left(278 - \frac{159}{16}\,s_\phi\right)/10 \qquad \Delta\alpha_2 = \left(299 - \frac{65}{8}\,s_\phi\right)/10 \qquad \Delta\alpha_3 = \left(323 - \frac{10}{3}\,s_\phi\right)/10\ .$$

We note that the first term inside each parenthesis makes up the set a_i from the *Khaṇḍ*. expressed in sexagesimals in 134:4. Further, the second terms are precisely the *Khaṇḍ*. set of Δe_i as in 133:11. Finally, recalling that 1 day-second = 0;6° = 1/10 time-degrees, enables us to conclude that this is exactly the *Khaṇḍ* rule, except that the results now are given in time-degrees rather than day-seconds.

If the factor 1/10 is distributed inside the parentheses the denominators of the fractions become 160, 80, and 30 (135:6). This trivial variant is the rule of the Shahriyārān zīj, a synonym for the Zīj al-Shāh (Section 39). This establishes strong dependence of at least one version of the Sassanian royal canon upon the *Khaṇḍ*.

107. Two Versions from the Commentaries (135:8 - 136:4)

This says

(135:9)
$$\Delta e_1 = s_\phi \cdot (114/105)$$
$$\Delta e_2 = s_\phi \cdot (13/16)$$
$$\Delta e_3 = s_\phi \cdot (1/3)\ ,$$

which gives for the numerical example

(135:12)
$$\Delta\alpha_1 = 27;53^\circ - 5;21 \times 1;5,11 = 27;53 - 5;49 = 22;4^\circ$$
$$\Delta\alpha_2 = 29;54^\circ - 5;21 \times 0;48,45 = 29;54 - 4;21 = 25;33^\circ$$
$$\Delta\alpha_3 = 32;13^\circ - 5;21 \times 0;20 = 32;13 - 1;47 = 30;26^\circ.$$

The sentence beginning at 135:13 we do not understand. The fraction associated with Taurus is 13/16. To make either the denominator or the numerator 100 and obtain a $\Delta\alpha_2$ = 27;41 would require a change in the other number also.

However, it is not difficult to restore all the numbers in the remainder of the paragraph. Switching digits in 105 to obtain 150, there results $\Delta\alpha_1$=27;53 - 5;21(114/150) = 27;53 - 4;3,58 ≈ 23;49 as in 135:19.

The term "diameter of heaven" for 114 means nothing to us.

The second scheme says that, assuming Δe_1 known,

(136:2) $\Delta e_2 = \Delta e_1 \cdot (9/11)$ and $\Delta e_3 = \Delta e_1 \cdot (4/11)$.

Bīrūnī accepts the *Khaṇḍ* value Δe_1 = 5;19 from 133:15 so that

(136:4)
$$\Delta e_1 = 5;19^\circ$$
$$\Delta e_2 = 5;19 \times 0;49,6 = 4;21^\circ$$
$$\Delta e_3 = 5;19 \times 0;21,49 = 1;56^\circ$$

108. Daylight Differences in Two Indian Handbooks (136:5-13)

Next are the rules found in two *karaṇas*, Sanskrit works resembling zījes. The first is the Karaṇasāra, known only from occasional mention by Bīrūnī. It was written by Vateśvara, the son of Bhadatta, in Nāgarapūra in 899. (See *India*, transl., vol.2, p.306; *Transits*, transl., p.142; *Rai*, and *Shastri*.)

The rule is

(136:6)
$$\Delta e_1 = 10s_\phi \text{ day-sec.} = s_\phi \text{ time deg.} = 5;21^\circ$$
$$\Delta e_2 = 8s_\phi \text{ day-sec.} = (8/10)s_\phi \text{ time deg.} = 4;16,48^\circ$$
$$\Delta e_3 = 3\tfrac{1}{3}s_\phi \text{ day-sec.} = (3\tfrac{1}{3}/10)s_\phi \text{ time deg.} = 1;47^\circ$$

Note that Vateśvara's results are very close to those of the *Khaṇḍ*. (The numbers he gives are those adopted also by the second Bhāskara in the twelfth century (Siddhāntaśiromaṇi, Grahagaṇita 2,50). D.P.)

The other document is the *Karaṇatilaka*, mentioned in Section 80. Its rule is

(136:8)
$$2\Delta e_1 = 20s_\phi$$
$$2\Delta e_2 = 16s_\phi$$
$$2\Delta e_3 = 7s_\phi$$

All the other sets of numbers are given in terms of the differences of half the daylight differences; these are for the whole daylight differences. To make them compatible they must be halved. Then the only difference between these and the Karaṇasāra set is for Δe_3, where, as Bīrūnī says,

(136:13) $$7/2 - 3\frac{1}{3} = \frac{1}{6}\ .$$

The same set of numbers is attested in Bīrūnī's Arabic translation of the *Karaṇatilaka* (*Islamic Culture*, vol.18 (1964), p.206).

Converting to time-degrees as before

(136:12) $\Delta e_3 = (\frac{7}{2})s_\phi$ day-sec. $= (\frac{7}{2})(\frac{1}{10}) \times 5;21 = 1;52,21°$.

The Sanskrit word ***tilaka*** means "a caste-mark pasted on the forehead". It can hardly be put into Arabic as a single word, and was translated as *ghurra*, meaning among other things: a blaze upon a horse's forehead, whiteness of complexion, the best, a choice part.

(Vijayanandin's numbers are close to those of the original Pauliśasiddhānta, which had 20, $16\frac{1}{2}$, and 6 (*Pañca.*, 3, 10-12). D.P.)

It is claimed that the units of the result are *ghaṭīs*, in the text transliterated into Arabic characters as كهرى, or perhaps Persian گهرى. This may have been a misreading of Sanskrit *vighaṭī*, a day-second, for *ghaṭī* is a day-minute.

In this connection, Mr. Alain Brieux has shown us the plate of an astrolabe, in the center of which appears the value of ϕ for which it has been laid out. Beside this is the maximum length of daylight for this latitude given as 12;48 hours. Under both is the inscription كهرى in Arabic characters, followed by the Persian plural termination ها and the number 33. That the latter is day-minutes is easily verified by noting that

$$12;48^h \times 2;30 \text{ hrs./day-min.} = 33 \text{ day-minutes.}$$

Thus *kaharī*(?), the Muslim version of *ghaṭī*, has appeared in a context independent of Bīrūnī. The astrolabe itself is the property of the Time Museum, Rockford, Illinois, and is of late Indian provenance. It will be published by Brieux and Maddison as "École de Lahore, No.5".

109. Another Indian Scheme (136:14 - 137:2)

This may be expressed as

(136:16)
$$\Delta e_1 = (159/160)s_\phi$$
$$\Delta e_2 = (7/10)s_\phi$$
$$\Delta e_3 = (1/3)s_\phi \quad ,$$

where the units are time-degrees. Here only the middle

number differs from the *Khaṇḍ.*, it being

(136:19) $\Delta e_2 = 0;42 \times 5;21 = 3;44,42.$

The name of the person who uses this we transliterate as Yalṭabān the Indian, but the word is not Sanskrit, and it occurs in no other known context. The man is said to have worked with chords of the circle.

The same set of numbers has turned up in another Arabic manuscript (*Hāshimī*, comm., Section 44.1), but with no indication as to origin.

110. A Linear Zig-Zag Scheme (137:3-12)

An anonymous document prescribes computing the "base"

(137:4,10) $b = s_\phi(1+\frac{1}{3}+\frac{1}{5}) = \frac{23}{15}s_\phi = 1;32 \times 5;21 = 8;12,12°$.

Twice the text says "a third and a fifth of a third", (137:3,8) but our emendation is justified by the numerical results in the text.

Then

(137:12) $\Delta\alpha_1 = 30° - b = 30° - 8;12,12° = 21;47,48°$,

and

(137:5,12) $\Delta\alpha_2 = \Delta\alpha_1 + \frac{2}{5}b = 21;47,48° + 0;24 \times 8;12,12° = 25;4,41°$,

and

(137:12) $\Delta\alpha_3 = \Delta\alpha_2 + \frac{2}{5}b = 25;4,41° + 0;24 \times 8;12,12° = 28;21,34°$.

This is an example of a doctrine found in the cuneiform sources. In it

$$\Delta\alpha_1 = 30° - b = \Delta\alpha_{12}$$
$$\Delta\alpha_2 = \Delta\alpha_1 + \Delta = \Delta\alpha_{11}$$
$$\Delta\alpha_3 = \Delta\alpha_2 + \Delta = \Delta\alpha_{10}$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta\alpha_6 = \Delta\alpha_1 + 5\Delta = \Delta\alpha_7 \ ,$$

where $\Delta = \frac{2}{3}b$ (see *Neugebauer*, 1, pp.253-255). See the discussion in Section 112 below.

111. An Anonymous Persian Rule (137:13 - 138:2)

This says simply

(137:14) $\Delta e_1 = S_\phi$ and $e_6 = -S_\phi$

(137:15) $\Delta e_2 = S_\phi - (2+\frac{1}{2}+\frac{1}{3})$ and $e_5 = -\Delta e_2$

(137:17) $\Delta e_3 = S_\phi - 3$ and $e_4 = -\Delta e_3$

For the numerical example the results will be

$\Delta\alpha_1 = 27;53° - 5;21° = 22;32°$

(137:19) $\Delta\alpha_2 = 29;54° - (5;21° - 2;50°) = 27;23°$

$\Delta\alpha_3 = 32;13° - (5;21° - 3;0°) = 29;52°$

Notice that here the Δe's are found by subtracting various amounts from S_ϕ, while the Indian rules multiply by successive fractions.

112. A Babylonian Scheme (138:3-13)

Using the notation of Section 110 above for this passage also,

(138:4,10) $b = S_\phi \cdot (25/18) = 1;23,20S = 7;25,50.$

(138:5,10) $\Delta = [60-2(30-b)]/5 = \frac{2}{5}b = 0;24 \times 7;25,50 = 2;58,20.$

So that

$\Delta\alpha_1 = 30° - b = \quad = 22;34,10°$

(138:7,11) $\Delta\alpha_2 = \Delta\alpha_1 + \Delta = 22;34,10 + 2;58,20 = 25;32,30°$

$\Delta\alpha_3 = \Delta\alpha_2 + \Delta = 25;32,30 + 2;58,20 = 28;30,50°$

$\Delta\alpha_4 = \Delta\alpha_3 + \Delta = 28;30,50 + 2;58,20 = 31;29,10°$

$\Delta\alpha_5 = \Delta\alpha_4 + \Delta = 31;29,10 + 2;58,20 = 34:27,30°$

$\Delta\alpha_6 = \Delta\alpha_5 + \Delta = 34;27,30 + 2;58,20 = 37;25,50°$

The linear zigzag approximation to rising times, here and in Section 110, is found in ancient Babylonian astronomy. Its appearance in an Islamic context, and with the name Babylon attached, is a curious example of the survival of "arithmetic" methods after the

development of trigonometry. However, the use of the equinoctial shadow to fix the rising times is not characteristic of the cuneiform sources, which relied rather on M and m, maximum and minimum lengths of daylight respectively.

It can be shown that for such systems

$$M = 2(\Delta\alpha_4 + \Delta\alpha_5 + \Delta\alpha_6) = 180° + \frac{18}{5}b,$$

and

$$m = 2(\Delta\alpha_1 + \Delta\alpha_2 + \Delta\alpha_3) = 180° - \frac{18}{5}b,$$

where M and m are measured in time degrees (see *Neugebauer* 1).

We can make no plausible conjectures as to what guided the unknown originators of these rules, beyond the general realisation that S_ϕ and M increase together.

For the rule of this section, $b = (25/18) \cdot S_\phi$, so

$$M = 180° + \frac{18}{5} \cdot \frac{25}{18} S_\phi = 180° + 5\,S_\phi,$$

which is a particularly simple transformation. But in Section 110, $b = (23/15) \cdot S_\phi$, which gives

$$M = 180° + \frac{138}{25} S_\phi.$$

For both rules the resulting $\Delta\alpha$'s in Bīrūnī's numerical example are in the neighborhood of the trigonometrically computed values.

CHAPTER 23

TIME OF DAY BY SHADOWS

113. The Pauliśasiddhānta Rule (138:16 - 140:18)

This chapter, like the preceding one, is long and of great historical interest. The author states that he will not make use of the sine function, this being a treatise on shadows (the tangent and cotangent functions). Roughly speaking, the chapter divides itself into two parts. Approximate arithmetical rules are described in the first, accurate trigonometric methods in the second. The material was first studied by *Davidian*.

The first rule given is

(139:8) $$t = 6L / (S - S_n + 12) ,$$

where L is the length of daylight in day-minutes, S is the shadow in digits at the time of the observation, and t is the time in day-minutes since sunrise (if in the forenoon), or until sunset (if in the afternoon).

The rationale is primitive indeed, consisting of four arithmetic operations defining a function which has correct values at its middle and end points and which is continuous in between. At sunrise and sunset S is infinite, so $t = 0$. At noon $S = S_n$, the denominator becomes 12, and $t = L / 2$; half the daylight has elapsed.

The rule may be from the original Pauliśasiddhānta, and is to be found in *Pañca.* IV,48.

In 139:13 - 140:18 Bīrūnī seeks to explain how the method was worked out. It is true that to convert to time-degrees from day-minutes one multiplies by six, and he thinks that this is the reason for the factor of six in the numerator. The twelve in the denominator, he feels, is not from the twelve-digit gnomon, but from the same six, multiplied by two because of the doubling of the noon half length of daylight.

However, in the next section a rule from a different source is given in which the gnomon length appears explicitly as an addition to the denominator. Hence we feel that this is the case here also. If this is so, then a multiple of six in the numerator is the only factor there which will force a correct answer at noon.

The rest of the discussion is plausible enough. The author presumes that the inventor of the rule hypothesized an inverse relation between shadow length and time since sunrise. More specifically,

(140:10) $$\frac{t}{L/2} = \frac{S_n}{S} .$$

But, realizing that for some times and places $S_n = 0$, he chose to operate with shadow differences rather than shadows themselves to obviate the breakdown of the rule under such circumstances.

114. A Rule of Brahmagupta (140:19 - 141:10)

In *Brāhmasp.* 12(not 13,) 52 is a verbal statement which is expressible as

(141:1) $$t = \frac{L/2}{S+1} ,$$

the inverse of which is

(141:4) $$S = \frac{L/2}{t} - 1 .$$

In the Sanskrit it is clear that the gnomon length is to be taken as unity, so Bīrūnī's suspicions about the translation were well founded.

In order that this expression give a time of 1/2 at noon it is necessary that $S_n = 0$. That is, it required that the sun culminate in the zenith, an event which at many places never occurs, and at localities where it does, it happens only twice a year. So the rule is quite bad.

Pṛthūdkasvāmin wrote a commentary, partially extant, to the *Brāhmasp.* at Kurukṣetra shortly before 864. (*Pingree, 3* and *4*).

115. An Early Abbasid Translation (141:11 - 142:6)

The rule given in this passage may be expressed as

(141:14) $t = (2/5)\cdot(6L)/(S + 12 - S_n)$.

Bīrūnī's comment is clear and complete. Observe that if the rule of 139:8 is multiplied by two fifths the present expression will result. But since 60 day-minutes = 24 hours, multiplying minutes of the day by 2/3 converts them into hours, and these are the units yielded by this rule. The text's *ghūlijāt* is a strange transliteration of *ghaṭikā*, like *ghaṭī* a word for day-minute. The text says *ghūlijāt niṣf al-nahār*, "the *ghūlijāt* of half the day,(or of noon)" which would seem to imply $L/2$ rather than L, but this must be a slip.

The Abbasid dynasty came to power in 749, and under its patronage the first serious study of science by Muslims was commenced.

116. Shadow Lore in Arabic Verse (142:7 - 146:4)

The Arabic word Sindhind is a corruption of Sanskrit *siddhānta*. The name was attached to a category of zījes based ultimately on the *Brahmasp.* These were among the earliest scientific treatises in Arabic, and they continued to be used by some devotees after the superior Hellenistic tradition represented by the *Almagest* had penetrated into the Islamic world.

It is true that Sanskrit scientific works were composed in verse, and the doggerel Bīrūnī passes on is alleged to be in imitation. However the verse form used in Sanskrit is not the *śloka*. Nowadays the word is popularly used for poetry in general, and perhaps this was the case also in Bīrūnī's time. Apparently both passages are from the astronomical ode of al-Fazārī (Section 39), and are the only extant fragments of this work. He is supposed to have written also a book called the Great Sindhind (see *Fazārī* and *Ya'qūb*).

The poesy can be compressed into the formula

(142:12, 143;12) $t = 72 / (S - S_n + 12)$,

for the time in unequal hours from sunrise to the observation, or from the observation to sunset.

An unequal hour (*al-sā'at al-mu'awaja*, lit. crooked hour, 145:1) or a temporal hour (*al-sā'at al-zamāniya*) is one twelfth of the time from sunrise to sunset. The unequal hours vary with the season and the geographical latitude.

At noon $S = S_n$ and the expression above becomes $t = 72/12 = 6$. At sunrise and sunset $t = 0$, so this rule, like the one at 139:8, returns correct results at these times of day, but at no other. In fact, except for the units, they are the same, for the L of 139:8 is the daylight length, which is twelve unequal hours. Substitution of twelve for L in the first gives the second.

Figure 42 is simply a geometric representation of the calculations involved in using the rule. The technique is that of the "gnomon" of Greek geometrical algebra (*Heath*, vol.1, pp. 78, 151, 379). Since the construction

(145:9) rectangle GH = rectangle MH = 72,

BE = rectangle $MH / KH = 72 / (S - S_n + 12) = t$.

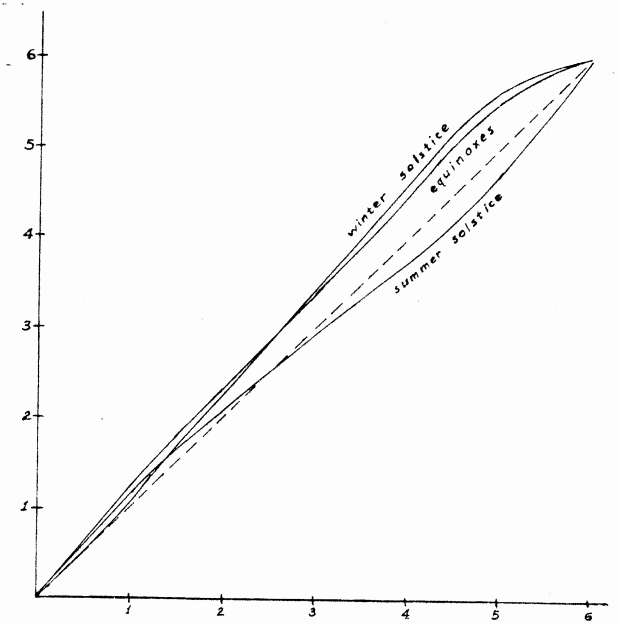
The passage beginning at 146:1 is simply a device for avoiding a fractional result. If $S > 72$, then calculate $60 \times 72 / S$ to obtain the time in minutes of unequal hours. The procedure should have been,if $S - S_n + 12 > 72$, calculate $60 \times 72 / (S - S_n + 12)$. The presumption is that less than an hour has elapsed since sunrise.

117. Precision of the Approximate Schemes

Figure C8 exhibits graphically the accuracy of the arithmetical rule

$$t = 72 / (S - S_n + 12)$$

For $\phi = 24°$ and three cases, $\delta = \varepsilon, 0,$ and $-\varepsilon$, an analemma was used to determine graphically the denominator of the fraction above for each of the six unequal



Plot of $72/(s-s_n+12)$ *against unequal hours,* $\varphi = 24°$.

Figure C8

hours from sunrise to noon. The result of the division, *t*, was plotted to the same scale as the hours, so that a measure of the accuracy of the various cases is the divergence between the straight diagonal shown dotted and the various curves.

The best fit is exhibited by the summer solsticial curve, probably because then, for the ϕ chosen, the sun

culminates very near the zenith. Indeed, for so crude an arrangement, the results are quite good, being exact at about $2;30^h$ after sunrise and with a maximum error of about a quarter of an hour. For both of the other assumed situations the results are much worse. The calculated results invariably lead the actual time except at the end points, and for the winter solstice the maximum error attains about three quarters of an unequal hour.

118. Time from Shadow Length Directly (158:10 - 159:7)

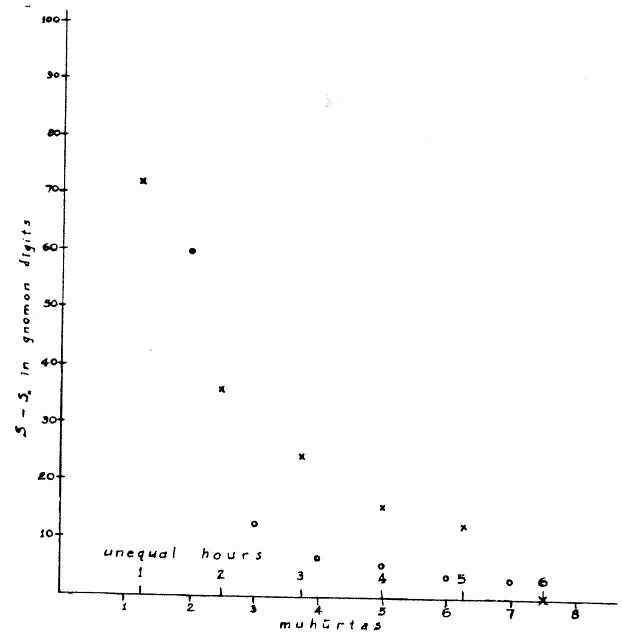
This displaced passage in the edition gives two very primitive timekeeping arrangements of unknown origin. The first is displayed below in a table such as Bīrūnī has in the text for the second:

Unequal hours since sunrise or before sunset	1	2	3	4	5	6
$s - s_n - 12$	60	24	12	2;50	0	
$s - s_n$	72	36	24	14;50	12	0

Both have been plotted in Figure C9, to the same scale, although no gnomon units are given for the second. There the unit of time is a *muhūrta*, a fifteenth part of the time from sunrise to sunset. Since 15 muhūrtas = 12 unequal hours, one such hour is one and a quarter muhūrtas, as Bīrūnī says (159:7).

(This same scheme appears also in the *India* (transl., vol. 1 p. 339), and in a first or second century Buddhist text, the Sārdūlakarnāvadāna, edited by S. Mukhopadhyaya, Santiniketan, 1954, reprinted in P.L. Vaidya, Divyāvandāna, Buddhist Sanskrit Texts 20, Darbhanga, 1959. However, in the Buddhist text the numbers associated with 6 and 7 muhūrtas are 4 and 3 respectively, instead of 3 and 2 as with Bīrūnī. D. P.)

O. Neugebauer has investigated a table giving the time of day as a function of integer shadow lengths and



Primitive Shadow Schemes

Figure C9

the season. This table originated in fifth century B.C. Athens and spread to Alexandria, thence up the Nile, into medieval Greek and Latin treatises, and even as far as Ethiopia. Bīrūnī's tables seem to be independent of these. (See *Neugebauer*, *2* and *3*.)

119. The Rule from the Hārūnī Zīj (159:8-11)

Expressed symbolically, this passage says that the time in (unequal) hours from sunrise to the observation, or from the observation to sunset, is

(159:9) $t = \text{arc Sin}_{150}\ (150 \cdot \text{Csc}\ h_n / \text{Csc}\ h)$,

which, in terms of modern functions is

$$t = \text{arc sin}\ (\sin h / \sin h_n),$$

provided the unit of arc length is the ***kardaja***. In some documents, notably the *Khanḍ.* 1,30, the sine function was calculated in increments of fifteen degrees and with $R = 150'$. Each of these arcs came to be known in the early Islamic astronomy as a ***kardaja***, from Sanskrit *kra-majyā*, the taking of sines in order. Now in an hour the celestial sphere rotates fifteen degrees, hence one ***kar-daja***.

Like previous approximate rules, the expression returns correct answers at sunrise, noon, and sunset, but unles $\phi = \delta = 0$, at no other time. For $t = 0$ when $h = 0$, and at noon $h = h_n$, so $t = \text{arc Sin}_{150}\ 150 = 90° = 6$ kardajas = 6 hours.

For variant uses of the term kardaja see ***Hāshimī***.

Mention of this zīj, the Hārūnī, has been encountered nowhere else in the literature (see the ***Survey***, p. 136). If the rule is typical of the rest of the document, it is based on Indian or Sasanian techniques.

120. Derivation of the Trigonometric Expression (159:12 - 160:4, 146:4 - 147:15)

The author now kills two birds with one stone by using Figure 43 to obtain a precise rule for t, simultaneously showing that the Hārūnī rule is invalid. Our version of the figure has the same letters as that of the text, but otherwise bears little relation to it. First, by similar triangles

(146:6) $EY/EC = OE/OF$ and $EC/EK = LH/(HE=OE)$.

Multiplying these two proportions together gives

(146:8) $EY/EK = LH/OF = HS/OT$.

Hence

(146:12) $HS = EY \cdot OT/EK = \mathrm{Csc}_g h_n \; \mathrm{Vers}_\rho \; d / \mathrm{Csc}_g h$,

where ρ = Cos δ = OD, the radius of the day circle. So

(146:14) $\mathrm{Vers}_\rho r = OT - HS = \mathrm{Vers}_\rho d - \mathrm{Csc}_g h_n \; \mathrm{Vers}_\rho \; d / \mathrm{Csc}_g h$,

where r is the hour angle. Or

$\mathrm{Vers}\ r = \mathrm{Vers}\ d\ (1 - \mathrm{Csc}\ h_n / \mathrm{Csc}\ h) = \mathrm{Vers}\ d\ (1 - \mathrm{Sin}\ h / \mathrm{Sin}\ h_n)$,

with any suitable parameter. In terms of the modern functions,

$$t = d - r = d - \mathrm{arc\ vers}\ [\mathrm{vers}\ d\ (1 - \sin h / \sin h_n)]$$

This expression, or its equivalent, was well known in Islamic astronomy. See, e.g., *Nadir*. Bīrūnī derives a formula equivalent to it in the *Taḥdīd* (comm., p. 113).

Bīrūnī goes on to remark at 147:3 that the author of the Hārūnī zīj acts as though *TO* were the radius of the day circle, whereas this is only the case at an equinox, when the day circle merges into the celestial equator. Hence the Hārūnī rule in general is invalid.

121. A Rule of Yaʿqūb b. Ṭāriq (147:15 - 148:13)

The verbal rule of Yaʿqūb (see Section 39) is equivalent to putting

(147:16) $$\mathrm{Sin}\ t = \frac{1800}{\mathrm{Csc}\ h} \cdot \frac{150}{\mathrm{Sin}\ h_n}$$

Assuming that R = 150' and g = 12 and transforming into modern functions, we have

$$150' \sin t_2 = \frac{1800}{12 \csc h} \cdot \frac{150'}{150' \sin h_n} = 150' \frac{\sin h}{\sin h_n}$$

or
$$t = \mathrm{arc\ sin}\ (\sin h / \sin h_n),$$

which is the same as the Hārūnī approximation of Section 119 above. Assuming that the resulting t is in degrees, division by fifteen is the equivalent of using kardajas to obtain hours. The right answer is obtained at noon only, unless $\phi = \delta = 0$. Bīrūnī's remarks are equivalent to ours, but he again refers to Figure 43.

122. The Rule of the Shāh Zīj (148:13 - 150:8)

This passage presents us with another fragment from one version of the Sasanian royal canon. First calculate

(149:1) $1800/\mathrm{Sin}\ h (= 12 \cdot 150 / \mathrm{Sin}_{150} h = 12 / \sin h = 12 \csc h)$
$= \mathrm{Csc}_{12}\ h$.

Then determine

(149:2) $\mathrm{arc\ Sin}\ [150 - (\mathrm{Vers}\ d - \mathrm{Vers}\ d\ \mathrm{Csc}\ h_n / \mathrm{Csc}\ h)]$.

If this is compared with expression (146:14) above, it will be noticed that the part inside the parentheses is Vers r. Recalling the definition of the versed sine function, and the fact that here R = 150, subtraction of Vers r from 150 will give Cos r. But the rule says find the arc *sine*. The answer will be $\bar{r}$ instead of the desired r. This is the error to which Bīrūnī objects, and which he discusses at length in 149:7 - 150:8.

Note the close resemblance between this procedure and the rule of the *Khaṇḍ.* in the section just below, including the parameter R = 150. The word *muḥṣā*, here translated as "computed" is unusual and probably a relic of early Islamic mathematical nomenclature.

123. The Same Rule in the Khaṇḍakhādyaka (150:9 - 152:2)

The passage which prescribes

(150:10) $r = (1/6)\ \mathrm{arc\ Vers}\ (\mathrm{Vers}\ d - \mathrm{Vers}\ d \cdot \mathrm{Csc}\ h_n / \mathrm{Csc}\ h)$

is, except for the added clause "what comes out is the equation", a faithful rendition of what is in *Khaṇḍ.* 3,16(15). It is the same as the accurate expression derived at 146:4 except for the factor 1/6, to convert from degrees into day-minutes (Section 66).

The next verse in the original of the *Khaṇḍ.* (transl., p.76) says, in effect,

$$t + e = \mathrm{arc\ Sin}_\rho\ (HS + SM)\ ,$$

the letters referring to Figure 43. After this verse the chapter ends. The substance of this is contained in the quotation at 151:10. The other passages Bīrūnī quotes are not in the original Sanskrit, and must be additions made in the Arabic version he was using. One such has, in terms of Figure 43:

If $TO - HS > R$,

calculate

(150:16) $r = \text{arc Sin}\,[(TO - HS) - R] + 90°$,

where $90° = 90 \times 60 = 5400'$.

At 151:7 Bīrūnī gratuitously adds a limiting special case. He says if $TO - HS = 0$, then $t = e$. But then, in fact $t = d = e+90°$. What he probably intended was: if $(TO - HS) - R = 0$, then $t = e$. Note that in our version of the figure the day circle is assumed to be south of the celestial equator, whereas in Bīrūnī's example it is north.

124. The Two Karaṇas Again (152:3 - 153:8)

We recall from Section 108 that Vateśvara was the author of the non-extant Karaṇasāra. The Sanskrit means "Epitome of the Zījes", but the Arabic at 153:13 says *kāsir*, annihilator or breaker.

The rule from it is

(152:3) $\text{Vers}\ r = \text{Vers}\ d \cdot (\text{Csc}\ h - \text{Csc}\ h_n) / \text{Csc}\ h$,

which is the relation derived in 146:14 except that here the common factor Vers d has been extracted.

Our text now takes up the inverse problem, given r, calculate h. From the *Karaṇatilaka* is the procedure

(152:8) $\text{Vers}\ d = R + \text{Sin}\ e$,

then

(152:10) $\text{Csc}\ h = \text{Vers}\ d \cdot \text{Csc}\ h_n / (\text{Vers}\ d - \text{Vers}\ r)$.

If this equation is solved for Vers r the result is 152:3, so the rule is a valid inverse. It is indeed given in Bīrūnī's translation of the *Karaṇatilaka*.

Concerning the Arabic translation of the name Karaṇaṣāra, see Section 108.

The same rule appears in *Khaṇḍ*. 3, 15(13).

According to Bīrūnī, Brahmagupta adds, presumably in the *Khaṇḍ*., that if $r > 15$ day-minutes (=90°) then use

$$\text{Vers}\ r = 2R - \text{Vers}\ (180° - r)$$

in place of Vers r. Apparently the versed sine was thought of as being defined only for arcs in the first quadrant, and this is an extension of the definition.

(This rule is given in the Uttarakhaṇḍa of the *Khaṇḍ*. as quoted by Amarāja in *Khaṇḍ*. 3,15 (pp. 112-113 Miśra). D.P.)

(Several of the ślokas of the commentator Balabhadra (153:1) are quoted by Utpala on the second chapter of the Bṛhatsaṃhitā; the formula given here is not among them. D.P.)

Balabhadra composed a commentary to the *Brahmasp*., probably at Kānyakubja, in the eighth century (*Pingree* 3 and 4).

His rule is, presumably when $r > 90°$, put

$$\text{Vers}\ r = R + \text{Sin}\ (r - 90°) \ ,$$

which will give the same result.

The *Karaṇasāra* method of finding h in terms of r is to calculate

(153:4) $\text{Csc}\ h = \text{Vers}\ d\ \text{Csc}\ h_n / [\text{Sin}(t-e) + \text{Sin}\ e]$.

Now, referring to Figure 43 and recalling that there e is negative, $t - e = \bar{r}$, and $d - e = 90°$. Therefore $\text{Sin}(t-e) = \text{Cos}\ r$ and $\text{Sin}\ e = (d-90°) = -\text{Cos}\ d$. Substituting these in the denominator of (153:4) above,

$$\text{Csc}\ h = \text{Vers}\ d\ \text{Csc}\ h_n / (\text{Cos}\ r - \text{Cos}\ d)$$
$$= \text{Vers}\ d\ \text{Csc}\ h_n / [R - \text{Cos}\ d - (R - \text{Cos}\ r)]$$
$$= \text{Vers}\ d\ \text{Csc}\ h_n / (\text{Vers}\ d - \text{Vers}\ r),$$

which is the valid expression (152:10). Hence the *Karaṇasāra* inverse rule is also correct.

(It is given by Brahmagupta in *Khaṇḍ*. 3, 16(14). D.P.)

CHAPTER 24

APPLICATION OF AZIMUTHS

125. Altitude from Azimuth (153:11 - 154:16)

In the horizon system, the celestial coordinates are altitude and azimuth. Thus far the author has concentrated on the first of the two. He now takes up the second, remarking that azimuths may play an analogous role to altitudes. When he says that the one determines the other he is assuming that the locality (ϕ) and the season (λ) are known.

This he proceeds to demonstrate, using Figure 44. The text has four different figures to illustrate various situations, but in fact one will suffice. The object is to calculate h, *OH* on the figure, in terms of the azimuth, declination, and local latitude. Now he evinces no reluctance to use the sine function, which in previous chapters he avoided.

Bīrūnī remarks (in 154:5) that, *KM* having *Z* as its pole, it is the complement of the angle at *Z*. This may be seen by noting that if the great circle between *M* and *Z* is drawn (not shown on the figure), spherical angle *MZH* is a right angle and arc *MK* is measured by *MZK*. Angle *MZE*, Bīrūnī's angle *Z*, is the complement of *MZK*. In other respects the figure is self-explanatory.

Application of the Rule of Four (Section 101) to triangles *EMK* and *EDG* gives

(154:7) $\text{Sin}\ (EM = \overline{az})\ /\ \text{Sin}\ MK = \text{Sin}(ED = 90°)\ /\ \text{Sin}\ (DG = \bar{\phi})$,

so *MK* can be calculated.

By the law of sines applied to the right spherical triangle *ZHE*,

(154:9) $\text{Cos}\ MK\ (=\sin Z)\ /\ \text{Sin}\ E = \text{Sin}\ (EH = az)\ /\ \text{Sin}\ ZH$,

from which *ZH* can be computed. If $\delta = 0$ the altitude would be *ZH*, hence it is called the "mean altitude".

Another application of the Rule of Four, this time to the triangles *ZSA* and *ZOL*, gives

(154:11) $\text{Sin}(ZS = \overline{ZH})\ /\ \text{Sin}(SA = \phi) = \text{Sin}\ ZO\ /\ \text{Sin}\ (OL = \delta)$

This gives *ZO*, the "equation of the altitude", which added algebraically to the mean altitude is $HO = h$, the desired altitude.

126. The Time Since Sunrise, in Terms of the Azimuth (154:17 - 155:15)

Given the same quantities, ϕ, δ, and the azimuth, it is possible to calculate t. The demonstration proceeds as follows.

By the law of sines applied to the right spherical triangle *EHZ*, still in Figure 44,

(154:18) $\text{Sin}\ (EH = az)\ /\ \text{Sin}\ EZ = \text{Sin}\ Z\ /\ \text{Sin}\ (H = 90°)$

The angle at *Z* having been determined in the section above, the arc *EZ* can be calculated. Now apply the Rule of Four to the triangles *OZL* and *OTS* to obtain

(155:1) $\text{Sin}\ OZ\ /\ \text{Sin}\ ZL = \text{Sin}\ (OT = \bar{\delta})\ /\ \text{Sin}\ (TS = \bar{\phi})$.

The arc *OZ*, the equation of altitude, was determined at the end of the preceding section. Hence *ZL* can be found. The "mean ascension", arc $EL = ZL + EZ$ is now known, and algebraic addition to it of the equation of daylight, $e = EC$ gives $t = CL$. As usual, Bīrūnī examines special cases. If $az = 0$, then $EZ = 0$, and if $\delta = 0$, $ZL = 0$. (155:7)

In the *Taḥdīd* (comm., p. 115), and by using much the same methods, he solves the same problem, except that there he solves for the hour angle rather than the time from sunrise (or until sunset).

127. Azimuth from Altitude (155:16 - 158:9)

The author now takes up a problem which can be regarded as the inverse of that solved in Section 125: given h,δ, and ϕ, calculate the azimuth. For this he uses a new figure, 45. Now he operates on plane triangles inside the sphere; with Figure 44 he dealt with arcs and angles on the surface of the sphere.

The text's description of the figure is clear and calls for no remark beyond noting a slight anomaly. Bīrūnī calls *KZM* the "day triangle", whereas it is what he usually designates as the "time triangle" (see Section 77).

As for the demonstration, applying the law of sines to the plane right triangle *KZM*,

(157:9) KZ (= Sin h) / ZM = Sin ($\angle M = \bar{\phi}$) / Sin ($\angle K = \phi$) ,

so ZM can be found. We note incidentally that since $\sin \bar{\phi} / \sin \phi = \cos \phi / \sin \phi = \cot \phi$, Bīrūnī here has passed up an opportunity to save himself a division operation by utilizing a shadow function.

Again, as in Section 93, he assumes that the reader is capable of computing the rising amplitude, $w = BF$, in terms of δ and ϕ, so that Sin $w = ML$ can be regarded as known. Hence the argument of azimuth,

(157:12) $$ZL = ZM + ML$$

can also be obtained. The addition is algebraic, since as we would put it, either or both of the two components may be negative.

The text now has, based on the similarity of triangles *EZL* and *EHY*,

(157:17) $$(EZ = \text{Cos}\ h)^2 \ / \ \overline{ZL}^2 = (EH = R)^2 \ / \ \overline{HY}^2$$

This is correct, but there is no need whatever to square everything in sight. There would be some justification for it if one of the elements in the proportion were obtained by means of the Pythagorean Theorem, as a square. This happens just below, but not here. Solution of this expression gives

(158:1) $$\text{Sin}\ az = HY = \sqrt{\overline{ZL}^2 \cdot R^2 / \text{Cos}^2\ h} ,$$

from which the desired azimuth may be found.

The azimuth of the shadow is, of course, equal but opposite in direction to the azimuth of the sun (155:16, 158:3).

Bīrūnī now describes how to obtain the azimuth reckoned from the north (or south) point. It is $AH = \bar{HB}$; one need only take the complement of the arc already found. Perhaps the author has in mind to calculate it

directly, before *BH* has been found. He prescribes (after we have restored a missing clause in the text)

(158:4) $$\text{Sin}\ \widehat{AH} = HC = \sqrt{(\text{Cos}^2\ h - \overline{ZL}^2) \cdot R^2 / \text{Cos}^2\ h} = \sqrt{\overline{LE}^2 \cdot R^2 / \text{Cos}^2\ h} ,$$

which follows from the similarity of triangles *HCE* and *ZLE*. Here there is some excuse for the squarings, since *LE* appears already squared. But by all odds the preferable choice would have been to take the square root of $\overline{LE}^2$ as soon as it emerges, and then square nothing.

In the *Taḥdīd* (comm., p.61) Bīrūnī solves the same problem. His method is closely related to this, but he does not assume that *w* has been calculated.

....

We remind the reader that the next passage following the subject of these remarks, 158:10 - 160:4, is an intrusion in the printed text. It is commented upon in Sections 118-120.

....

CHAPTER 25

DEFINITIONS OF THE MUSLIM TIMES OF PRAYER

128. Times Forbidden for Prayer (160:8-19)

It is incumbent upon every Muslim to pray five times daily; moreover, the proper times for two of these prayers are defined in terms of shadows. Hence it is most natural that a treatise on shadows written by a Muslim should contain a discussion of these matters. In this chapter our author presents the sources from which the sometimes conflicting definitions are derived. In the next chapter he describes instruments for determining the prayer times.

After discussing the times at which prayer is especially interdicted, he exhibits in detail the traditions and Quranic passages from which the prescribed times have been inferred. The elaborations and variants advocated by the four orthodox legal schools and the principal Islamic sects are cited, followed by brief enumeration of the prayers of other religions. The chapter closes with a description of the technical qualifications for a muezzin, one who calls the faithful to prayer.

Prayer is prohibited at the instants of sunrise, noon, and sunset. This rule, like the prohibition of intercalation in the lunar calendar, was part of the process of making Islam distinct from competing religions.

Of the three faiths mentioned in this passage, the Ḥarrānians, also known as the Sabians, were adherents of a pagan sect tolerated by Islam. Centered at Ḥarrān on the upper Euphrates, they worshipped the planetary dieties (*EI*, vol. 4, p. 21). Their fixed times of prayer were sunrise, noon, and sunset (*Chron.*, transl., p. 188).

The orthodox Hindus likewise engage in devotions thrice during the day, precisely at the juncture (*saṃdhyā*, whence the name of the ceremony) of the three divisions

of the day: forenoon afternoon, and night. So the three times are again sunrise, noon and sunset. (see *Srinavasan*, p. 161, and *Kane*, p. 701).

The Magians, or Zoroastrians were accorded a tolerated status by Islam along with the Jews and the Christians (*EI*, vol. 3, p. 97). The Zoroastrian custom of greeting the sunrise by a fanfare from the roof of the royal palace survived well into modern times in Muslim Iran.

The notion of the sun rising between the horns of Satan is attributed by a tradition to the Prophet himself (*Wiedemann and Frank*, p.7).

129. Quotations from the Authorities (161:1 - 164:13)

In this passage the somewhat vague and occasionally contradictory traditions concerning the prayer times are reproduced verbatim. The same material, greatly condensed, is displayed in tabular form on page 135 below. The times themselves are shown graphically with the line diagram of Figure C10.

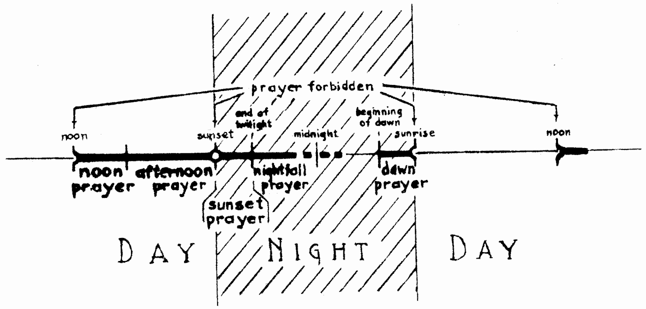


Figure C10

In the report involving the archangel Gabriel and his praying with the Prophet at the cubical building in Mecca which houses the sacred black stone (161:12), the shadow's being like a rope is taken to imply its thinness. Hence the time is noon, the time of minimum shadow, s_n.

The requirement that the shadow of a thing shall be equal to itself (161:14) is that $s = g$, where g is the gnomon length.

The times specified for the first and second days are taken to define the initial and terminal points respectively for the time spans allowable for the several prayers.

The ʿUmar quoted in 162:2 was the second caliph. A great administrator, he was the founder of the Arab empire. (*EI*, vol. 3, p. 982).

A *farsakh* is a unit of distance equal to three miles (*Taḥdīd*, comm., p. 134).

Abū Mūsā al-Ashʿarī (162:7) was a companion of the Prophet and sometime governor of Baṣra. Not to be confused with his descendant, Abū al-Ḥasan al-Ashʿarī, he was known as a student of the Qur'ān and was one of two arbiters in the dispute between the caliphal claims of Mu'āwiya and ʿAlī. (*EIne*, vol. 1, p. 695).

Jaʿfar al-Ṣādiq (d. 765) was the last of the Shīʿa imāms to be recognized by the two main divisions of the sect. The hours he names in 163:8 are, of course, of the seasonal variety (cf.Section 116), and the times they give correspond only very crudely to the times of prayer as usually defined. (*EIne*, vol. 2, p. 374).

It is strange that quotations from the Qur'ān, the primary authority (163:11 - 164:13), should follow statements attributed to humans. Perhaps it is because of the vagueness of the Quranic material.

130. Etymology of the Prayer Names (164:14 - 166:9)

An attempt to link the Qur'ānic word *dulūk* (decline) with its triliteral root *dalaka* (to rub with the hand) leads naturally to additional etymological remarks concerning *ẓuhr* (noon) *ʿaṣr* (afternoon), and *ʿishā'* (nightfall).

The Faḍāla of 165:5 was a Successor (*tābiʿī*) of the companions of the Prophet. Ibn ʿAbd al-Barr (in *al-Istīʿāb*) gives the tradition cited in our text, but says that the *nisba* of the one who transmitted it was al-Laythī, not al-Zahrānī.

SOURCES		DAY PRAYERS: 1st الظهر - NOON	DAY PRAYERS: 2d العصر - AFTERNOON	NIGHT PRAYERS: 1st المغرب - SUNSET	NIGHT PRAYERS: 2d العشاء (العتمة) - NIGHTFALL	NIGHT PRAYERS: 3d الصبح - MORNING
The QURĀN	xi,114 (163:12), xx,130 (164:2), lii,48 (164:8), xvii,78 (164:11)	pray at the sun's decline	before sunset upon arising	at the extremes of the day at twilight	the first part of the night; when the stars disappear (?)	at the extremes of the day before sunrise; upon arising at dawn
Tradition concerning GABRIEL (161:13)	1st day (from)	when the shadow is like a rope	$s = g$	sunset, breaking of the fast	disappearance of twilight	dawn, commencement of fasting
	2d day (to)	when $s = g$	$s = 2g$	"	passage of a third of the night	"
The Caliph ʿUMAR	to the governors 17 AH. (162:1)	from: the shadow a cubit; to: $s = g$	when the sun is high enough for a 2 or 3 farsakh ride	sunset	from: disappearance of sunlight; to: passage of a third of the night	when the stars are barely visible
	to Abū Mūsā al-ASHʿARĪ (162:7)	when the sun declines	when the sun is pure white, not yet yellow	"	when the nightfall is not yet finished (?)	when stars are visible and there is time to recite two long sūras
JAʿFAR al-ṢĀDIQ (162:15)	verbally	"	from: $s = g$; to: $s = 2g$; or as long as the sun is white	"	end of twilight	beginning of dawn
	in hours	7th	9th	1st	3d	11th
THE LEGAL SCHOOLS	Abū ḤANĪFA	to: $s = 2g$ (better) $= s_n + 2g$; according to a rare tradition $= s_n + g$ (168:10)	from: $s = s_n + 2g$; in the rare tradition $= s_n + g$ (169:1) (182:6)	from: sunset (167:3); to: end of the whiteness	from: disappearance of twilight; to: beginning of dawn	from: the beginning of dawn (the white along the horizon)
	Abū Yūsuf and Muḥ. b. al-Shaybānī	to: $s = s_n + g$	from: $s = s_n + g$ (182:6)	to: end of the redness (167:3)	"	"
	MĀLIK		may delay until darkness (also Ṭāʾūs & ʿAṭā, 169:5)	to: dawn (hence overlap, 167:3)	"	"
	al-SHĀFIʿĪ	from: $s = s_n$ + less than a cubit (168:2); to: $s = s_n + g$ (168:15)	from: $s = s_n + g$ (182:6)	to: end of the redness (167:3)	"	"
	Ibn ḤANBAL			to: end of red in open air, " " white in built-up regions (167:6)		

The Ḥumayd b. Thaur al-Hilālī quoted in 165:10 was a seventh century poet, one of whose patrons was the caliph Marwān (*EIne*, vol. 3, p. 573).

At 166:6 is the first mention of the founders of two of the four orthodox Muslim legal schools. Abū ʿAbdallāh Muḥammad b. Idrīs al-Shāfiʿī (d. 820) was raised in Mecca, studied in Medina under Mālik (see the next paragraph), and taught in Baghdad and Egypt. His eclecticism made him a sort of middleman between the traditionalists and the advocates of independent juridical rulings. (See *EI*, vol. 4, p. 252).

Mālik ibn Anas (d. 795) was the author of the earliest extant book on Muslim law. Most of his life was spent in Medina, and his views tended to be the consensus of legal opinion in that city. (See *EI*, vol. 3, p. 205).

131. Final Definitions - Nocturnal Prayers (166:10 - 168:1)

Each of the five prayers is now taken up systematically, commencing with the nocturnal ones. The names of four additional jurists are cited here.

Abū Ḥanīfa, al-Nuʿmān b. Thābit (d. 767), founded the third legal school, that of ʿIrāq. He lived in Kūfa, but eventually died in prison in Baghdad. His task was to systematize legal theory. (167:3, see *EIne*, vol. 1, p. 123).

Abū Yūsuf, Yaʿqūb b. Ibrāhim al-Anṣārī (d. 798) was a disciple of Abū Ḥanīfa although he was more dependent upon tradition than his master. He also studied under Mālik. (*EIne*, vol. 1, p. 164).

Another follower of Abū Ḥanīfa and sometime student of Mālik was Muḥammad b. al-Ḥasan al-Shaybānī (d. 805). His own teaching made tradition fundamental. (*EI*, vol. 4, p. 271).

The head of the fourth legal school was Aḥmad ibn Ḥanbal (d. 855) who studied in Baghdad under Abū Yūsuf. As a traditionalist he opposed the rationalist Muʿtazilite sect. (*EIne*, vol. 1, p. 272).

All these individuals agree that the interval for the first nocturnal prayer, that of sunset, commences as soon as the sun's body has disappeared below the horizon.

Al-Shāfiʿī alone requires that the prayer begin immediately after sunset. The others agree that the time may be extended throughout the duration of twilight, there being individual differences concerning the limiting color of the waning sunlight (167:5).

The time during which the nightfall prayer must be prayed begins with the end of twilight and ends at dawn (167:12). This is the second nocturnal prayer.

The time for the third and last, *al-ṣubḥ*, commences with the true dawn and lasts until sunrise (167:15, cf. *Wiedemann and Frank*, p. 768[12]).

132. Final Definitions — Daytime Prayers (168:2 - 169:9)

At 168:2 the discussion of the time for the dawn prayer is abruptly cut off and the new paragraph breaks in with the presentation of the opinions concerning the noon prayer already underway. Evidently there is a hiatus, probably short, in our single manuscript source.

The matter being considered is the breakdown in the naive injunction appearing repeatedly in the traditions (e.g. 161:13) that the time for the noon prayer expires (and that for the afternoon begins) when the shadow of an object equals itself ($s = g$). This is the time in the afternoon when the declining sun passes through an altitude of $h = 45°$. As remarked in 168:3, there are many latitudes for which this requirement cannot be fulfilled, at least for part of the year.

To demonstrate this, recall that on any day, max h occurs at noon. For any latitude, the minimum noon altitude occurs on the first day of winter, when the day circle of the sun is the celestial tropic of Capricorn (on Figure C11). On that day $h_n = \bar{\phi}-\varepsilon$, and the condition above will fail whenever $\bar{\phi}-\varepsilon<45°$, or $90°-\phi-\varepsilon<45°$, or

$$\phi > 45° - \varepsilon \approx 45° - 23\frac{1}{2}° = 21\frac{1}{2}°$$

This fact had been realised long before Birūnī's time, and by Abū Yūsuf, Muḥammad al-Shaybānī, and al-Shāfiʿī the impractical injunction had been replaced by a rule depending implicitly upon both the latitude and

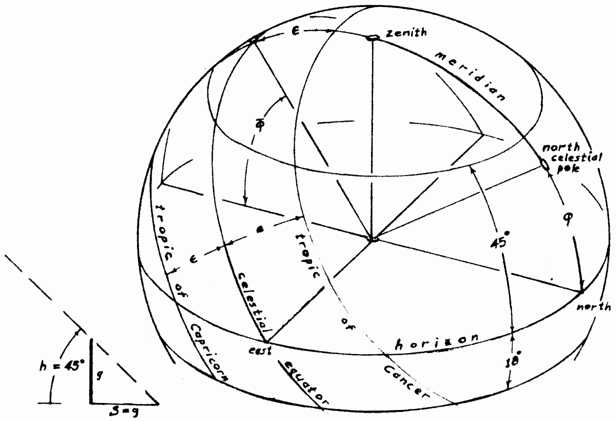


Figure C11

the season. It is that the time for the noon prayer ends when the horizontal afternoon shadow of a vertical gnomon equals the gnomon length plus its noon shadow. Symbolically

$$s = s_n + g \ ,$$

a requirement which can be applied at all latitudes outside the arctic regions.

Abū Ḥanīfa (according to most traditions) did the same thing with the primitive rule

$$s = 2g \ ,$$

and put for the end of the noon prayer

$$s = s_n + 2g \ ,$$

giving a time later each afternoon than that of the other three schools.

At 169:5 the opinion of three jurists is given to the effect that it is permissible to delay the afternoon prayer through the twilight. Of these, Mālik has been introduced in Section 130 above. ʿAṭā' b. abī Rabaḥ (d. c. 732) was prominent as a teacher in Mecca (*EIne*, vol. 1, p. 730). Abū ʿAbd al-Raḥmān Ṭā'ūs b. Qaysān al-Hamdānī was an authority on tradition (*ḥadīth*) and jurisprudence who died in Mecca in 725 (*Ibn Khallikān*, vol. 1, p. 642).

133. Miscellany (169:10 - 174:14)

In addition to the compulsory prayers, the Muslim may carry out extra, optional devotions. The reason for sometimes forbidding them (in 169:15) is probably to insure that they will not carry over into the prohibited times when the sun is on the horizon.

In the tradition quoted at 169:18 the commentator mistook the word *sharaq* (choking) for *sharq* (sunrise, or east), written in the same way. Hence he thought the remark was restricted to the morning prayer, whereas, as Abū Rayḥān explains, all the prayers are intended. The general idea seems to be that within the allowable time span of a particular prayer the earlier it is prayed the better. There is no trace in this text of the practise of dividing each span into five parts in descending order of desirability. This development may have taken place after Bīrūnī's time. (Cf. *Wiedemann and Frank*, p. 1).

Abū ʿUbayd (169:19) al-Qāsim b. Salām (d. 834) was a grammarian, expert in the Qur'ān, and judge, who lived in ʿIrāq, Asia Minor, and Arabia (*EIne*, vol. 1, p. 157).

The Shīʿa (170:17) make up one of the two major subdivisions of Islam. They do not recognise the first three orthodox caliphs, holding that the succession passed from the Prophet directly to his cousin and son-in-law ʿAlī ibn abī Ṭālib (172:17). (See *EI* vol. 4, p. 350; *EIne*, vol. 1, p. 381).

The Zaydites (171:4) in turn are a subdivision of the Shīʿa, powerful in the Yemen, who follow Zayd b. ʿAlī, Zayn al-ʿĀbidīn (d.c. 740, *EI*, vol. 4, p.1196).

Abū ʿAbdallāh Muḥammad b. Karrām al-Sijistānī (171:14, d.c. 870) was the founder of a sect, now extinct, which flourished in Khurāsān. The Karrāmīs attempted to fit elements of Aristotelian philosophy into the tenets of the Qur'ān (*EI*, vol. 2, p. 773).

Of course it is true that half the difference in shadow lengths does not correspond to half the difference in the respective times (171:18).

All agree that the noon prayer is designated the "first". Each of the prayers includes a fixed number of *rakʿas* as shown below, a *rakʿa* being a prescribed set of genuflections with accompanying recitations.

1. Noon	4
2. Afternoon	4
3. Sunset	3
4. Nightfall	4
5. Dawn	2

The numbers are those given in order in the couplet quoted at 172:10.

There is general disagreement as to which prayer is to be known as the "middle" one. At least one supporter is named for each of the five except the nightfall prayer, Bīrūnī's own preference being for the afternoon (173:18).

Of the individuals named in this passage, Ibn ʿAbbās (d.c. 687), the founder of Quranic exegesis, was the greatest of the first generation of Muslim scholars (*EIne*, vol. 1, p. 40).

Qatāda (d. 735) was another learned divine, resident in Baṣra (*Nicholson*, p. 294).

A certain Abū Bakr b. Mujāhid al-Tamīmī (d. 936) was in his time the chief of the Qur'ān readers of Baghdād (*GAL*, suppl. vol. 1, p. 320). But his relation, if any, to the Mujāhid of 172:17 is not established.

Jābir b. ʿAbdullāh al-Anṣārī (173:6) was one of the Companions of the Prophet. He was a great authority on the traditions (*ḥadīth*) (*Istīʿāb*, vol. 1, p. 219).

ʿAbdallāh b. ʿUmar (d.c. 750) was a son of the caliph ʿUmar II. He served as a provincial governor

(*EIne*, vol. 1, p. 53).

Qubayṣa b. Dhuwayb (173:13) was another of the Companions of the Prophet (*Istīʿāb*, vol. 3, p. 1272).

ʿArafāt and Muzdalifa (173:2) are the names of two localities near Mecca where special ceremonies take place during the annual pilgrimage (*EIne*, vol. 1, p. 604; *EI*, vol. 3, p. 800).

The word *qunūt* (173:3) is a technical term used in the Qur'ān. Its meaning is disputed (*EI*, vol. 2, p. 1118).

134. The Prayers of Other Religions (174:15 - 175:11)

Abū Rayḥān's information about the Jewish prayer times may have been garbled in transmission. *Maimonides* (pp. 98 - 100), writing somewhat later than Bīrūnī, states that there are two obligatory services of prayer daily, morning and afternoon, with eighteen benedictions each. He adds that on days of additional offering there is a third, an optional evening service which, however, has been adopted by all Israelites as obligatory. So the total number is the same as our source reports. But the distinguishing of a white thread from a black associated with the third prayer at 174:19 is an ancient criterion for the beginning of the dawn prayer. Perhaps this should have followed immediately after the mention of dawn. And perhaps mention of the afternoon prayer as the third has fallen out of the original text.

As for the Christians, Professor A. Vööbus writes that Bīrūnī's information is based ultimately upon the Syrian tradition, the so-called Order of the Apostles, being published by him as a part of the Syriac "Synodicon of the West Syrian Tradition, I".

We find no information on the Manichean (see Section 76) prayers to confirm or supplement the statements of the text, tabulated on the next page.

For the initiates			For the laymen	
No.	Time	Genuflections	No.	Time
1	noon	37(two less on Mondays)	1	noon
2	afternoon	21	2	nightfall
3	nightfall	25	3	dawn
4	a half hour later	25	4	sunrise
5	midnight	30		
6	dawn	50		
7	beginning of day	26		

Concerning the Zoroastrians, see 160:8 and Section 128.

135. Qualifications of a Muezzin (175:12 - 180:11)

At each mosque there is an official charged with the duty of announcing from the minaret the times of prayer. Each day he must ascertain the instant of the sun's culmination and the gnomon's shadow length, s_n, at that time. Bīrūnī seems to exaggerate the difficulties involved. Of course, in the absence of reliable mechanical clocks the intervention of clouds, as he reports at Ghazna (176:12), would be frustrating.

There was no need to obtain a daily observation of s_n, however. Many zījes have tables showing $h_n = \bar{\phi} + \delta$ as a function of the solar true longitude, λ. It would not be difficult to use such material to calculate a table of $s_n(t) = \text{Cot } h_n$, where the argument t stands for a calendar date. In order that the table serve for more than one year the underlying calendar must be solar (Byzantine = *Rūmī* = Julian), not lunar (176:4, cf. Section 28). Further, the person who calculates the table must be able to convert from t to $\bar{\lambda}_s$ (mean longitude) to λ_s, that is, he must take into consideration the variation in solar angular velocity (176:6). Once in possession of s_n, the determination of the shadow length for the afternoon prayer is

immediate, whether one follows Abū Ḥanīfa or the others.

As for the fixing of noon, the shadow length being minimum at that time, its rate of change then vanishes, and the instant is difficult to observe. The rate of rotation of the shadow does not vanish, and the best method of finding local apparent noon seems to be to note when the shadow crosses the previously marked meridian line, Bīrūnī's second method (177:5). Meridian determinations have already been discussed, in Chapters 18 through 21.

In the event of cloudy weather there was no escape, except guesswork, from the use of clepsydras or hour glasses (178:13), however unsatisfactory. Such a device is described in *Wiedemann and Frank*, p. 3. Even so there would remain the problem of converting from shadow length on a particular day to a corresponding time, in equal or unequal hours. This in turn entails a knowledge of daylight lengths, which depend upon t, ϕ, and ε in addition to the solar complications mentioned above.

So the poetic peroration with which Bīrūnī ends the chapter is on the whole justified. The relations involved are complicated combinations of trigonometric functions, and a competent *muwaqqit* (timekeeper) would indeed need a reasonable command of the ancient authors named in 180:4. Such practitioners did exist in profusion, and their passion for elegant and precise computations resulted in the *ʿilm al-mīqāt* (science of timekeeping). Their massive achievements, only now being systematically explored, are impressive by any standards. (See *King*, 1, 2, and 3).

The Idrīs mentioned at 180:8 appears twice in the Qur'ān and is usually identified with the Biblical Enoch. Since in legend he is supposed to have made a trip through the celestial spheres on his way to paradise, it is appropriate to invoke his name in a paragraph stressing the importance of astronomy for religion. (*EIne*, vol. 4, p. 350; *Bouché-L.*, p. 606).

CHAPTER 26

DETERMINATION OF THE PRAYER TIMES WITH INSTRUMENTS

136. Determining the Noon and Afternoon Prayer Times (180:14 - 183:1)

After repeating the doctrine that the noon prayer may commence at any time after noon has passed, the text gives the rule (181:3):

if $s_2 < s_1$ it is the forenoon,

but if $s_2 > s_1$ it is the afternoon,

where the subscripts indicate the order in which the two shadow observations are taken.

The author finds this easy to demolish, noting that we may in fact have $s_2 = s_1$ if the two times happen to be spaced symmetrically with respect to noon. Or $s_2 < s_1$ may hold and yet the second observation may be in the afternoon by a time less distant from noon than was the first observation. What is most serious is the slow variation in shadow length in the vicinity of noon (cf. Section 135 above). For these reasons he recommends abandoning the rule and the use instead of a meridian line marked on the ground at the foot of the gnomon (182:3).

For the beginning of the afternoon prayer one need only wait in the afternoon until the gnomon's shadow attains the length s_n+g, or s_n+2g if the opinion of Abū Ḥanīfa is preferred.

If it is more convenient to work with solar altitudes rather than shadow lengths, the same time will be obtained by the rule quoted from Ḥabash. Expressed in symbols it is

(182:13)
$$h = \begin{cases} 90° - \text{arc Tan}_{60}(\text{Tan}_{60} h_n + 120), \text{according to Abū Ḥanīfa} \\ 90° - \text{arc Tan}_{60}(\text{Tan}_{60} h_n + 60), \text{according to others}, \end{cases}$$

since for Ḥabash $g = 60$.

137. Prayer Curves on the Astrolabe Plates (183:2 - 184:9)

For a general description of the astrolabe the reader may consult Section 50 above. To use it for determining the prayer-times it is necessary to have suitable curves engraved either on each of the plates inside the mater or on the back. Here Bīrūnī describes the layout of the former.

It is necessary to have at hand tables of h_s, the solar altitude, for the times in question. The independent variable is λ_s with domain $-90° \leq \lambda_s \leq 90°$, with the parameter ϕ fixed for each plate. Such tables are to be found, e.g., in the work of Abū al-Ḥasan al-Marrākushī (*Sedillot, J.J.*).

The curves are drawn on the lower left-hand quadrant, below the map of the horizon. It would be more natural to have them above, but the nocturnal portion was probably chosen because it is unencumbered by the net of coordinate curves which cover the diurnal part.

To determine a point on the curve, choose a value of λ_s and look up the corresponding h_s. Rotate the rete (*ʿankabūt*, Hartner's "spider") until the point on the ecliptic circle corresponding to λ_s falls upon the almuncantar corresponding to h_s on the plate below. then make a mark on the plate under the point on the ecliptic scale corresponding to $\lambda_s+180°$, the sun's "opposite point" (183:12). Each such point will fall below the horizon, since the sun has been placed on an almucantar above it. The set of all such points makes up the curve in question. (Cf. *Wiedemann and Frank*, p.26).

Representations of plates with these curves may be seen, e.g., in Figures 57-63 of Plates 16-19 in the back of *Sedillot, L.A.* An actual astrolabe having them is the one made by Muḥammad b. al-Baṭṭūṭī, M-35 in the collection at the Hayden Planetarium, Chicago. But the prayer curves appear only on the plates for ϕ=36° and ϕ=36;40°, probably having been added sometime after the instrument was made, and by a craftsman less skillful than the original maker.

On all these the curves are either dotted, as Bīrūnī suggests (183:17) or having very short cross-hatchings to distinguish them from the lines of the unequal hours.

A quick method of approximating the prayer curves and those of the unequal hours is to plot only three points on each, and to pass a circle through each triple of points. The curves of the equal hours are indeed circular arcs, drawn as though the horizon had rotated about the plate center. For them no loss of precision is involved if this technique is used (184:1-9).

138. Using the Prayer Curves on the Plate (184:10-19)

The curves having been laid out, they may be applied as follows. For any particular day mark the corresponding λ_s + 180° on the rete. Then the amount of rotation of the rete required to carry the marked point from the relevant curve to the horizon (or the meridian) is a measure of the time between the prayer-time and sunset (or from noon to the prayer-time).

Conversely, at any time of the afternoon take the solar altitude, h_s. Mark the proper λ_s on the rete and rotate the latter until it falls on the almucantar for h_s. Now mark λ_s + 180°, and note the relative positions of the mark and the appropriate curve. If the mark falls on the curve the time is the beginning of the prayer span; if the mark is between the meridian and the curve the time for prayer has not yet arrived, and so on.

139. Curves of Solar Depression and the Unequal Hours (185:1 - 186:1)

A solar depression of 18° below the horizon is taken by our text as marking the beginning of dawn if in the east and the end of twilight when in the west.

Other values for the critical depression were also in use, and some astronomers, including Bīrūnī, used different amounts for the eastern and western horizons. (***Wiedemann and Frank***, pp. 13-24).

Whatever the value of the parameter, the sun's crossing marked the beginning of the time for the dawn prayer and the end of the time for the sunset prayer, when in the east and west respectively.

To this end Bīrūnī prescribes the marking of the almucantar for h = -18° on each plate. Its use is analogous to that of the other almucantars. Since the sun is below the horizon at both these times it cannot be used for direct observation, but the astrolabe can be employed to calculate mechanically the time, say, from sunset to the end of twilight as a function of λ_s.

The ʿUmar mentioned at 185:4 was the second caliph having that name (see Section 129). He reigned only from 717 to 720, but his saintly character made his brief rule memorable (***Nicholson***, p. 204).

This passage simply points out that since the definitions of the times of prayer and the hours are essentially different, it is erroneous to identify any particular prayer-time limit with any particular hour. This goes also for the similar attempt on the part of the imām Jaʿfar al-Ṣādiq noted at 163:8.

The rules reported in 185:11 - 186:1 seem to be, as Abū Rayḥān infers, a case of confusing two meanings of the word *aṣbaʿ*. The original denotation is "finger", like the Latin *digit*. The same word means also a unit, a twelfth of a gnomon of arbitrary length. In 185:11 the statement probably intends "when the shadow is increased by a fingerbreadth", i.e., a little bit. The little bit is to make it certain that noon has actually passed. The thirteen in 185:13 may mean "twelve digits (the gnomon length) plus a finger breadth".

140. Prayer Curves on the Astrolabe Back (186:2 - 188:3)

The object of the remarks in 186:2-5 is to insure that the alidade be so constructed that one of its edges will be a radius of the astrolabe back. Then when the instrument is suspended in a vertical plane and an object is viewed through the alidade sights the edge of the alidade will cross the protractor scale on the back at the altitude of the object. Two types of such alidades are illustrated in ***Hartner***, p. 2550.

Instead of using the plates at the front of the instrument as described in Section 137 above, it is possible to lay out curves for the prayer times on one quadrant of the back. Three such are needed, for the determination of

(1) noon,

(2) the beginning of the afternoon prayer time, and

(3) the same, but according to the opinion of Abū Ḥanīfa.

Two variables are involved in the location of each point on each curve:

(a) the solar longitude, λ, and

(b) the shadow, cast by a gnomon at the time and locality in question.

It is implicit that this set of curves will be valid for only one ϕ.

For λ a scale is laid out as shown in Figure 47 along a horizontal radius. From each graduation of the scale a quadrant is drawn concentric with E, the center of the astrolabe back. For the shadows a scale is thought of as extending from G in the direction of C. To any particular λ there corresponds a unique arc, and three shadows, say GY, GK, and GL for phenomena (1), (2), and (3) respectively. Draw EY, EK, and EL intersecting the circle in points M, S, and O. As λ runs through its domain of values, M, S, and O will trace out the three required curves.

Once drawn, the curves are used as follows: Some time in the afternoon take the solar altitude. This will set the alidade along a particular radius through E. Now note which member of the family of circles corresponds to that particular day. If the circle, the alidade edge, and the curve through S, say, are all three concurrent at a single point it is the beginning of the permissible time for the afternoon prayer. If the altitude is less

the time has already commenced, and conversely. The sentence at 180:2 is a reminder of the uncertainty surrounding observations in the vicinity of noon. If the observation happens to be in the forenoon the rule above does not hold, for then the altitude is increasing with time.

Photographs of actual astrolabes having these curves engraved on the back are Plates IV, X, and XIII in ***Mayer***.

141. Timekeeping with Astrolabes (188:4 - 191:10)

The paragraph 188:4-8 mentions but does not describe a quadrant on the astrolabe back used for telling time. Many astrolabes indeed have one quadrant inscribed with a characteristic family of six circles tangent to the horizontal at the center of the back. Explanation of their use, together with a proof of the construction's validity, may be found in ***Cittert***, p. 44. Examples of such astrolabes are shown in ***Mayer***, Plates XIV and XVII.

The rather primitive device described in the next passage, 188:9 - 190:3 consists of a scale having unequally spaced graduations laid out on the alidade as indicated in Figures 48 and C12. Marks S, O, F, Y, and M are made on the edge TK at S such that $TS = \mathrm{Tan}_{12}\ 15°$, $TO = \mathrm{Tan}_{12}\ 30°$, and so on, where the scale is such that the width of the alidade sight, TH, is taken as twelve digits. The integers 1 through 5 (for the forenoon hours) and 11 down to 7 (for the afternoon) are inscribed on these five marks respectively, in pairs as shown in Figure 48.

To make use of the arrangement, set the alidade at the noon solar altitude, h_n, for the day in question. Then suspend the astrolabe as usual when taking an observation, so that the face is in the vertical plane passing through the sun. Read the time in unequal hours by noting where, on the scale described above, the edge of the shadow of the sight falls. Thus, whenever $\theta = 45°$ the scale will read three hours of the forenoon or nine in the afternoon. The results will be accurate at noon, but in general at no other time. For at noon, $\theta = 90°$ and the whole alidade will be in shade (cf. 190:1). But at sunrise, which should read zero on the scale,

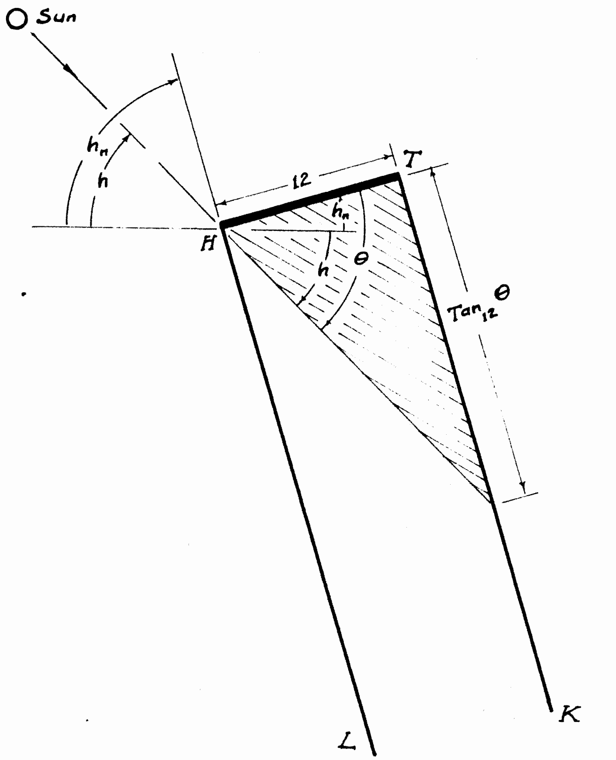


Figure C12

h = 0° and hence $\theta = h_n$. So, only on the rare occasions when the sun culminates in the zenith (making h_n=90°) does the scale work at both endpoints. There is no reason for supposing it to be any better in between.

Instead of laying out the scale graphically, this may be done numerically with the aid of the table of $Tan_{12}\theta$, θ=15°, 30°, 45°, 60°, 75° given in the text at 190:12. Except for the trivial restorations noted in the critical apparatus, all entries have been calculated correctly to the one fractional sexagesimal place shown.

We do not find this table in either of the extant zījes of *Ḥabash* (*Berlin* or *Istanbul*, see 190:12).

The tangent function increases ever more rapidly as the argument approaches 90°, hence successive graduations on the scale diverge sharply. To damp this effect the scale may be projected on the diagonal, *TL*, as shown in Figure 48. The name *sāq al-jarāda* (locust's thigh, 191:5) is probably from the resemblance between the resultant triangle on the alidade and the sharply tapering leg of the grasshopper. The same name is attached to a different type of sundial in *Sedillot, J.J.*, vol. 2, p. 440, plate X.

142. A Semi-cylindrical Alidade and Other Instruments (191:11 - 192:11)

A variant of the construction described just above is the rather elegant application of elementary geometry pictured in Figure C13. It has been elaborated from Figure 50 in the text. A strip of material is bent into a half cylinder and mounted on the alidade. Since the scale is now circular it follows that each length on it is twice the angle θ formed by the shadow-edge and the tangent to the upper limb of the strip. Hence the distances between any pairs of successive hour lines are equal. As the drawing indicates, it is used in the same manner as the locust's leg, and is subject to all its shortcomings.

The astrolabe can be employed as a portable sundial by putting a folding gnomon on one of its surfaces and laying out the proper curves on the surface. As Bīrūnī indicates in 192:7, the traces of the end point of the gnomon's shadow make up a family of hyperbolas bounded by the hyperbolas for the solstices. It was customary to engrave only the solsticial curves

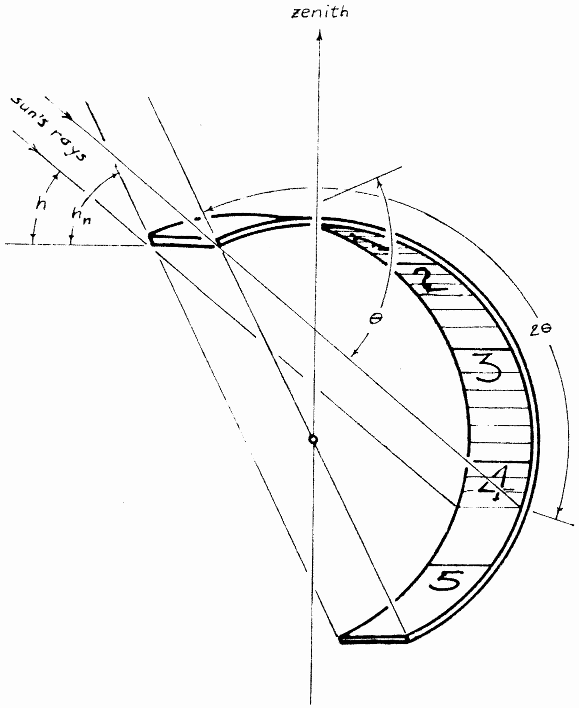


Figure C13

on the plate, to mark on each pair the points indicating the beginnings of the hours, then to join corresponding pairs of points with straight lines. Numerous examples may be seen in, e.g., ***Sedillot, J.J.***, vol. 2, figures 85-112.

143. Types of Sundials (192:12 - 194:18)

The types of plane sundial are said to be those with the plate in the plane of the

1) horizon,
2) meridian,
3) prime vertical,
4) celestial equator.

All four cases can be treated as one if the plate is regarded as a horizon and the particular ϕ appertaining to it determined. For 2) the ϕ is taken as zero; for 3) it is the $\bar{\phi}$ of the locality, and for 4) it is 90°.

The fact that the family of conics for 4) becomes the set of circles having the gnomon base as center is implied in 193:9.

In 193:15 - 194:8 the instruments mentioned in this concluding section of the chapter are insufficiently described to give a complete notion of their appearance, and we have not encountered their names in other contexts. The ruler with a movable gnomon in some ways resembles the ***mukḥula*** described by ***Livingston***. The use of "Byzantine", i.e., Julian months (as in the instrument destroyed by the muezzin in 37:7) is simply an application of a solar calendar to identify segments of the ecliptic.

CHAPTER 27

THE TANGENT FUNCTION APPLIED TO SPHERICAL ASTRONOMY

144. From Complete Quadrilateral to Spherical Triangle (194:11 - 199:4)

Since, as the author himself says, he has already worked over (in Chapter 10) the relation between the shadow functions and the sine, as well as the effect on the shadow functions of shifting parameters, it is difficult to fathom his objective in the opening passage of this chapter. With Figure 51 he regards *ZD* and *GB* as corresponding sides of circumscribed and inscribed regular polygons. Thus each side of the outer polygon is a doubled tangent, and of the inner a doubled sine. This greatly restricts his choice of arcs, and he makes no use of it in what follows. He notes that

(195:8) $BY/YE = TH/TE = DA/AE$,

and that $TH = \mathrm{Tan}_g AB$ and $AD = \mathrm{Tan}_R AB$.

Moreover, the regular polygons he has hypothesized can only exist if the adjoining arcs are equal. For instance, since $\widehat{LG} \neq \widehat{AG}$, $\widehat{LG}$ cannot be used for the adjacent sides.

Attention is now transferred from the plane to the sphere, and the material presented is clear and significant. Figure 52 is used to state Menelaos' theorem, proved in *Almagest* 1, 13. In the complete spherical quadrilateral shown there,

(196:12) $\mathrm{Sin}\, EB / \mathrm{Sin}\, AE = (\mathrm{Sin}\, ZD / \mathrm{Sin}\, AZ)\cdot(\mathrm{Sin}\, GB / \mathrm{Sin}\, DG)$

Let $AB = BG = AD = EG = 90°$ Then the angles at *E*, *B*, and *D* will be right angles, and the angles at *A* and *G* are measured by arcs *BD* and *EB* respectively. It was shown above, at 57:7 (Section 43), that

(196:16) $\mathrm{Sin}\, EB / \mathrm{Sin}\, (EA = \overline{EB}) = \mathrm{Tan}_g EB / g$,

and $\mathrm{Sin}\, DZ / \mathrm{Sin}\, (ZA = \overline{DZ}) = \mathrm{Tan}_g DZ / g$.

Substitute these expressions in (196:12) to obtain

(197:1) $\mathrm{Tan}_g EB / g = (\mathrm{Tan}_g DZ / g)\cdot(\mathrm{Sin}_R(GB{=}90°) / \mathrm{Sin}_R DG)$.

For $g = R$ this becomes, upon inverting,

(197:6) $R / \mathrm{Tan}\, EB = (R / \mathrm{Tan}\, DZ)\cdot(\mathrm{Sin}\, GD / R)$,

whence

(198:4) $\mathrm{Tan}\, DZ / \mathrm{Tan}\, EB = \mathrm{Sin}\, GD / R$.

This we would do by cancelling *R* from both sides and rearranging what is left. Bīrūnī is constrained to justify it by manipulations with compound proportions. In any event, since $EB = \angle G$, he has now proved the relation between the legs and one acute angle which subsists for any right spherical triangle. This he announces verbally in 198:5. In our notation it is

$$\tan A = \tan a / \sin b,$$

where *C* is the right angle, and small letters denote sides opposite the angles bearing the letter as a capital.

In triangle *ZDG*

(198:9) $\mathrm{Tan}\, ZD / \mathrm{Sin}\, DG = \mathrm{Tan}\, \angle G / R$

and, employing the other acute angle,

(198:10) $\mathrm{Tan}\, DG / \mathrm{Sin}\, DZ = \mathrm{Tan}\, \angle Z / R$

The author remarks in two places (197:8 and 199:1) that he has been dealing with the reversed shadow (the tangent function) rather than the direct shadow (the cotangent). The latter can always be introduced by using the identity $\tan\theta \equiv 1/\cot\theta$, or $\tan\theta \equiv \cot\bar{\theta}$, but to do so in the triangle relation would introduce a quantity not found in the actual triangle.

We note that the theorem represents a step in the emergence of trigonometry proper. The subject commenced with a single function, the chord, and its application to relations involving six arcs on a complete spherical quadrilateral. It evolved in the direction of relations connecting functions (principally the sine, cosine, and tangent) of sides and angles of triangles. The theorem above is just such a relation (see the *Overview*).

145. Declinations in Terms of Longitudes (199:5 - 200:11)

Now this and other relations are applied to the standard problems of spherical trigonometry. If in Figure 52 *AZD* and *AEB* are made the ecliptic and equator respectively, and *Z* is an arbitrary ecliptic point, then $BD = \varepsilon$ and $EZ = \delta$. This is the "first declination"; if the arc from *Z* is made perpendicular to the ecliptic, instead of the equator, the resulting arc length is the medieval "second declination" (199:12).

To calculate the (first) declination in terms of the longitude (λ), use

(199:8) $\sin \delta = \sin ZE = \sin AZ \sin BD / R = \sin \lambda \sin \varepsilon / R$

This follows from a tacit application of the Rule of Four.

To calculate δ_2, the second declination, consider Figure 53, where the configuration is that of the preceding figure, except that the ecliptic and equator have been interchanged, and all four sides of the quadrilateral have been extended. Also the great circle *LOHT*, with pole at *Z*, has been added.

Application of the Rule of Four to triangles *AHT* and *AKM* gives

(199:16) $\sin (AH{=}\overline{AE}) / \sin HT = \sin (AK{=}90°) / \sin (KM{=}\varepsilon)$.

AH and $AE = \lambda$ are complementary because *H* is the pole of circle *LGZE*, the angles at *E* and *L* being right angles.

Hence $\sin HT = \cos AE \sin KM / R$,

$$HT = \text{arc} \sin (\cos \lambda \sin \varepsilon / R)$$

Another application of the Rule of Four, this time to triangles *OHK* and *HLS*, yields

(199:18) $\sin(OH{=}\overline{HT}) / \sin(OK{=}\bar{\varepsilon}) = \sin(HL{=}90°) / \sin(LS{=}EZ{=}\bar{\delta}_2)$.

That *OH* and *HT* are complementary follows from the fact that *O* is the pole of *DZTM*. As for *LS* and *EZ*, $LS = \overline{GL}$, since *G* is the pole of *SKHAEB*. And $GL = \overline{GZ}$ because *Z* is the pole of *LOHT*. Finally, $GZ = \overline{ZE}$. The equation is equivalent to

$$\cos \delta_2 = R \cos \varepsilon / \cos HT .$$

Substituting the expression for *HT* above, we have

(200:2) $\cos \delta_2 = R \cos \varepsilon / \cos \text{arc} \sin(\cos \lambda \sin \varepsilon/R)$.

If, however, the theorem of (198:9) is applied to triangle *AEZ*, taking *A* as the acute angle, there results,

(200:8) $\sin(AE{=}\lambda) / \sin(AB{=}90°) = \tan(EZ{=}\delta_2) / \tan(BD{=}\varepsilon)$,

from which is the expression for δ_2,

(200:10) $\tan \delta_2 = \sin \lambda \tan \varepsilon / R$

which is certainly far easier to compute than (200:2).

146. Calculating the Arc of Daylight (201:1 - 202:5)

As an additional example of the utility of the tangent function, Bīrūnī poses the following problem. Given δ, *ZE* in Figure 54, and $\phi = GD$, calculate $e = AE$, the equation of daylight.

He writes

(201:6) $\sin(EZ{=}\delta) / \sin ZC = \sin(BD{=}\bar{\phi}) / \sin(DG{=}\phi)$,

from which *ZC* can be found. Bīrūnī does not notice that the equation is in fact equal to cot ϕ. Neither does he say how the expression was obtained, but application of the law of sines to triangles *ACZ* and *AEZ* gives

$$\sin \phi / \sin CZ = 1/\sin AZ = \sin \bar{\phi} / \sin \delta ,$$

which is equivalent to his equation.

Now

(201:9) $\sin(GZ{=}\bar{\delta}) / \sin ZC = \sin(GE{=}90°) / \sin(EA{=}e)$,

by the Rule of Four applied to triangles *GZC* and *GAE*.
Or

(201:10) $\mathrm{Sin}\ e = R \cdot \mathrm{Sin}\ ZC\ /\ \cos\ \delta$

$\doteq R\ (\mathrm{Sin}\ \delta\ \mathrm{Sin}\ \phi\ /\ \mathrm{Cos}\ \phi)\ /\ \mathrm{Cos}\ \delta.$

On the other hand, an application of theorem (198:9) to triangle *AEZ* , taking angle *DAB* as the acute angle employed, gives

(202:3) $\mathrm{Sin}(AE{=}e)\ /\ \mathrm{Sin}(AB{=}90°) = \mathrm{Tan}(ZE{=}\delta)\ /\ \mathrm{Tan}(DB{=}\bar{\phi})$,

whence

(202:1) $\mathrm{Sine}\ e = \mathrm{Tan}\ \delta \cdot R\ /\ \mathrm{Tan}\ \bar{\phi}$.

With the benefit of modern notation it is easy to see that grouping of the pairs of functions of δ and ϕ in (201:10) gives this immediately.

p. 159 CHAPTER 28

DISTANCES AND HEIGHTS BY USING SHADOWS

147. The Width of a Valley (202:8 - 203:7)

This chapter gives a number of problems in mensuration solvable with the astrolabe or with gnomons. They are applications of elementary plane geometry of a genre frequently encountered in books on the astrolabe. The first example follows.

The observer stations himself with an astrolabe at *A* (in Figure C14) on the accessible bank, and sights

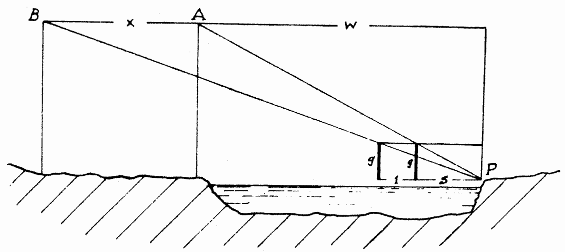


Figure C14

through the alidade pinnules *P* on the far bank. The number *S* which he then reads off the scale of horizontal shadows is the length in digits of a shadow which would be cast in light emanating from *P* by a gnomon set up as shown. Its position is such that the end of the shadow coincides with *P*. The observer then rotates the alidade to the setting for a shadow of *S* + 1 digits, and he retires along a horizontal line normal to the valley. He continues until he finds a second station, *B*, from which

P again appears when sighted through the pinnules. This implies that the gnomon would have to be moved one digit toward *B* in order that a shadow cast by light from there again terminate at *P*. The length *AB* being measured and found to be *x* units, the valley width is

(203:7) $$w = Sx$$

in the same unit. For, by invoking the proportionality of corresponding sides of similar triangles, $w / x = S / 1$.

The technique is in theory sound, but not very practical. The valley bank must be endowed with a providentially level stretch of sufficient length; the astrolabe shadow scale is essentially imprecise, and, as Bīrūnī says, *AB* should be elevated considerably above *P*.

148. The Height of an Object Whose Base Is Accessible (203:8 - 205:9)

A technique which requires no computation is for the observer so to station himself that the angle of elevation from him to the top of the object is 45°. There being then an isoceles right triangle, the distance to the base of the object equals its height.

A more general method is illustrated in Figure 55. The observer, from any convenient distance, rotates the alidade of his astrolabe until the sights are aligned on *A*, the summit of the object. He then reads *S*, the amount of the horizontal shadow, off the appropriate scale on the back of the instrument. He must also measure the distance *BG* from his station to the foot of the object. By similar triangles

$$AB / BG = g / S ,$$

where, as usual, *g* is the gnomon length. So

(204:10) $$AB = BG \cdot g / S .$$

If the observation is not taken from ground level, due adjustment must be made of the result.

The example allegedly from the *Brahmasp.* is rather an inverse of the problem as posed. Now, as illustrated in Figure 56, the distances are given, and the shadow is required. Again, by similar triangles,

$$GD / TG = S / g = BG / AM$$

whence

(205:4) $$S = BG \cdot g / AM = 110 \times 12 / 88 = 15$$

(The rule, but not the example, is given in *Brahmasp.* 12, 53 and 19, 14. However, the worked example is in the commentary of Pṛthūdakasvāmin. D.P.) (See Section 114).

149. Height of an Object - Base Inaccessible (205:10 - 208:6)

The situation is indicated on Figure 57, where the dimensions and dotted lines have been added by the translator; they do not appear on the text figure.

With the astrolabe the observer measures *S*, the horizontal shadow cast at *D* by a light source at *A*. He finds also ΔS, the increase in shadow length when the instrument is moved from *D* to *E*. The distance $DE = \Delta x$ is measured. Then the desired height is

(206:8) $$y = AB = g \cdot DE / \Delta S = 12 \cdot \Delta x / \Delta S$$

Further,

(206:10) $$x = DE \cdot S / \Delta S = x \cdot S / \Delta S$$

These also follow from the proportionality of corresponding pairs of sides of similar figures, e.g.,

(206:13) $$\Delta S / g = ED / AB = \Delta x / y .$$

Another rule of Brahmagupta is illustrated on Figure 58, where again the dimensions have been added to the text figure. Here the astrolabe is replaced by a gnomon at each of the stations.

Here, as shown,

(207:7) $S_2 = 18$ and $DE = 7$,

(207:9) the "base" $x_2 - x_1 = DZ = 7 + 18 = 25$,

(207:10) $S_2 - S_1 = 3$, so $S_1 = S_2 - 3 = 15$.

The rule is

(207:11) $x_1 = (x_2-x_1)s_1 / (s_2-s_1)$ $x_2 = (x_2-x_1)s_2 / (s_2-s_1)$

$= 25 \cdot 15 / 3 = 125$ $= 25 \cdot 18 / 3 = 150$,

and

(207:12) $y = g \cdot x_1 / s_1$ $y = g \cdot x_2 / s_2$

$= 12 \cdot 125 / 15 = 100$ $= 12 \cdot 150 / 18 = 100$

The final computation is not carried out in the text.

Bīrūnī demonstrates the validity of the rule by noting that, from the similar triangles *ZHE* and *ZAB*,

(207:18) $(BZ = x_2) / (ZE = s_2) = (AB = y) / (EH = g)$.

The text has the second ratio upside down. It is corrected here.

From the similar triangles *DTG* and *DAB*,

(208:1) $(GD = s_1) / (DB = x_1) = (GT = g) / (AB = y)$.

Hence $ZB / ZE = DB / GD$,

or

(208:3) $(ZB = x_2) / (DB = x_1) = (ZE = s_2) / (DG = s_1)$.

From this

(208:4) $(ZK=s_2-s_1) / (DG=s_1) = (DZ=x_2-x_1) / (DB=x_1)$,

and

(208:5) $(ZK=s_2-s_1) / (ZE=s_2) = (DZ=x_2-x_1) / (ZB-x_2)$.

Thus Brahmagupta's expressions for x_1 and x_2 at 207:11 have been established.

The final steps at 207:12 depend upon the proportions

- $y/g = x_1/s_1$ and $y/g = x_2/s_2$

already set down at 208:1 and 207:18.

(This rule is found in *Brahmasp.* 12, 54 and 19,5 and 15. The numerical example is from the commentary of Pṛthudakasvāmin. D.P.) (See Section 114.)

150. Special Cases (208:7 - 209:13)

As usual, Bīrūnī feels constrained to exhaust possible special cases. One possibility is that the base of the second gnomon coincide with the end of the first shadow so that $ED = 0$ as shown in Figure 59a. He does not discuss the consequences in general terms but alters the numerical example accordingly, keeping the same minaret and the same position for the first gnomon. Then $x_1 = 125$ as before,

but now $(s_2+x_1) / y = s_2 / g$

or $(s_2+125) / 100 = s_2 / 12$

so $12 s_2 + 1500 = 100 s_2$

and (208:10) $s_2 = 17\frac{1}{22}$,

as he says, but without explanation.

The second contingency occurs when the first shadow overlaps the second, by an amount *ED* as shown in Figure 59b. Again a specific assumption is made, that $ED = 7$ as before, but now *E* is to the left of *D*. We proceed to calculate $s_2 = EZ$, assuming as before that the minaret and the first gnomon are unchanged

$$(s_2 - ED + BD) / y = s_2 / g ,$$

$$(s_2 - 7 + 125) / 100 = s_2 / 12 ,$$

$$12s_2 + (118)12 = 100 s_2$$

$$s_2 = 177/11 = 16\frac{1}{11} = EZ .$$

The text says at 209:1 that $EZ = 9\frac{1}{12}$, which is absurd, since it would made s_2 shorter than s_1. We note that $DZ = s_2 - ED = 16 - 7 = 9$, which differs very slightly from the number in the text. Perhaps Bīrūnī had this in mind.

The chapter closes with a passage to the effect that the same technique may be used for measuring depths as well as heights and widths.

CHAPTER 29

CELESTIAL DISTANCES INVOLVING SHADOWS

151. Solar Distance by Means of an Orifice (210:3 - 212:1)

The language of this passage is sometimes obscure, as when, at 210:10, Bīrūnī speaks of the "shadow" cast by a beam of sunlight passing through an orifice. Nevertheless, the general tenor of the argument is clear.

The orifice, whether circular or a narrow slit is not stated, is of width EZ in an opaque plate, represented in Figure 60, fixed normal to the sun's rays. Lines AE and BZ are rays extending from the sun's limbs to the near edges of the orifice and striking, at H and T respectively, a plane surface parallel to EZ. L is the center of TH. BM is the ray from one limb of the sun's disk to the opposite edge of the orifice at E.

Assuming that LH, LM, EZ, and EK can be measured (210:11), the author proceeds to calculate the solar distance and apparent diameter in terms of these segments. Draw EO parallel to BT. Then, by similar triangles

(210:16) $$MO / ME = TM / MB .$$

Now $$MO = ML-LO = ML-(TO-TL) = ML-(EZ-LH) ,$$

$ME = \sqrt{EK^2+KM^2} = \sqrt{EK^2+(OM+ZF)^2}$, and $TM = MO+TO = MO+EZ$.

So the fourth term of the above proportion, MB, can be calculated.

It is now claimed that triangle TMB "is known as to sides". TM and MB are indeed known, but the third side, BT, has not been worked out in terms of the measured quantities, although this can now be done. This goes also for CT, the solar distance. Next

(210:18) $$TZ / (ZF=\frac{1}{2}(EZ-HT)) = TB / (BC=AN) ,$$

from which BC can be determined. Then the solar diameter is $AB = 2(BC + LH)$.

In exactly the same fashion, if DG were the apparent lunar diameter, the computation would proceed as before, except that DES would replace BEM, and so on, culminating in the lunar distance and diameter.

Bīrūnī returns to the sun, observing at 211:10 that the angles as well as the sides of triangle MTB can be calculated.

The geometry involved is unexceptionable, but the technique is quite impractical. The rays penetrating the hole EZ are very nearly parallel, yet the observer is required to note precisely the positions of points H and M, which is out of the question. The region TH is illuminated from the whole solar disk. At H the illumination begins to decrease, until at M no direct light is received at all. The three regions TH, HM, and from M on must merge imperceptibly into each other.

Bīrūnī gives no numbers at all, and his skepticism is implied at the end of the passage. In the next chapter he makes a weak attempt to apply the technique (see Section 156 below), but soon gives it up as a bad job.

152. Ptolemy on the Solar Distance (212:2 - 213:2)

Now, using Figure 61, the idea is to combine the previously determined lunar distance and diameter with the fact that the lunar and solar apparent diameters are almost equal, to calculate the solar distance and diameter.

The values of DG, EZ and $MZ = EZ - DG$ are at hand, and ET can be calculated in terms of these quantities. So,

(212:13) $$ET / EZ = TO(=TE+GE) / OK ,$$

in which OK is the only unknown, and can be calculated. Further, $HK = OK-OH$ can now be found, since OH is the known lunar radius. Draw SK parallel to EH. So in

(212:16) $$ZS\ (=ZE-KH) / SK\ (=EH) = ZE/ EA ,$$

EA is the only unknown and can be calculated. This is taken as the solar distance. Finally,

(212:18) $$EH / HO = EA / AB ,$$

where the only unknown is AB, the solar radius.

This completes the problem, and it is essentially the method explained in *Almagest* 5, 15, where , however, numerical results are carried through.

Our author, at 213:1, iterates his undertaking of 209:11 to compose a separate treatise on the determination of distances. There is no evidence that he in fact carried through on this.

153. Sinān b. al-Fatḥ and al-Kindī on the Lunar Distance (213:3 - 214:10)

The few lines quoted here seem to be the only extant fragment of the works of this minor scientist. Born at Ḥarrān on the upper Euphrates, Sinān (fl.c.925) wrote books on arithmetic and algebra (*Suter*, p.66).

Reworded, his rule amounts to this: Observe the (apparent) lunar altitude at culmination, h_n. Calculate h, its true (i.e. geocentric) altitude at the same time. The difference, $p = h - h_n$, is the parallax in the altitude circle. Then the lunar distance is

(213:5) $$D = R / p \ .$$

An independent derivation may illuminate both the rule and Bīrūnī's discussion. In triangle *TAE* of Figure C15, the altitude from *A* is approximately $r \cos h$, as shown. Also

$$\sin p \approx r \cos h / D \ .$$

Introducing the medieval functions and solving for D,

$$D \approx r \text{ Cos } h / \text{ Sin } p \ .$$

If we measure D in earth-radii, as customary, $r = 1$, and the rule becomes

(1) $$D \approx \text{Cos } h / \text{ Sin } p \ .$$

Suppose that the moon is on the horizon, or nearly so. This violates the condition of the rule, that it be culminating, but it is what Bīrūnī implies in saying (at 213:12, referring to Figure 62) that $p = \widehat{GD}$. He then writes

(213:13) DK (=Sin $\widehat{DG}$) / DE(=Sin 90°) = DK(=r=1) / DE(=the lunar distance).

(2) Or, in our notation, $D = R$ / Sin p.

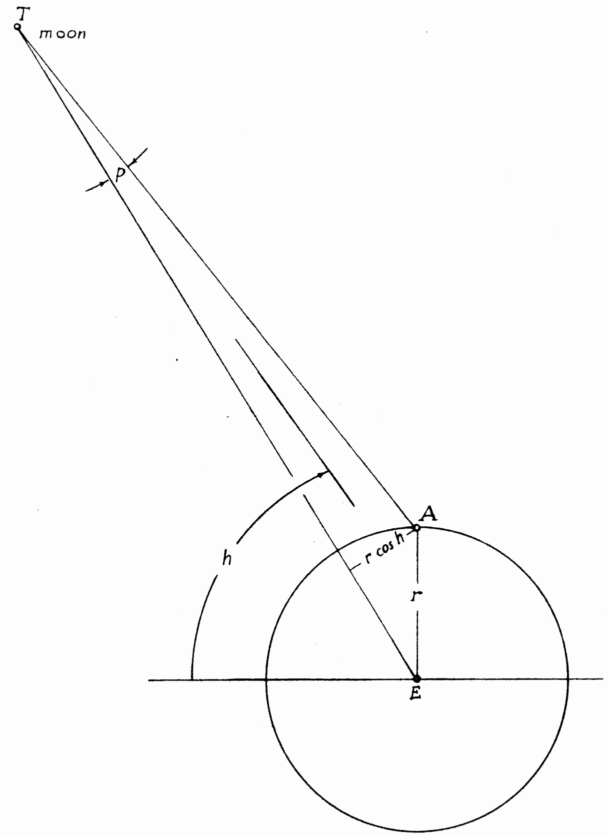


Figure C15

This is also the special case of expression (1) when $h = 0$, for Cos $0 = R$. And, as Bīrūnī says (214:5), if in Sinān's rule of 213:5, p is replaced by Sin p, equation (2) results.

Suppose, however that Sinān is using Āryabhaṭa's R = 57;18 = 3438', as did some of his contemporaries. Then for small p, the case here, Sin $p \approx p$, the angular measure being degrees (cf. *Overview*, p. 347). The general expression (1) then becomes

$$D \approx \text{Cos } h / p$$

and, for the special case $h \approx 0$,

$$D \approx R / p$$

which is precisely Sinān's rule.

The passage closes with a reference to a lost work of al-Kindī on the lunar distance. This is probably the source of his remarks on parallax previously noted (see Sections 12, 21, and 75). The word *qāthiṭ* is an Arabic transliteration of Greek κάθετος , "perpendicular".

154. Lunar Distance from Eclipses (214:11 - 216:5)

In this section Bīrūnī shows that the lunar distance can be obtained from observations of lunar eclipses. To this end, Figure 63 uses the shadow cone cast in space by the earth in the rays of the sun. In our version of the figure the cone is sketched in; in the text only the axis, *GB*, and an element *GA*, appear.

A pair of eclipses are observed, both in what can be thought of as the plane normal to *GB* at *D*. The reader may find it useful to follow the discussion on Figure C16, which depicts this plane. What we will call the first eclipse takes place when the lunar latitude $\beta_1 = DZ$. The magnitude *EH*, the portion of the lunar diameter which penetrates into the shadow, is taken to be 1/3 (215:4).

A second eclipse, portrayed dotted on Figure C16, has latitude $\beta_2 = DT$ and magnitude *EK* of 1/5 (215:6).

Since 1/3 - 1/5 = 2/15,

(215:8) 2/15 = $KH(=\Delta\beta=TZ)$ / (the apparent lunar diameter).

From this the lunar diameter can be calculated, of which *ZE* is a sixth.

Hence the shadow radius

(215:11) $$ED = ZD - EZ$$

can be determined.

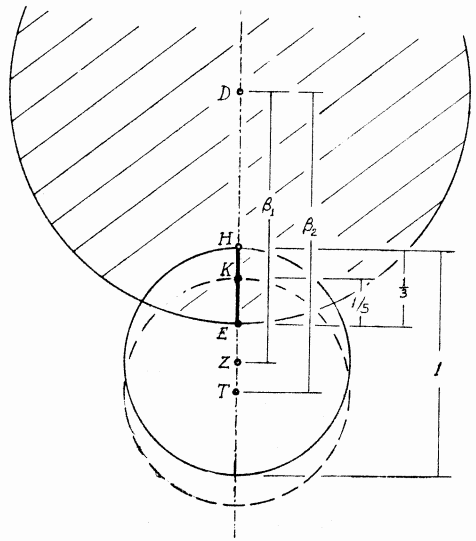


Figure C16

In the same manner, presumably by observation of a second pair of eclipses, both at distance *LB*, the corresponding shadow radius *LM* is calculated.

Now

(215:13) *DL* (=*BL-BD*) / *OE*(=*ED-ML*) = *GD* / *DE* .

So *GD* can be calculated. *BG* being known, and since

(216:2) $$BG / BA = GD / DE ,$$

AB can be calculated in units such that, presumably, *BG*=60. Finally, taking *AB* as a unit, express the lunar distances and the length of the shadow cone in terms of it.

In principle the method is valid, provided that observations be at hand of two pairs of eclipses which fulfill the conditions assumed.

CHAPTER 30

MISCELLANEOUS PROBLEMS

155. The Disappearing Umbrella Shadow (216:8 - 217:8)

The reader who considers seriously the disparate and sometimes whimsical topics with which the book closes can only agree with the author's verdict — they are troublesome indeed. At the same time, some involve lost documents which are of great historical interest.

As for the first problem, presumably the idea is to elevate a parasol sufficiently high in the direction of the sun that the vertex of the shadow cone cast by the parasol will coincide with the observer. This requires that

(1) $$D_u / X_u = D_s / X_s$$

where D stands for distance, X for diameter, u for umbrella, and s for sun (see Figure C17a).

The rule at 216:12 says

$$D_u = (10{,}000/4)\ X_u = 2{,}500\ X_u$$

so that in this case $D_u = 2{,}500 \quad 4 = 10{,}000$ cubits.

The word *ayuta* (at 216:13) is Sanskrit for ten thousand (*India*, transl., vol. 1, p. 175); the transliteration with Arabic *jīm*, *ajūta*, is doubtless a reflection of a local pronounciation.

Bīrūnī goes on to claim that the ratio

(216:16) $$X_s / D_s = 1 / 625 \ (= 4/2{,}500) \quad ,$$

where now, as in Figure C17b, the distance involved is the axis length of the earth's shadow cone. Two misconceptions are involved. First, the author acts as though the parasol's shadow cone were 2,500 cubits in length, whereupon dividing this by the parasol diameter would give the "convergence ratio" of 1/625. But the length of the cone is 10,000 cubits.

What is worse is his assumption, continued in the sequel, that the convergence ratio of the parasol's

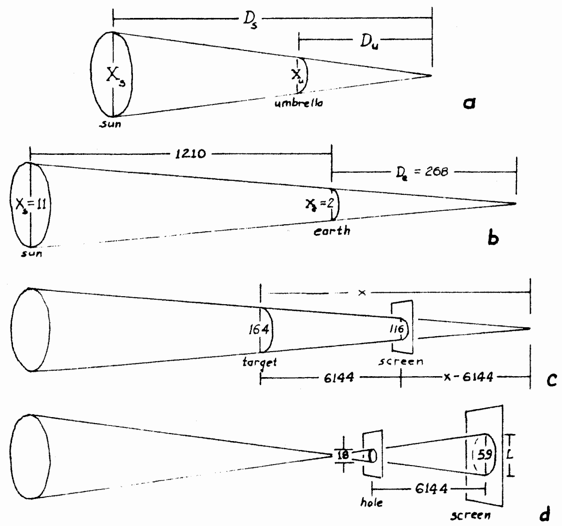


Figure C17

shadow equals that of the earth, which is by no means the case.

He now examines the situation with Ptolemaic parameters. With these the convergence ratio of the earth's shadow cone will be

(217:1) $$X_s / D_s = 11/(1210+268) \approx 1 / 134$$

as can be seen from Figure C17b. The various distances and dimensions are from *Almagest* 5,15 and 5,16.

Bīrūnī persists in thinking this applies to the parasol shadow also, so that to find the length of its shadow he puts

(217:3) $D_u = 134\ X_u = 134 \cdot 4 = 536$ cubits,

although the text writes 534, perhaps a scribal error. We do not understand the talk of division at 217:5 nor the origin of the number nineteen cited later. Perhaps it arises from the same sort of reasoning which led to expression (216:16).

156. An Abortive Attempt at the Solar Distance (217:9 - 218:7)

The author now reports on an experiment along the lines discussed in Section 151. First he poses a target of diameter 164 in some suitable units, the shadow of which is cast upon a screen distant from it by 6144 of the same units. The arrangement is schematized in Figure C17c. He observes that the diameter of the target's shadow is 116. To determine the distance at which the shadow disappears completely one may solve the equation

$$x / 164 = (x-6144) / 116 ,$$

as seen from Figure C17c. Its root is $x = 20992$ as the text says at 217:13. So in this cone the "ratio of convergence" is 164 / 20,992.

Now Abū Rayḥān begs the question by assuming the same ratio for the Ptolemaic $X_s = 11$ and $X_e = 1$. He puts $D_s / X_s = D_s / 11 = 20992/164$,

whence

(217:15) $D_s = 1408$ earth radii.

Also $D_e / X_e = D_e / 2 = 20992 / 164$, from which

(217:16) $D_e = 256$ earth radii.

So the earth-sun distance is 1408-256 = 1152 earth radii, and the difference between the Almagest value and the new determination is 1210-1152 = 58, as stated at 217:16.

Bīrūnī claims that in the same units, the sun's nearest distance would be 1163. We attempt to verify this by taking Ptolemy's eccentricity of 0;2,30 (from

Almagest 3,4), from which the nearest distance should be

$$1210 \times 57.5 / 60 = 1159.6,$$

which is close to the text value, but not quite the same. If the text result is accepted the difference becomes 1163 - 1152 = 11, although at 217:18 Bīrūnī calls it 10.

He now starts over again, using an orifice through which the sun shines, instead of the shadow of a target. The new situation is sketched on Figure C17d. This procedure is much closer to what he described in Section 151 and Figure 60, but he is unable to carry through completely.

The diameters of the hole and of the spot of light on the screen are 18 and 59 respectively, the distance between them being the same as that between the target and the screen in the preceding experiment. If the size of the hole were 164 units, like the target, the size of the spot would be

(218:1) $L = 59 \times 164 / 18 = 537\frac{5}{9}$ in diameter.

To avoid fractions he again changes scale, multiplying all dimensions by nine:

(218:3) $9 \times 6144 = 55,296$ for the ruler,

$9 \times 164 = 1,476$ the target and hole,

$9 \times 116 = 1,044$ the shadow,

and $9 \times 537\frac{5}{9} = 4,838$ the spot.

At this stage the futility of the operation overcomes the author. He throws up his hands, and consigns the data to anyone who is prepared to work with them. He adds, however, one item. From 217:8 he recalls that in standard units the length of the ruler bearing the target and the orifice is

(218:6) 5 cubits = 5×24 = 120 digits

(see *Luckey*, p.6).

Somehow or other he arrives at a result of 321,563,636 digits for the radius of the earth. Division by 24 gives

$$13,398,484\frac{5}{6} \text{ cubits}.$$

Two other estimates by Bīrūnī are:

in the *Canon* 12,851,269;50,42 cubits,

and in the *Taḥdīd* 12,803,337;2,9 cubits,

(*Taḥdīd*, comm., p. 143).

157. Al-Fazārī's Rule for the Lighted Portion of the Earth (218:11 - 220:15)

This passage gives a fragment from the non-extant zīj of Fazārī (Section 39). Consider the great circle on the earth determined by a plane through its center normal to the rays of the sun. The rule says that from any point on this circle, in a direction opposite the sun, the earth will be lighted by the sun to a distance of

(218:16) $(d/2)\cdot(6583/21600')$ farsakhs,

where d is the solar apparent diameter in minutes of arc. A farsakh contains three miles.

The rule is satisfactorily explained by Bīrūnī using Figure 64, and we paraphrase his remarks. Note that the method is essentially that used by Hipparchus, c.150 B.C. (*Dreyer*, p. 183). Let the circle *MTH* represent the earth with the line *ES* through the centers of the earth and the sun. Then on a large circle *SLZ* with center at *E* the apparent diameter of the sun will intercept an arc d bisected at S as shown. The inscribed angle *SZL* will be of magnitude $d/4$ since it intercepts an arc of magnitude $d/2$. *LZ* is an element of the shadow cone, and the sun's rays will reach around beyond the diameter (or great circle) *DK* illuminating the zone *KDTM* in addition to the hemisphere *KHD*. The portion of the great circle *MHT* illuminated in addition to the semicircle *KHD* will be $\widehat{KM} + \widehat{TD} = 2\,\widehat{TD}$. But because of the similarity of triangles *TED* and *DZE*, $\widehat{TD} = \angle TED = \angle EZD = d/4$. Hence, in linear rather than angular units,

$$2\cdot\widehat{DT} = 2\cdot(d/4)\cdot(c/360°),$$

where d is in degrees and c is the earth's circumference. But this is al-Fazārī's rule, since 216000' = 360° and 6583 is his c.

There follows a metrological discussion. Before commenting upon it we note that 6583 farsakhs is close to

the value $6597\frac{9}{25}$ attributed to Fazārī elswhere (*Taḥdīd*, comm. p. 147). On the other hand. Bīrūnī himself attempts to explain this parameter by saying that Fazārī increased Pulisa's earth circumference of 5026 yojanas by a quarter

(220:14) $5025 + (5026/4) = 5026 + 1256\frac{1}{2} = 6282\frac{1}{2}$

$\approx 6283.$

Perhaps, therefore, one digit in the 6583 of the rule (218:16) should be emended to 6283.

The equation

(220:3) 1 yojana = $2\frac{2}{3}$ farsakhs

is suspect. There is authority, in our text at 220:12, and in the *India* (transl. vol.1, p. 167) for putting

1 Indian farsakh = 16,000 cubits.

Then $2\frac{2}{3}$ farsakhs = $42{,}666\frac{2}{3}$ cubits,

not 38,000 as stated at 220:3. A closer approximation is

$2\frac{1}{3}$ farsakhs = $37{,}333\frac{1}{3}$ cubits, but the

exact conversion is

$2\frac{3}{8}$ farsakhs = 38,000 cubits.

Perhaps the text should be emended along these lines.

The relations

1 yojana = 8 *kroh* = 8 miles

appear also in the *India* (vol.1, p. 167).

Brahmagupta's values, accurately reported at 220:6 and in the *India* (transl., vol. 1, p. 168) are from *Brahmasp.* 21,32. They lead to

$$5000/1581 = 3\frac{257}{1581} = 3.1625\ldots \approx\sqrt{10}\ ,$$

which is Brahmagupta's approximation to π.

Returning to the circumference of 5026 and diameter of 1600 yojanas attributed to Pulisa, in the original Pauliśasiddhānta the circumference was taken as 3200 (*Pañca.* III, 14; comm. p. 31). But in the *India* (transl., vol. 2, p. 67) Bīrūnī gives $5026\frac{14}{15}$ and 1600 again.

158. Operations from an Archaic Zīj (220:16 - 221:5)

The material in this passage most certainly did not originate in the *Almagest*, but from the early stages of Muslim astronomy before it was affected by Ptolemy.

The first procedure is, given the shadow, s, calculate

(221:2) $\sqrt{s^2 + 144} = \sqrt{s^2 + 12^2} = \sqrt{s^2 + g^2} = \mathrm{Csc}_{12}h$ (cf. Section 38)

This is to be divided into

$$41{,}265' = 12 \times 3438' = 12 \times 57;18 = g \cdot R,$$

the R being that of Āryabhaṭa. Hence the next step is to find

(221:3) $g \cdot R / \mathrm{Csc}_g\ h = R / \csc h = R \sin h = \mathrm{Sin}_{57;18}\ h.$

The result, which is simply the sine of the solar altitude, is called the solar "hoop" (*ṭawq*). The same term, associated with the same R, appears in the *Taḥdīd* (comm., p. 146). Bīrūnī's exegesis, at 221:9-14 is clear and concise.

The next operation, explained in 221:15-18, calculates

(221:4) $3438' - \mathrm{Vers}_{3438}\delta = \mathrm{Cos}_{57;18}\delta$,

which is simply the radius of the day circle.

The third procedure prescribes the calculation of

(221:6) $(23/60)\ \mathrm{Sin}\ h = (23/60)\ \mathrm{Sin}_{150}\ h$

$= (23 \times 150/60) \sin h$

$= (3450/60) \sin h = 57;30 \sin h$

$= \mathrm{Sin}_{57;30}\ h$

where the expression to the left of the first equals sign is the rule quoted from the anonymous zīj. All the rest is based on Bīrūnī's explanation in 221:19 - 222:5. From this it is clear that the objective is simply to determine the sine of the solar altitude, but with a change of parameter included. It is tacitly assumed that the sine table at hand is one in which $R = 150'$, as in the *Khaṇḍ.*, and the result is transformed into a sine in which $R = 57;30 = 3450'$. This parameter has been encountered

nowhere else in the literature. As Bīrūnī says, it is distinct from Brahmagupta's $R = 54;30 = 3270'$ and Āryabhaṭa's $R = 57;18 = 3438$, of which it may be a rounding (cf. Section 32).

The language of the quotation contains technical terms: "hoop", "solar distance" (for declination), and "minutes of the chord", which were no longer current in Bīrūnī's time.

159. The Sindhind and Shahriyārān Zījes (222:6 - 223:12)

This quotation, from the same anonymous zīj cited above, purports to calculate a "difference" between the bases of two famous documents (or categories of documents) which have since disappeared.

The first is the Sindhind (Section 116) with base locality at the Cupola (*qubba*, sometimes written *qubbat arīn*, for Ujjain). Its coordinates are taken to be: $\Lambda = 90°$ (222:7) and $\phi = 0°$. The latter is not stated explicitly, but this can be inferred from what follows. For it

(222:8) $\mathrm{Cos}_{150}\ \phi_c = \mathrm{Sin}_{150}\ 90° = 150,$

the term "chord" here meaning sine. The subscripts C and B will denote Cupola and Baghdad respectively. The same transformation is then carried through here as in the preceding section to give

$$\mathrm{Cos}_{57;30}\ \phi_c = \mathrm{Cos}_{57;30}\ 0° = 150 \cdot 23 \div 60 = 57;30.$$

These coordinates for the Cupola are consistent with what Bīrūnī says about it in the *Canon* (vol. 1, pp. 502-5). Since the point called the Cupola is the apex of a hemisphere, the inhabited half of the terrestrial globe, the notion of a dome was appropriate. He states that the idea was taken over by the Iranians from Indian books, not named.

The second zīj is the Shāh (Section 39), called here alternatively Shahriyārān. Its base is Babylon with coordinates given as: $\Lambda = 78°$, $\phi = 36°$.

No source available to us has precisely these coordinates for Babylon. In *Almagest* 2,13 the latitude characterizing the fourth climate is indeed 36°. But

Ptolemy, both in the Handy Tables and the Geography, gives $\Lambda = 79°$ and $\phi = 35°$ for Babylon. He is followed by ***Battānī*** (***zīj***, vol. 2, p. 43, see also ***Geogr.Tables***), but on p.35 Battānī also has "Babilūniya regio Bābil", $\Lambda = 78°$, $\phi = 32°$.

For the Shāh Zīj also

(222:11) $\text{Cos}_{150}\ \phi = \text{Sin}_{150}\ h_{\phi} = \text{Sin}_{150}\ \bar{\phi} = \text{Sin}_{150} 54° = 122.$

This is reasonably accurate, since 150 sin 54° = 150 × .8090 = 121.35. The same transformation gives

$$\text{Cos}_{57;30}\ \phi = (23/60)\cdot 122 = 46;46 \text{ as in the text.}$$

Next

(222:12) $\text{Cos}\ \phi_C - \text{Cos}\ \phi_B = 57;30 - 46;46 = 10;44,$

whereupon the text says "find its arc" by calculating

$$(11/7) \times 10;44 = 16;52.$$

Then

$$\text{Sin}_{150'} 16;52° = 0;43,16, \text{ called "the retained".}$$

(The parameter $R = 150'$ is not stated, but is clearly intended, for 150 sin 16;52 = 43;36,20, and the 16 for 36 may be a scribal confusion of *ya'* = 10 for ***lām*** = 30).

The text now turns ϕ_B into hours by computing

(222:15) $\phi_B/15 = 36/15 = 2\frac{2}{5}^h \quad (= 0;6^d).$

Underlying the operation is the relation $360° = 24^h = 1^d$ for the daily rotation. But it makes no sense whatever for latitudes. Since the daily mean motion of the sun is 0;59,8,20°, and 0;6 × 0;59,8,20 = 0;5,54,50, it is correct to say (at 222:16) that in $2\frac{2}{5}$ hours the sun travels 0;5,55°.

This last result is added to the retained to obtain (0;43,16 + 0;5,55≈) 0;49. The result is, as it were, a difference due to latitude. The text now sets about calculating an analogous difference due to longitude which, as it happens, is legitimate. For

(222:18) $(4/5)^h = (\Lambda_C - \Lambda_B) / 15 = (90° - 78°) / 15 = 0;2^d.$

In this time the sun travels 0;2 × 0;59,8,20 = 0;1,58,17°. The text has 0;1,57,36. This is the difference due to longitude. The latitude and longitude differences are next added to produce, at 222:19,

(0;1,57,36 + 0;49 = 0;50,57,36≈) 0;51, the final result.

It is, of course, sheer nonsense — that is why Bīrūnī has exhibited it. The anonymous author felt, in some vague way that both latitude and longitude should be taken into consideration in shifting from one zīj to another. His result does so, at whatever cost to the facts of astronomy.

The first part of the operation, the calculation of $\Delta\text{Cos}\ \phi$ at 222:12, is susceptible of interpretation. It gives the difference between the day radii of the two localities, and exhibits some relation to an Indian operation explained in the ***Taḥdīd*** (comm., p. 147) for finding the distance between two localities. Hence, as in the Taḥdīd, the coefficient by which Cos ϕ is multiplied, here 23/60, may involve a shift to some sort of terrestrial unit. Bīrūnī's reasonable conjecture of a change of R may have been wrong.

Nonsensical as it is, the passage is useful in that it gives reliable coordinates for the base locality of one version each of the lost Sindhind and Shahriyārān zījes.

160. The Final Passages (223:13 - 226:19)

In the little that remains of the book there is even less that requires comment, since Bīrūnī continues his recital of the foolishness of others.

Māshā'allāh (fl. 780) was, after Abū Maʿshar, perhaps the best known astrologer of the Middle Ages. An Iranian of the Jewish persuasion, his name went into Latin as Messahala. He served the Abbasid caliph al-Manṣūr (see ***Suter***, p. 7).

The mythical Hermes Trismegistos (226:3), the Hellenistic appellation of the Egyptian god Thoth, entered the Islamic world as a hero (or three heroes) of ancient times, author of books on philosophy, science, and magic (***EIne***, vol. 3, pp. 463-65).

The "Eighty-five Chapters" is referred to in various places in the literature, and a copy is extant in manuscript. However, it has not been studied in modern times (see ***Ullmann***, p. 292).

The practise of assigning "years" to the planets was well established in astrology. But the set given at 223:18 differs from any seen by us. It is

Saturn	32
Jupiter	64
Mars	48
Sun	72
Venus	50
Mercury	51
Moon	33

See *Neugebauer & Van Hoesen*, p. 10.

We find no other mention in the literature of the Aḥmad b. Salmān named at 225:8.

161. The Date of the Shadows

The colophon at 226:18 fixes the date at which this particular copy was finished as between 28 August and 26 September of 1234. It says nothing about when the work itself was composed. In the *Canon*, vol. 3, p. vii, Barani asserts that this took place in Ghazna in 1022. He gives no source for the statement, but Abū Rayḥān himself mentions Ghazna in the text, at 176:12. It is not difficult to show that Barani's date is reasonable, and cannot be in error by more than a few years. Bīrūnī's own bibliography (see *Aufsätze*, vol. 2, p. 490) lists the *Shadows*. Hence it must have been in existence by 1035. There is some reason for making the *Shadows* earlier than the *Canon*, since in the latter the modern tangent function is tabulated ($g = 1$, see Section 45 above), whereas the *Shadows* makes no mention of this practise by Bīrūnī himself. The *Canon* in turn was dedicated to Sulṭān Masʿūd, presumably shortly after he came to power in 1030. As for a *terminus post quem*, Abū Rayḥān had no opportunity to study Indian civilization and to become acquainted with the Sanskrit sources until after he had been hauled off to Ghazna by Sulṭān Maḥmūd in 1015 or thereabouts. And by the time he wrote the *Shadows* he had accumulated a great deal of information on these subjects, a task which must have occupied him for some years.

BIBLIOGRAPHY

Each entry below has its full title preceded by a short title or abbreviation. In the commentary, bibliographical references are given by citing the short title in italics.

ALMAGEST: Ptolemäus, ***Handbuch der Astronomie***, Deutsche Übersetzung und erläuternde Anmerkungen von K. Manitius. Vorwort und Berichtigungen von O. Neugebauer. B.G. Teubner, Leipzig, 1963.

ĀRBYABHAṬĪYA: Āryabhaṭa, *Āryabhaṭīya*, ed. by H. Kern, Leiden, 1874; transl. by W.E. Clark, University of Chicago Press, 1930.

AUFSÄTZE: Eilhard Wiedemann, ***Aufsätze zur arabische Wissenschaftsgeschichte***, 2 vols., Hildesheim - New York, 1970. (This is a collection of papers which originally appeared in the *Sitzungs-Er.)*

AL-BATTĀNĪ, BIOGR.: Willy Hartner, article on al-Battānī in *DSB*, vol. 1, pp. 507-516.

AL-BATTĀNĪ, ZĪJ: ***Al-Battānī sive Albatenii Opus Astronomicum***, ed. and transl. by C. A. Nallino, 3 vols., Milan, 1899-1907.

BEAL: Samuel Beal, ***Chinese Accounts of India***, vol. 1, Susil Gupta Ltd., Calcutta, 1957.

BOILOT: D. J. Boilot, ***L'Oeuvre d'al-Beruni***, Essai bibliographique, ***Mélanges de l'Institut Dominicain d'Études Orientales***, 2(1955), pp. 161-255.

BOSWORTH: Clifford Edmund Bosworth, ***The Ghaznavids, their Empire in Afghanistan and Eastern Iran, 994-1040***, Edinburgh University Press, 1963.

BOUCHÉ - L.: A. Bouché-Leclercq, ***L'Astrologie Grecque***, Paris, E. Leroux, 1899.

BRAHMASP.: Brahmagupta, ***Brāhmasphuṭasiddhānta***, edited by Sudhākara Dvivedin, Benares, Medical Hall Press, 1902.

CANON: Al-Bīrūnī, *Al-Qānūn'l-Mas'ūdī (Canon Masudicus)*, 3 vols., Osmania Oriental Publications Bureau, Hyderabad-Dn., India, 1954-1956.

CARRA: Carra de Vaux, "L'Almageste d'Abū'lwéfa Albūzdjānī", *Journal Asiatique*, 19(1892), pp. 408-471.

CHORDS: Al-Bīrūnī, *Maqāla fī istikhrāj al-awtār fī'l-dā'ira...*, extant in four MSS. The Patna version was published as the first of the *Rasā'il*. The Leiden version was translated into German by H. Suter in *Bibliotheca Mathematica*, vol. 11(1910), pp. 11-78. Both versions were used in a Russian translation by B.A. Rosenfeld and L.A. Karpova with a commentary by B.A. Rosenfeld and S.S. Krasnova in *Iz istorii nauki i tehniki v stranah Vostoka*, vyp. 3, Moscow, 1963.

CHRON.: Al-Bīrūnī, *Al-āthār al-bāqiya min al-qurūn al-khāliya*, ed. by E. Sachau as *Chronologie orientalischer Voelker von Alberuni*, Leipzig, 1878; reprinted Leipzig, 1923; reprinted Baghdad, 1963. Transl. by E. Sachau as, *The Chronology of Ancient Nations*, London, 1879.

CITTERT: Pieter Hendrick van Cittert, *Astrolabes, a Critical Description of the Astrolabes, Noctilabes, and Quadrants in the Care of the Utrecht University Museum*, Leiden, E. J. Brill, 1954.

COMM. VOL.: *Al-Bīrūnī Commemoration Volume, A. H. 362 - A. H. 1362*, Calcutta, Iran Society, 1951.

DAVIDIAN: M. L. Davidian, "Al-Bīrūnī on the Time of Day from Shadow Lengths", *JAOS*, vol. 80(1960), pp. 330-335.

DE CAELO: *Aristotle, On the Heavens*, text and transl. by W. K. C. Guthrie (Loeb Classical Library), London, 1939.

DE SENSU: pp. 207-277 of *Aristotle, On the Soul, Parva Naturalia, On Breath*, with an English translation by W. S. Hett, Harvard University Press, Cambridge, Mass., 1935.

DREYER: J. L. E. Dreyer, *A History of Astronomy from Thales to Kepler* (formerly titled *History of the Planetary Systems from Thales to Kepler*), 2d. ed., Dover Publications, Inc., 1953.

DSB: *Dictionary of Scientific Biography*, Charles Scribner's Sons, New York, 1970- ....

EI: *The Encyclopaedia of Islam*, 4 vols., Leyden and London, 1913-1934.

EINE: *The Encyclopaedia of Islam*, New edition, Leiden and London, 1960- ....

FAZĀRĪ: David Pingree, "The Fragments of the Works of al-Fazārī", *JNES*, vol. 29(1970), pp. 103-123.

FIHRIST: Ibn al-Nadīm, Muḥammad b. Isḥāq, *Kitāb al-Fihrist*, mit Anmerkungen, ed. by Gustav Flügel, 2 vols., Leipzig, 1871-72.

FRANK: Josef Frank, "Die Verwendung des Astrolabes nach al Chwārizmī", *Abhandlungen zur Geschichte der Naturwissenschaften und der Medizin*, Erlangen, vol. 3(1922), pp. 1-32.

GAL: Carl Brockelmann, *Geschichte der arabischen Literatur*, 2nd. ed. Leiden, 1943.

GEOGR. TABLES: Fuad I. Haddad and E.S. Kennedy, "Geographical Tables of Medieval Islam", *Al-Abhath*, vol. 24(1971), pp. 87-102.

ḤABASH, BERL.: Berlin, Preussische Staatsbibliothek, MS Ahlwardt 5750, Zīj by Aḥmad b. ʿAbdallāh Ḥabash al-Ḥāsib.

ḤABASH,BIOGR.: Sevim Tekeli, article on Ḥabash in the *DSB*, vol. 5, pp. 612-620.

ḤABASH, ISTANB.: Istanbul, Yeni Cami Kutuphanesi, MS 784,2°, *al-Zīj al-Dimashqī* by Aḥmad b. ʿAbdallāh Ḥabash al-Ḥāsib.

HARTNER: Willy Hartner, "The Principle and Use of the Astrolabe", Chapter 57 in *A Survey of Persian Art*, vol. VI, pp. 2530-2554, Oxford Univ. Press, 1938-9; reissue published 1964-5.

AL-HĀSHIMĪ: *The Book of the Reasons Behind Astronomical Tables (Kitāb fī ʿIlal al-Zījāt)* by ʿAlī b. Sulaymān al-Hāshimī, a facsimile edition with a translation by Fuad I. Haddad and E. S. Kennedy, and a commentary by David Pingree and E. S. Kennedy, to appear.

HEATH: Thomas L. Heath, *A History of Greek Mathematics*, 2 vols., Oxford, 1921.

HERON, BIOGR.: A. G. Drachmann, "Hero of Alexandria," article in the ***DSB***, vol. 6. pp. 310-314.

H & K: Heinrich Hermelink and E. S. Kennedy, "Transcription of Arabic Letters in Geometric Figures", ***JAOS***, vol. 82(1962), p. 204.

INDIA: Al-Bīrūnī, ***Kitab fī taḥqīq mā li'l-Hind...***, ed. by C.E. Sachau, London, 1888; ed. Hyderabad-Dn., 1958, transl. by C. E. Sachau as ***Alberuni's India***, 2 vols. London, 1910.

AL-ISTIʿĀB: Abū ʿUmar Yūsuf b. ʿAbdallāh b. ʿAbd al-Barr, ***Al-Istiʿāb fī Maʿrafat al-Aṣḥāb***, Cairo, 4 vols., no date.

JAOS: Journal of the American Oriental Society.

JHA: Journal for the History of Astronomy.

JNES: Journal of Near Eastern Studies

KANE: Pandurang Vaman Kane, ***History of Dharmaśāstra***, vol. II, part 1, Bhandarkar Oriental Research Institute, Poona, 1941.

KARAṆATILAKA: Vijayanandin, ***Karaṇatilaka***. The Sanskrit original is not extant. Transl. into Arabic by Bīrūnī as ***Ghurrat al-zījāt***, extant in a MS in the Dargah Library of Pir Muhammad Shah, Ahmadabad. Part of the Arabic text, with Engl. transl. and commentary publ. by Sayyid Samad Husain Rizvi in ***Islamic Culture***, vol. 37(1963), pp. 112-130, 167-187, 223-245; vol. 38(1964), pp. 47-74, 195-213; vol. 39(1965), pp. 1-26, 137-180. Another text, transl. and commentary, by M.F. Qureishi exist in typescript, but have not been published.

IBN KHALLIKAN: Ibn Khallikan's Biographical Dictionary, transl. from the Arabic by B^{n} Mac Guckin de Slane, 4 vols., Oriental Translation Fund of Great Britain and Ireland, Paris, 1842.

KHAṆḌ.: Brahmagupta, ***The Khaṇḍakhādyaka***, transl. by Prabodh Chandra Sengupta, University of Calcutta, 1934.

Al-KHWĀRIZMĪ,ALG.: The Algebra of Mohammed ben Musa, ed. and transl. by Frederic Rosen, London, 1831.

AL-KHWĀRIZMĪ, BIOGR.: G. J. Toomer, article on al-Khwārizmī in the ***DSB***, vol. 7, pp. 358-365.

AL-KHWĀRIZMĪ, ZĪJ: H. Suter, ***Die astronomischen Tafeln des Muḥammed ibn Mūsa al-Khwārizmī..., Kgl. Danske Vidensk. Skrifter***, 7. R., Hist. og filos. Afd. 3,1, Copenhagen, 1914. Engl. transl. by O. Neugebauer, ***The Astronomical Tables of al-Khwārizmī, Hist. Filos. Skr. Dans. Vid. Selsk.*** 4, no. 2, Copenhagen, 1962.

KING, 1: D. A. King, Ibn Yūnus' "Very Useful Tables" for Reckoning Time by the Sun, ***Archive for History of Exact Sciences***, vol. 10(1973), pp. 343-394.

KING, 2: D. A. King, "Al-Khalīlī's Auxiliary Tables for Solving Problems of Spherical Astronomy", ***JHA***, vol. 4(1973) pp. 99-110.

KING, 3: D. A. King, "On the Astronomical Tables of the Islamic Middle Ages", ***Studia Copernicana*** (Proceedings of the Colloquia Copernicana), Torun, Poland,1973.

KING, 4: David A. King, "Studies in Astronomical Timekeeping in Medieval Islam", to appear.

K & M: E. S. Kennedy and Ahmad Muruwwa, "Bīrūnī on the Solar Equation", ***JNES***, vol. 17(1958), pp. 112-121.

KUNITZSCH: Paul Kunitzsch, ***Untersuchungen zur Sternnomenklatur der Araber***, Otto Harrassowitz, Wiesbaden, 1961.

LESLEY: Mark Lesley, "Bīrūnī on Rising Times and Daylight Lengths", ***Centaurus***, vol. 5(1957), pp. 124-141.

LINDBERG, 1: David C. Lindberg, "The Theory of Pinhole Images from Antiquity to the Thirteenth Century", ***Archive for History of Exact Sciences***, vol. 5(1968/9), pp. 154-176.

LINDBERG, 2: David C. Lindberg, "Alkindi's Critique of Euclid's Theory of Vision", ***Isis***, 62(1971), pp. 469-489.

LIVINGSTON: John W. Livingston, "The Mukḥula, an Islamic Conical Sundial", ***Centaurus***, vol. 16(1972), pp. 199-308.

LUCKEY: Der Lehrbrief über den Kreisumfang (al-Risāla al-Muḥītiya), von Ǧamšīd b. Masʿūd al-Kāšī, übersetzt und erläutert von P. Luckey, Akademie Verlag, Berlin, 1953.

MAHĀB.: Bhaskara I, *Mahā-bhāskariya*, ed. and transl. by Kripa Shankar Shukla, Lucknow University, 1960.

MAIMONIDES: *The Mishneh Torah by Maimonides*, ed. with an English transl. by Moses Hyamson. vol. 2. Boys Town Jerusalem Publishers, 1962.

MĀSHĀ'ALLĀH: E. S. Kennedy and David Pingree, *The Astrological History of Māshā'allāh*, Harvard University Press, Cambridge, Mass., 1971.

MAYER: L. A. Mayer, *Islamic Astrolabists and their Works*, Geneva, 1956.

MERID., 1: E. S. Kennedy, "Bīrūnī's Graphical Determination of the Local Meridian", *Scripta Mathematica*, vol. 24(1959), pp. 251-255.

MERID., 2: E.S. Kennedy, "Al-Bīrūnī on Determining the Meridian", *The Mathematics Teacher*, vol. 50(1963), pp. 635-637. (see also, Hermelink, "Bestimmung der Himmelsrichtungen aus einer einzigen Schattenbeobachtung nach al-Bīrūnī, "*Sudhofs Archiv*, vol. 44 (1960), pp. 329-332.

METEOROLOGICA: Aristotle, *Meteorologica*, with an English translation by H. D. P. Lee, (the Loeb Classical Library), Harvard University Press, Cambridge, Mass., 1951.

MOOSA: Matti Moosa, "A New Source on Aḥmad ibn al-Ṭayyib al-Sarakhsī: Florentine MS Arabic 299", *JAOS*, vol. 92(1972), pp. 19-24.

MR: *Mathematical Reviews*.

NADIR: Nadi Nadir, "Abū al-Wafā' on the Solar Altitude", *The Mathematics Teacher*, vol. 52(1960), pp. 460-463.

NEUGEBAUER, 1: O. Neugebauer, "On Some Astronomical Papyri and Related Problems of Ancient Geography", *Transactions of the American Philosophical Society*, N. S. vol. 32(1942), pt. 2.

NEUGEBAUER, 2: O. Neugebauer, "Über griechische Wetterzeichen und Schattentafeln", *Österreichische Akademie der Wissenschaften, Phil.-Hist. Klasse, Sitzungsberichte*, 240. Band, 2. Abhandlung, Wien 1962, pp. 29-44.

NEUGEBAUER. 3: O. Neugebauer, "Notes on Ethiopic Astronomy", *Orientalia*, vol. 33(1964), pp. 49-71.

NEUGEBAUER & VAN HOESEN: O. Neugebauer and H. B. Van Hoesen, *Greek Horoscopes, Memoirs of the American Philosophical Society*, vol. 18, Philadelphia, 1959.

NICHOLSON: Reynold A. Nicholson, *A Literary History of the Arabs*, Cambridge University Press, 1956.

OVERVIEW: E. S. Kennedy, "The History of Trigonometry, an Overview", pp. 333-359 in the 31st *Yearbook of the National Council of Teachers of Mathematics*, Washington, D.C. 1969.

PAÑCA.: O. Neugebauer and D. Pingree, *The Pañcasiddhāntikā of Varāhamihira*, Part 1, Text and Translation, Part 2, Commentary; *Hist. Filos. Skr. Dan. Vid. Selsk.* 6, 1, Copenhagen, 1970, 1971.

PARALLAX: E. S. Kennedy, "Parallax Theory in Islamic Astronomy", *Isis*, vol. 47(1956), pp. 33-53.

PAULIŚASIDD.: David Pingree, "The Later Pauliśasiddhānta", *Centaurus*, vol. 14(1969), pp. 172-241.

PETERS: F. E. Peters, *Aristoteles Arabus, the Oriental Translations and Commentaries on the Aristotelian Corpus*, E. J. Brill, Leiden, 1968.

PETRAITIS: Casimir Petraitis, *The Arabic Version of Aristotle's Meteorology*, Librarie Orientale, Beyrouth, 1967.

PINES: S. Pines, "The Semantic Distinction Between the Terms Astronomy and Astrology According to al-Bīrūnī", *Isis*. vol. 55(1964), pp. 343-349.

PINGREE, 1: David Pingree, "The Mesopotamian Origin of Early Indian Mathematical Astronomy", *JHA*, vol. 4(1973), pp. 1-12.

PINGREE, 2: David Pingree, "Sanskrit Astronomical Tables in the United States", *Transactions of the American Philosophical Society*, N.S. vol. 58(1968), pt. 3.

Pingree, 3: David Pingree, "Brahmagupta, Pṛthudaka, and al-Bīrūnī", to appear.

PINGREE, 4: David Pingree, "Al-Bīrūnī's Knowledge of Sanskrit Astronomical Texts", to appear.

PLINY: Pliny, ***Natural History***, with an English translation in ten volumes, vol X, by D.E. Eichholz, The Loeb Classical Library, Harvard University Press, Cambridge, 1962.

QUR'AN: ***The Holy Qur'an***, Text, Translation, and Commentary, 3d. ed., by Abdullah Yusuf Ali, New York, 1938.

RAI: R. N. Rai, "Karaṇasāra of Vateśvara", ***Indian Journal of the History of Science***, vol. 6(1971), pp. 27-35.

RASĀ'IL: ***Rasā'ilu'l-Bīrūnī***, by Abū Rayḥan Muḥ. b. Aḥmad al-Bīrūnī, containing four tracts ... edited and published by the Osmania Oriental Publications Bureau, Hyderabad-Dn., 1948.

RONCHI: Vasco Ronchi, ***The Nature of Light, an Historical Survey***, Harvard University Press, Cambridge, 1970.

ROSENTHAL: Franz Rosenthal, ***Aḥmad b. aṭ-Ṭayyib as-Saraḫsī, a Scholar and Litteratéur of the Ninth Century***, (American Oriental Series, Volume 26) American Oriental Society, New Haven, Connecticut, 1947.

SABRA: A. I. Sabra, Article on Ibn al-Haytham in the ***DSB***, vol. 6, pp. 189-210.

SAIDAN: A. S. Saidan "The Rasā'il of Bīrūnī and Ibn Sinān, a Rearrangement", ***Islamic Culture***, 34(1960), pp. 173-175.

SANJARĪ ZĪJ: ***Kitāb jāmiʿ al-tawārīkh li'l-Sanjarī***, Vatican cod. Arab. 761.

SAYILI: Aydin Sayili, ***The Observatory in Islam***, Türk Tarih Kurumu Basimevi, Ankara, 1960.

SCHMIDT: Fritz Schmidt, ***Geschichte der geodätischen Instrumente und Verfahren im Altertum und Mittelalter***, Neustadt an der Haardt, 1935.

SCHOY: C. Schoy, "Abhandlung über die Ziehung der Mittagslinie, dem Buch über das Analemma entnommen, samt dem Beweis dazu von Abū Saʿīd aḍ-Ḍarīr", ***Annalen der Hydrographie...***, vol. 10(1922), pp. 265-271.

SÉDILLOT. J.J.: ***Traité des instruments astronomiques des arabes*** composé au treizieme siécle par Aboul Hhassan Ali, de Maroc, intitulé Jāmiʿ al-mabādī w'al-ghāyāt, traduit de l'arabe... par J.J. Sédillot... et publié par L. AM Sédillot, 2 vols., Paris, 1834-5.

SÉDILLOT. L.A.: ***Mémoire sur les instruments astronomiques des arabes***, par M.L. AM. Sedillot. Extrait du tome premier des memoires présentés par divers savants à l'Académie Royale des Inscriptions et Belles-Lettres, Paris, 1842.

SHADOWS: Al-Bīrūnī, ***Ifrād al-maqāl fī amr al-ẓilāl***, the second of the ***Rasā'il***. The present publication presents a translation and a commentary.

SHĀH: E. S. Kennedy, "The Sasanian Astronomical Handbook ***Zīj-i Shāh*** and the Astrological Doctrine of Transit", ***JAOS***, vol. 78(1958), pp. 246-262.

SHASTRI: Shastri, T.S. Kuppanna, "The System of the Vateśvara Siddhānta", ***Indian Journal of the History of Science***, 4 N. 1-2, 1969, pp. 99-106.

IBN SINĀN: ***Rasā'il ibn Sinān: li'l-ʿalāma Ibrāhīm bin Sinān bin Thābit bin Qurra***, Oriental Publications Bureau, Hyderabad-Dn., 1948.

SITZUNGS-ER.: ***Sitzungsberichte der Physikalisch-Medizinischen Sozietät zu Erlangen.***

SRINIVASAN: Doris Srinivasan, "Saṃdyā: Myth and Ritual", ***Indo-Iranian Journal***, vol. 15(1973), pp. 161-178.

SURVEY: E. S. Kennedy, "A Survey of Islamic Astronomical Tables", ***Transactions of the American Philosophical Society***, N.S. 46, Pt. 2, Philadelphia, 1956.

SŪRYASIDD.: K. S. Shukla (ed.), ***The Sūrya-Siddhānta with the Commentary of Parameśvara***, Lucknow, 1957. Transl. by E. Burgess, ***JAOS*** (1860), University of Calcutta reprint, 1935.

SUTER: H. Suter, "Die Mathematiker und Astronomen der Araber und ihre Werke", *Abhandlungen zur Geschichte der mathematischen Wissenschaften...*, X Heft, Leipzig, 1900.

TAḤDĪD: Al-Bīrūnī, *Taḥdīd nihāyāt al-amākin litaṣḥīḥ masāfāt al-masākin;* critical ed. of the text publ. by P.G. Bulgakov in *Majalla maʿhad al-makhṭūṭāt al-ʿarabiya,* Cairo, 1962. Russian transl. and comm. by P.G. Bulgakov is *Abu Reihan, Izbrannye Proizvedeniya III (Geodezia)*, Akademiya Nauk Uzbekskoi SSR, Tashkent, 1966. English transl. by Jamil Ali, *The Determination of the Coordinates of Cities, Al-Bīrūnī's Taḥdīd al-Amākin,* American University of Beirut, 1967. Comm. by E. S. Kennedy, *A Commentary upon Biruni's Kitāb Taḥdīd al-Amākin,* American University of Beirut 1973.

THAʿĀLIBĪ: *The Laṭā'if al-Maʿārif of Thaʿālibī*, translated with introduction and notes by C. E. Bosworth, The University Press, Edinburgh, 1968.

THAʿĀLIBĪ, KHĀṢṢ: *Kitāb Khāṣṣ al-Khāṣṣ*, Cairo, 1908.

THAʿĀLIBĪ, YATĪMA: Al-Thaʿālibī, *Yatīmat al-dahr*, Cairo, 1956,

TRANSITS: Al-Bīrūnī, *Tamhīd al-mustaqarr li taḥqīq maʿna al-mamarr.* The text was published as the third of the *Rasā'il.* *Al-Bīrūnī on Transits* is an English translation with commentary by M. Saffouri, A. Ifram, and E. Kennedy, The American University of Beirut, 1959.

ULLMANN: Manfred Ullmann, *Die Natur- und Geheimwissenschaften im Islam*, Leiden, E. J. Brill, 1972.

VIJAYANANDIN, BIOGR.: David Pingree, article on Vijayanandin in the *DSB*.

WAGNER: Ewald Wagner, *Abū Nawās, Eine Studie zur arabischen Literatur der frühen ʿAbbāsidenzeit,* Akademie der Wissenschaften und der Literatur, Veröffentlichungen der orientalischen Kommission, Band XVII, Wiesbaden, 1965.

WIDENGREN: Geo. Widengren, *Mani and Manichaeism,* Weidenfeld and Nicolson, London. 1965.

WIEDEMANN & FRANK: Eilhard Wiedemann and Josef Frank, "Die Gebetszeiten im Islam",pp. 757-788 in vol. 2 of *Aufsätze*; published first in *Sitzungs-Er.*, vols. 58, 59, (1926, 1927), pp. 1-32.

YAʿQŪB: David Pingree, "The Fragments of the Work of Yaʿqub ibn Ṭāriq", *JNES*, vol. 26(1960), pp. 97-125.

YĀQŪT: Yāqūt al-Rūmī, Shihāb al-Dīn, *Muʿjam al-udabā',* 20 vols., Cairo, 1936-38.

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Here, and in the indices of parameters which follow, references consisting of a pair of numbers separated by a colon are to the page and line respectively of the Hyderabad edition of the Arabic text. The same pages and lines are indicated in our translation. Page references preceded by the letter *s* are to the passage of the text published by mistake in the book by Ibn *Sinān*. The reader is reminded that other passages of the text were printed out of order. In the translation their proper order has been restored, but the pagination of the printed text has been retained. A full specification of the displaced passages has been given in the preface.

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