Path Planning for Groups Using Column Generation*

Marjan van den Akker, Roland Geraerts, Han Hoogeveen, and Corien Prins

Institute of Information and Computing Sciences, Utrecht University
3508 TA Utrecht, The Netherlands
{marjan,roland,slam}@cs.uu.nl,
C.R.Prins@students.uu.nl

Abstract. In computer games, one or more groups of units need to move from one location to another as quickly as possible. If there is only one group, then it can be solved efficiently as a dynamic flow problem. If there are several groups with different origins and destinations, then the problem becomes $\mathcal{NP}$-hard. In current games, these problems are solved by using greedy *ad hoc* rules, leading to long traversal times or congestions and deadlocks near narrow passages. We present a centralized optimization approach based on Integer Linear Programming. Our solution provides an efficient heuristic to minimize the average and latest arrival time of the units.

1 Introduction

Path planning is one of the fundamental artificial intelligence-related problems in games. The path planning problem can be defined as finding a collision-free path, traversed by a unit, between a start and goal position in an environment with obstacles. Traditionally, this problem and its variants were studied in the field of robotics. We refer the reader to the books of Choset *et al.* [3], Latombe [11], and LaValle [12] for an extensive overview.

The variant we study is the problem of finding paths for one or more *groups* of units, such as soldiers or tanks in a real-time strategy game, all traversing in the same (static) environment. Each group has its own start and goal position (or area), and each unit will traverse its own path. We assume that the units in a group are equal with respect to size and speed. The objective is to find the paths that minimize the average arrival times of all units.

Current solutions from the robotics field can be powerful but are in general too slow for handling the massive number of units traversing in the ever growing environments in real-time, leading to stalls of the game. Solutions from the games field are usually fast but greedy and *ad hoc*, leading to long traversal times or congestions and deadlocks near narrow passages, in particular when two groups meet while moving in opposite directions. Obviously, such solutions have a negative impact on the gameplay.

* This work was partially supported by the ITEA2 Metaverse1 (www.metaverse1.org) Project.

R. Boulic, Y. Chrysantou, and T. Komura (Eds.): MIG 2010, LNCS 6459, pp. 94–105, 2010.

One of the first solutions for simulating (single) group behavior was introduced by Reynolds in 1987 [17]. His influential boids model, comprising simple local behaviors such as separation, cohesion and alignment, yielded flocking behavior of the units. While this model resulted in natural behavior for a flock of birds or school of fish moving in an open environment, they could get stuck in cluttered areas. Bayazit *et al.* [2] improved this model by adding global navigation in the form of a roadmap representing the environment's free space. While the units did not get stuck anymore, they could break up, losing their coherence. By following a point that moves along a backbone path centered in a two-dimensional corridor, coherence was guaranteed by the method proposed by Kamphuis and Overmars [9]. In their method, the level of coherence was controlled by two parameters, namely the corridor width and the group area.

When multiple units are involved, possible interference between them complicates the problem, and, hence, some form of coordination may be required to solve the global problem. From the robotics field, two classes of methods have been proposed. Centralized methods such as references [18,19] compute the paths for all units simultaneously. These methods can find optimal solutions at the cost of being computationally demanding, usually making them unsuitable for satisfying the real-time constraints in games. Decoupled methods compute a path for each unit independently and try to coordinate the resulting motions [15, 21]. These methods are often much quicker than centralized methods but the resulting paths can be far from optimal. Also hybrid methods such as references [7,13] have been proposed. A variant to solving the problem is called prioritized motion planning [14,23]. According to some prioritization scheme, paths are planned sequentially which reduces the problem to planning the motions for a single unit. It is however not clear how good these schemes are.

Our main contribution is that we propose a centralized as well as computationally efficient solution for the path planning planning problem with groups. This solution translates the problem into a dynamic multi-commodity flow problem on a graph that represents the environment and uses column generation to identify promising paths in this graph. More concretely, it provides the division of characters at each node (and each time step). A series of divisions can be considered as a *global* path. The assignment of such a path to a specific unit and performing *local* collision avoidance between units is handled by an external local method such as the Predictive model of Karamouzas *et al.* [10] or the Reciprocal Velocity Obstacles of van den Berg *et al.* [22]. When characters follow the same global path, Kamphuis' method [9] can be used to introduce coherence if desired. Our solution can be used to handle difficult situations which typically occur near bottlenecks (e.g. narrow passages) in the environment. It is efficient because it provides a global distribution of the paths. As far as we know, it is the first method that combines the power of a centralized method with the speed and flexibility of a local method.

Our paper is organized as follows. In Section 2, we show that path planning for one group can be solved to optimality as a dynamic flow problem. Computing the distribution of paths for multiple groups is more difficult (i.e. $\mathcal{NP}$-hard). We propose a new heuristic solution that solves the corresponding dynamic multi-commodity flow problem in Section 3. We conduct experiments on some hard problems in Section 4 and show that they can be solved efficiently. In Section 5, we discuss the applicability of this technique for path planning in games, and we conclude our paper with Section 6.

2 Path Planning for One Group

We are given one group of units, who all need to move from their origin p to their destination s. We assume that all units have equal width and speed. Our goal is to maximize the number of units that have reached q for each time t; using this approach, we automatically minimize both the average arrival time and the time by which all units have reached the destination. We further assume that the environment in which the units move is static.

To solve the problem, we first need a directed graph that resembles the free space in the environment. There are several ways to create such a graph. One possibility is to use tiles but this may lead to unnatural paths. A better alternative is to use a waypoint graph [16] in combination with a navigation mesh [6]. No matter how the graph has been constructed, we determine for each arc (i, j) in the graph its traversal time $l(i, j)$ as the time it takes to traverse the arc, and we determine its capacity c_{ij} as the number of units that can traverse the arc while walking next to each other. For instance in [6], the traversal time can be computed as the edge length divided by the maximum velocity and the capacity by the minimum clearance along the arc divided by the character's width. We choose the time unit as the time a unit has to wait until it can leave after the previous one. The path planning problem can then be modeled as a *dynamic flow problem* for which we have to determine a so-called *earliest arrival flow* from the origin to the destination. This problem can be solved by a classic algorithm due to Ford and Fulkerson [4], with a small adaptation due to Wilkinson [24]. The algorithm by Ford and Fulkerson computes a dynamic flow in an iterative version: given an optimal dynamic flow for the problem with $T - 1$ periods, an optimal dynamic flow for the T-period problem is constructed. Even though we do not have a deadline T but a number of units that have to go to the destination, we can use this algorithm by increasing the deadline each time by one time unit until all units have arrived.

When an optimal solution to the T-period problem has been constructed, Ford and Fulkerson's algorithm [4] splits it up in a set of *chain-flows*, which can be interpreted as a set of compatible paths in the graph. The flow (units in our case) are then sent through the graph following the chain-flows, where the last unit leaves the origin such that it arrives at the destination exactly at time T. Although the decomposition in chain flows maximizes the number of units that have arrived at the destination at time T, this solution does not need to be optimal when it is cut off at time t, even though their algorithm did find it as an intermediate product. Wilkinson [24] described a way to store the intermediate information of the algorithm to find an earliest arrival flow.

3 Path Planning for Multiple Groups

In this section, we consider the path planning problem for multiple groups of units. For each group, we are given the origin, the destination, and the size of the group. The goal is to minimize the average arrival time of all units. We assume that the graph that we use to model the problem is directed, that the capacities are constant over time, and that all units are available at time zero. At the end of this section, we describe what to do if these assumptions do not hold.

Since there are different groups with different origins and/or destinations, we do not have a dynamic flow problem anymore, but a *dynamic multi-commodity flow problem*, which is known to be $\mathcal{NP}$-hard in the strong sense. We present a new heuristic for the problem that is based on techniques from (integer) linear programming. We refer the reader to reference [25] for a description of this theory. The basic idea is that we formulate the problem as an integer linear program (ILP), but we restrict the set of variables by eliminating variables that are unlikely to get a positive value anyway. In this way, we make the problem tractable, without loosing too much on quality.

Instead of using variables that indicate for each arc at each time the number of units of group k that traverse this arc (an arc formulation), we use a formulation that is based on *paths* for each origin-destination pair. A path is described by the arcs that it uses and the times at which it enters these arcs. Here we require that the difference in the entering times of two consecutive arcs (i, j) and (j, k) on the path is no less than the traversal time $l(i, j)$ of the arc (i, j); if this difference is larger than $l(i, j)$, then this implies that there is a waiting time at j. Initially, we assume that there is infinite waiting capacity at all vertices. The advantage of using a formulation based on path-usage instead of arc-usage is twofold. First of all, we do not have to model the 'inflow = outflow' constraints anymore for each arc, time, and group. Second, we can easily reduce the number of variables by ignoring paths that are unlikely to be used in a good solution.

Suppose that we know all 'possibly useful' paths for each origin-destination pair. We can now model our path planning problem as an integer linear programming problem as follows. First, we introduce two sets of binary parameters to characterize each path $s \in S$, where S is the set containing all paths. The first one, which we denote by d_{ks}, indicates whether path s does connect origin/destination pair k (then d_{ks} gets value 1), or does not (in which case d_{ks} has value 0). The second set, which we denote by b_{ats}, keeps track of the time t at which arc a is entered by s: it gets value 1 if path s enters arc a at time t, and it gets value 0, otherwise. Note that these are parameters, which are fixed in advance, when the path s gets constructed. More formally, we have

$$d_{ks} = \begin{cases} 1 \text{ if path } s \text{ connects origin/destination pair } k \\ 0 \text{ otherwise} \end{cases}$$
$$b_{ats} = \begin{cases} 1 \text{ if path } s \text{ enters arc } a \in A \text{ at time } t \\ 0 \text{ otherwise.} \end{cases}$$

As decision variables we use x_s for each path $s \in S$, which will denote the number of units that follow path s. We use c_s to denote the cost of path s, which is equal to the arrival time of path s at its destination. We formulate constraints to enforce that the desired number y_k of units arrive at their destination for each origin/destination pair k and to enforce that the capacity constraints are obeyed. We define K as the number of origin/destination pairs, and we denote the capacity of arc $a \in A$ by u_a. We use T to denote the time-horizon; if this has not been defined, then we simply choose a time that is large enough to be sure that all units will have arrived by time T. This leads to the following integer linear program (ILP):

$$\min \sum_{s \in S} c_s x_s \qquad \text{subject to}$$

$$\sum_{s \in S} d_{ks} x_s = y_k \qquad \forall k = 1, \ldots, K$$

$$\sum_{s \in S} b_{ats} x_s \leq u_a \qquad \forall a \in A; t = 0, \ldots, T$$

$$x_s \geq 0 \text{ and integral} \qquad \forall s \in S.$$

Obviously, we do not know the entire set of paths S, and enumerating it would be impracticable. We will make a selection of the paths that we consider 'possibly useful', and we will solve the ILP for this small subset. We determine these paths by considering the LP-relaxation of the problem, which is obtained by removing the integrality constraints: the last constraint simply becomes $x_s \geq 0$ for all $s \in S$. The intuition behind taking the relaxation is that we use it as a guide toward useful paths, since the problems are so close together that a path which will be 'possibly useful' for the one will also be 'possibly useful' for the other. The LP-relaxation can be solved quickly because there is a clear way to add paths that improve the solution. We solve the LP-relaxation through the technique of column generation, which was first described by Ford and Fulkerson [5] for the multi-commodity flow problem.

Column Generation

The basic idea of column generation is to solve the linear programming problem for a restricted set of variables and then add variables that may improve the solution value until these cannot be found anymore. We can start with any initial set of variables, as long as it constitutes a feasible solution.

Given the solution of the LP for a restricted set of variables, we check if the current solution can be improved, and, if this the case, which paths we should add. It is well-known from the theory of column generation, in case of a minimization problem, that the addition of a variable will only improve the solution if its *reduced cost* is negative; if all variables have non-negative reduced cost, then we have found an optimal solution for the entire problem. In our case, the reduced cost of a path s, characterized by the parameters d_{ks} $(k = 1, \ldots, K)$ and b_{ats} $(a \in A; t = 0, \ldots, T)$ has reduced cost equal to $c_s - \sum_{k=1}^{K} \lambda_k d_{ks} - \sum_{a \in A} \sum_{t=0}^{T} \pi_{at} b_{ats}$, where λ_k $(k = 1, \ldots, K)$ and π_{at} $(a \in A; t = 0, \ldots, T)$ are the shadow prices for the corresponding constraints; these values follow from the solution to the current LP. The reduced cost takes the 'combinability' of the path s into account with respect to the current solution.

Since we are testing whether there exists a feasible path with negative reduced cost, we compute the path with minimum reduced cost. If this results in a non-negative reduced cost, then we have solved the LP-relaxation to optimality; if the outcome value is negative, then we can add the corresponding variable to the LP and iterate. The problem of minimizing the reduced cost is called the *pricing problem*.

We break up the pricing problem into K sub-problems: we determine the path with minimum reduced cost for each origin/destination pair separately. Suppose that we consider the problem for the lth origin/destination pair; we denote the origin and destination by p and q, respectively. Since we have $d_{ls} = 1$ and $d_{ks} = 0$ for all $k \neq l$, the term $\sum_{k=1}^{K} \lambda_k d_{ks}$ reduces to λ_l, and we ignore this constant from now on. The resulting objective is then to minimize the adjusted path length $c_s - \sum_{a \in A} \sum_{t=0}^{T} \pi_{at} b_{ats}$. We will solve this as a *shortest path* problem in a directed acyclic graph.

We construct the following graph, which is called the *time expanded* graph. The basis is the original graph, but we add a time index to each vertex: hence, vertex i in the original graph corresponds to the vertices $i(t)$, with $t = 0, \dots, T$. Similarly the arc (i, j) with traversal time $l(i, j)$ results in a series of arcs connecting $i(t)$ to $j(t + l(i, j))$. We further add waiting arcs $(i(t), i(t + 1))$ for each i and t. The length of the arc is chosen such that it corresponds to its contribution to the reduced cost. As c_s is equal to the arrival time of the path s in the destination, this term contributes a cost $l(i, j)$ to each arc $(i(t), j(t+l(i, j)))$ and cost 1 to each waiting arc. With respect to the term $-\sum_{a \in A} \sum_{t=0}^{T} \pi_{at} b_{ats}$, suppose that arc a corresponds to the arc (i, j). Then b_{ats}, with $a = (i, j)$, is equal to 1 if the path uses the arc $(i(t), j(t + l(i, j)))$ and zero otherwise; and, therefore, this term contributes $-\pi_{at}$ to the length of the arc $(i(t), j(t+l(i, j)))$, given that $a = (i, j)$.

Summarizing, we put the length of the arc $(i(t), j(t+l(i, j)))$ equal to $l(i, j) - \pi_{at}$, where $a = (i, j)$; the waiting arcs $(i(t), i(t + 1))$ simply get length 1. The path that we are looking for is the shortest one from $p(0)$ to one of the vertices $q(t)$ with $t \in \{0, \dots, T\}$. We use the A^* algorithm [8] to solve this problem. We compute the reduced cost by subtracting λ_l. If this path has negative reduced cost, then we add it to the LP. Since we know the shortest paths from $p(0)$ to each vertex $q(t)$, we do not have to restrict ourselves to adding only the path with minimum reduced cost, if there are more paths with negative reduced cost. If in all K sub-problems the shortest paths have non-negative reduced cost, then the LP has been solved to optimality.

Obtaining an Integral Solution

Most likely, some of the variables in our optimal LP solution will have a fractional value, and, hence, we cannot follow this solution, as we cannot send fractions of units along a path. We use a heuristic to find a good integral solution. Since we have generated 'useful' paths when we solved the LP, it is a safe bet that these are 'useful' paths for the integral problem as well. Hence, we include all these paths in the ILP. Since there is no guarantee that these paths will enable a feasible solution to the ILP, we add some paths, which we construct as follows. First, we round down all decision variables, which leads to an integral solution that satisfies the capacity constraints, but in which too few units will arrive at their destination. For the remaining units, we construct additional paths using Cooperative A* by Silver [20]. These paths are added to the ILP, which is then solved to optimality by the ILP-solver CPLEX [1].

Extensions

Time constraints on the departure and arrival. It is possible to specify an earliest departure and/or latest possible arrival time for each group of units. These can be incorporated efficiently in the paths by restricting the time expanded graph. The only possible drawback is, if we put these limits too tight, that we may make the problem infeasible. Since the problem of deciding whether there is a feasible solution is $\mathcal{NP}$-complete, we apply a computational trick. We replace the y_k in the constraint that y_k units have to move from the origin to the destination by $y_k + Q_k$, where Q_k is an artificial variable measuring the number of units of group k that did not reach their destination. We now add a term $\sum_{k=1}^{K} w_k Q_k$ to the objective function, where w_k is a large penalty weight, which makes it unattractive for the units not to reach their target.

Changes in the environment. A change in the environment may lead to a change in the capacity of an arc (for example that it drops to zero if the arc gets closed) or to a change in the traversal time in a certain period. If we know the changes beforehand, then these are incorporated efficiently in our model. A change in the capacity can be modeled by making the capacity of arc a time dependent; the right-hand side of the capacity constraint then becomes u_{at} instead of u_a. A change in the traversal times can be modeled by changing the arcs in the graph that we use to solve the pricing problem.

Undirected edges in the graph. An undirected edge can be traversed both ways, which makes it much harder to model the capacity constraint. If, for example, the traversal time is l and we want to send x units through the edge at time t, then this is possible only if the number of units that start(ed) to traverse the edge from the other side at times $t - l + 1, t - l + 2, \ldots, t + l - 1$ does not exceed the remaining capacity. To avoid having to add this enormous number of constraints, we split such an edge e in two arcs, e_1 and e_2, which have a constant capacity over time. We do not fix the capacity distribution beforehand, but we make it time-independent by putting the capacities equal to u_{e_1} and u_{e_2}, which are two non-negative decision variables satisfying that $u_{e_1} + u_{e_2}$ is equal to the capacity of the edge. We can modify this time-independent capacity distribution a little by making u_{e_1} or u_{e_2} equal to zero until the first time it can be reached from any origin that is part of an origin/destination pair which is likely to use this arc. Similarly, in case of edges with capacity 1, we can fix $u_{e_1} = 1$ or $u_{e_2} = 1$

during given periods in time. Splitting an undirected edge results in the addition of two variables and one constraint.

4 Experiments

In this section, we will describe the experiments we have conducted. In particular, we investigated the efficiency of our solution on three difficult problems. The solution from Section 3 was implemented in C++ using the ILOG CPLEX Concert Technology library version 11.100 for solving the LPs and ILPs [1]. All the experiments were run on a PC (CentOS linux 5.5 with kernel 2.6.18) with an Intel Core 2 Duo CPU (3 GHz) with 2 GB memory. Only one core was used.

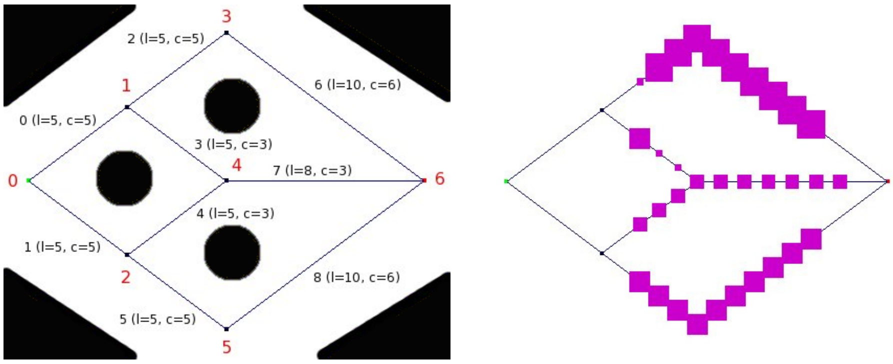
Each experiment was deterministic and was run a small number of times to obtain an accurate measurement of the average integral running times (in ms). These times include the initialization of the algorithm (such as the data structures, CPLEX, heuristics, building the initial LP), the column generation, the solving of the LP and ILP, path finding, and making an integer solution.

One Group

Fig. 1(a) shows the problem where one group moved from node 0 to node 6. The experiment was carried out for a single group with 100 through 500 units. Because it may be inefficient to let all units use the shortest path (e.g. when the capacity of the shortest path is low), it may be better to let some units take an alternative path. Indeed, as is shown in Fig. 1(b), the group (with 100 units) was split to minimize the average arrival times. The algorithm took 10ms for 100 units, 40ms for 200 units, 100ms for 300 units and 250ms for 400 units. Even with 500 units the algorithm took less than half a second. Note that these times can be distributed during the actual traversal of the paths, yielding real-time performance. In a game situation, the units should already start moving when the algorithm is executed to avoid stalls.

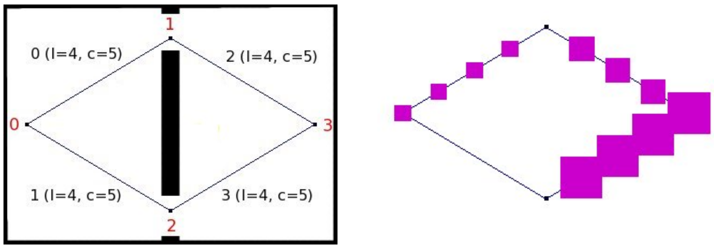
Two Groups Moving in Opposite Directions

In the following case, as is displayed in Fig. 2(a), two groups moved in opposite directions while switching their positions. One group started at node 0 and the other one at node 3. Since the arcs had limited capacities, the units had to share some arcs. There were two different homotopic paths between these two nodes, and both paths could be used by only 5 units per timestep. On the left side we placed 10 units and on the right side we had 50 units. Computing the solution took only 10ms. Also other combinations were tested, e.g. 20 versus 50 units (20ms), 20 versus 100 units (40ms), 40 versus 100 units (70ms), and 40 versus 200 units (230ms). The latter case is visualized in Fig. 2(b). Here, most units from the right side used the lower path, while some used the upper path. All the units from the left side used the upper path. Again, these running times were sufficiently low for real-time usage.



(a) Test environment (b) Output algorithm

Fig. 1. (a) The environment used for testing the division of units among the arcs. The (large) red numbers show the node numbering and the black numbers show the arc numbering. For every arc we give the length l and capacity c. (b) The output of the algorithm for 100 units at timestep 16. The pink squares symbolically represent the units, and the width of a square is proportional to the number of units.

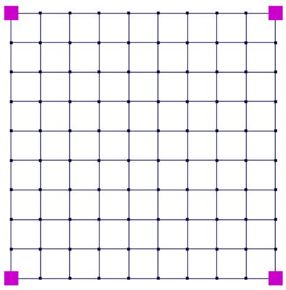


(a) Test environment (b) Output algorithm

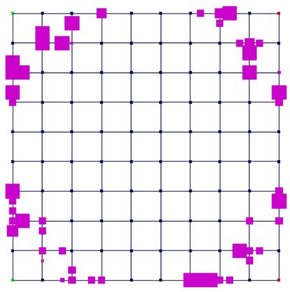
Fig. 2. (a) The environment and graph used for testing two groups moving in opposite directions, i.e. one group starts at node 0 and the other one starts at node 3. (b) The output of the algorithm for 40 versus 200 units at timestep 3.

Four Groups with Many Units Moving in a Big Graph

We created a large graph whose structure was a raster with arcs between the raster points. The length of these arcs was set to 3 with capacity 20. We refer the reader to Fig. 3 for an illustration of this graph and the output of the algorithm. In every corner we placed 100 units that needed to move to their diagonally opposite corners. Computing their paths took $510ms$. We also tested the algorithm with 1000 units placed at each corner (where the capacities were scaled with the same proportion), which took $480ms$. The results clearly illustrated the scaling power of the algorithm as it did not slow down when both the number of units and the capacities were scaled proportionally.



(a) Timestep 0



(b) Timestep 9

Fig. 3. (a) The test environment is empty and the graph's nodes lie on a raster. (b) The output of the algorithm is displayed at timestep 0 and 9.

5 Applicability in a Game

There are three main challenges that we face when we want to apply our technique in a game. First, all motions should be computed in real-time. Second, the environment may change while the units are moving, thereby making our solution infeasible. Third, how to satisfy game-specific constraints? For example, in combat games, it may be undesirable that a single unit (or a small troop of units) follows a path that separates them from the main group. Moreover, there can be other units with different sizes and velocities (for example soldiers and tanks). Below we will describe how we can tackle these issues, at the expense of a small loss in quality of the solution.

There are several ways to reduce the running time. The first one is to find a feasible solution by rounding instead of by solving the ILP. Since rounding down will reduce the number of units for which we find paths, we artificially increase the number of units that need to move from each origin to destination. An additional advantage of this increase is, if we have more routes for the units than we need after rounding, that we can select the best paths in a post-processing step. If necessary, we can reduce the running time even further by quitting the column generation before we have solved the LP-relaxation to optimality. In any case, the remainder of the computations can be done while the first units start moving to avoid stalls.

As mentioned in Section 3, changes in the environment are incorporated easily in the algorithm, because we can usually reuse the large part of the solution that is not affected by the change. Hence, in such a case, we initialize with the former set of paths, adjust the capacity constraints, and add additional paths in the column generation phase. We let the units depart from their current position as much as possible, but a group that is spread over an edge is artificially grouped in one or both of the edge's endpoints to reduce the number of origins; this is not a major violation of the truth since most units cannot leave from this origin immediately due to a limited capacity, and during this waiting time they can move to the origin from their current position. If there are obvious targets for sabotage, like bridges, then we can already incorporate these possibilities by computing additional paths circumventing these edges while the current solution is being executed in the game.

Avoiding isolated units can be achieved by post-processing a solution in which too many units move. If this is not satisfactory, then we can remove these isolated paths from the set of available paths and resolve the LP. If there are other types of units like tanks in the game, then we could first find paths for the tanks, and, given this solution, find paths for the units subject to the remaining capacity.

6 Conclusion

We have presented a centralized method based on techniques from ILP for path planning problems involving multiple groups. The crux is that the LP-relaxation can be solved quickly by using column generation. The solution to the LP-relaxation can be used as a basis to construct a heuristic solution. We have described a method to find a good approximation by solving a restricted ILP. If the instance is so big that solving the ILP would require too much time, then we can still use the solution to the LP-relaxation to find a solution by clever rounding. The units can then already start moving according to this solution while a good solution for the remaining units is determined in the meantime. We further have described ways to address the special constraints that are posed upon us by a game.

In future work, we will integrate two efficient local collision-avoidance models [10, 22] to test whether our solution leads to visually pleasing motions. We think that our solution enhances the gameplay in difficult path planning situations involving one or multiple groups.

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