Normal forms, reduction and persistence at the 1:2:6 resonance

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Abstract

We study a Hamiltonian system on $\mathbb{R}^6$ near an elliptic equilibrium whose normal frequencies are in 1:2:6 resonance, following the programme of Birkhoff normalisation, symmetry reduction and Kolmogorov–Arnol'd–Moser (KAM) persistence. The analysis shows a clean three-stage structure. The cubic normal form is *forced* to be integrable — the resonance admits a single indecomposable cubic coupling — but it is necessarily degenerate, its action-Hessian vanishing identically. The quartic normal form first carries the quadratic-in-action terms that can render an integrable truncation non-degenerate, and we exhibit a worked potential whose quartic normalisation is precisely such a non-degenerate integrable system. Beyond quartic order the third oscillator generically couples in, integrability is lost, and only the KAM skeleton survives. Throughout, the contrast is drawn with the 1:2:4 resonance of Hanßmann–Mazrooei-Sebdani–Verhulst [3], whose cubic normal form is already non-integrable. The geometric heart of the analysis is the reduced one-degree-of-freedom system on the singular "teardrop" surface, the standard reduced space of the 1:2 resonance [5, 6].

1 Introduction

A central question of local Hamiltonian dynamics is to describe the motion in a neighbourhood of an elliptic equilibrium. The linearised flow is a superposition of harmonic oscillators, and the qualitative fate of the nonlinear system is governed by the arithmetic relations — the resonances — among their frequencies. The systematic tool is the Birkhoff–Gustavson normal form: one applies a formal sequence of symplectic, near-identity transformations to simplify the higher-order Taylor terms as far as the resonance permits, and reads the dynamics off the truncated normal form, whose error is controlled in a neighbourhood whose size depends on the truncation order [4, 2, 7].

This essay carries out that programme for an equilibrium on $\mathbb{R}^6$ whose frequencies are in 1:2:6 resonance, the problem posed in [1]. The Hamiltonian is Taylor-expanded as $H = H_0^0 + H_1^0 + \cdots$, with quadratic part

$$H_0^0(x, y) = \sum_{j=1}^{3} \alpha_j \, \frac{x_j^2 + y_j^2}{2}, \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3) = (1, 2, 6), \tag{1}$$

so that the origin is an elliptic equilibrium in 1:2:6 resonance. The superscript records the normalisation step and the subscript the homogeneous order (so $H_k^0 \in G_{k+2}$, degree $k + 2$); the symbols $H_0^0 + H_1^0 + \cdots$ denote the *original* Hamiltonian, and normalisation will move data along the diagonal $H_k^0 \mapsto H_0^k$. Note that consequently H_0^k is an element of G_{k+2}, not of G_2: after the diagonal move the subscript 0 marks a fully normalised term and no longer reads off the degree, which is carried by the superscript.

The resonance 1:2:6 is a genuine second-order resonance occupying, in Hanßmann's classification of three-degree-of-freedom resonances [2, §2.2, Table 2.2], the block whose cubic normal form is integrable; the 1:2:4 resonance of the companion study [3] sits one block higher, where the cubic normal form already couples all three degrees of freedom and is generically non-integrable. Keeping that contrast in view, our aim is to answer four questions and to expose the structural reasons behind each: which terms survive normalisation; when the truncated normal form is integrable; when it is non-degenerate in Kolmogorov's sense; and what of the resulting picture persists to the true, non-normalised system. The answers organise into a three-stage arc — integrable but degenerate at cubic order, non-degenerate integrable at quartic order, generically non-integrable but KAM-robust beyond — whose geometric centrepiece is the reduced phase space of the 1:2 resonance.

The algebraic backbone is invariant theory: the normal form is a function of the basic invariants of the resonant circle action, and the reduced phase space is the variety those invariants cut out through their syzygies. We use the integrity-basis and Molien machinery as in [6] and the singular-reduction viewpoint of [5]; graded-ring language follows [8], and the normal-form and KAM theorems are those of [4, 2, 7].

2 The quadratic part and the resonant circle action

Hamilton's equations for (1) decouple into three planar rotations. Complexifying by $z_j = x_j + iy_j$ — so z_j is the complex amplitude-and-phase coordinate of oscillator j, with $\{z_\iota, \bar{z}_j\} = -2i\delta_{\iota j}$ and action $\tau_j = \frac{1}{2} z_j \bar{z}_j$ — turns each pair into the scalar equation $\dot{z}_j = -i\alpha_j z_j$, integrated by

$$\varphi^t\colon \quad z_j(t) = e^{-i\alpha_j t}\, z_j(0), \qquad j = 1, 2, 3. \tag{2}$$

Because the frequencies are integers, $\varphi^{2\pi} = \mathrm{id}$: the flow is genuinely 2π-periodic and defines an S^1-action on $\mathbb{R}^6$, rather than winding densely on a torus as in the non-resonant case. This circle is the geometric content of the phrase "1:2:6 resonance." It is the symmetry with respect to which the whole normalisation will be equivariant, and — crucially for §6 — it is the symmetry one reduces by; the integer commensurability has the further technical benefit, exploited silently throughout, that the eigenvalues driving normalisation are integers, so no small divisors arise at any finite order [2, App. C].

3 The homological operator and the canonical splitting

To normalise is to transform H by the time-one flow of a generator F; infinitesimally this acts on functions through the adjoint operator

$$\mathcal{A} := \{\,\cdot\,, H_0^0\}, \qquad \mathcal{A}\colon G_d \longrightarrow G_d. \tag{3}$$

In the monomial basis $M_m = z_1^{m_1} z_2^{m_2} z_3^{m_3} \bar{z}_1^{m_4} \bar{z}_2^{m_5} \bar{z}_3^{m_6}$, the single-factor rules $\{z_j, H_0^0\} = -i\alpha_j z_j$ and $\{\bar{z}_j, H_0^0\} = +i\alpha_j \bar{z}_j$, together with the Leibniz rule, give the eigenvalue at once,

$$\mathcal{A} M_m = -i\,\langle \alpha, a\rangle\, M_m, \qquad a = a(m) = (m_1 - m_4,\, m_2 - m_5,\, m_3 - m_6) \in \mathbb{Z}^3. \tag{4}$$

Here a is the *exponent vector*, recording for each oscillator its net number of z's over $\bar{z}$'s. The assignment $m \mapsto a$ is additive, $a(MN) = a(M) + a(N)$, so it is a $\mathbb{Z}_3$-grading of the polynomial algebra [8]; the entire normal-form problem reduces to the combinatorics of the sublattice $\langle \alpha, \cdot\rangle = 0$.

The spectrum of $\mathcal{A}$ is purely imaginary, and since H_0^0 is semisimple the operator is diagonalisable. This yields the splitting

$$G_d = \ker \mathcal{A} \oplus \operatorname{im} \mathcal{A}, \tag{5}$$

canonically — without the choice of an inner product that the non-semisimple case would require [2, App. C]. The homological equation $\{F, H_0^0\} + H_0^k = H_k^0$ is then solved termwise: every non-resonant monomial, with $\langle \alpha, a\rangle \neq 0$, lies in $\operatorname{im}\mathcal{A}$ and is removed by dividing by its eigenvalue $-i\langle \alpha, a\rangle$, while the resonant monomials, with

$$\langle \alpha, a\rangle = a_1 + 2a_2 + 6a_3 = 0, \tag{6}$$

span $\ker \mathcal{A}$ and survive into the normal form [4, §10]. Equivalently, $\ker\mathcal{A}$ is the algebra of functions invariant under the flow (2), and projection onto it is averaging over the S^1. The single operative principle of all that follows is therefore: *kernel terms stay, image terms go.*

4 Resonant monomials and the ring of invariants

Imposing (6) degree by degree determines $\ker\mathcal{A}$ and hence the generators of the ring of S^1-invariants [6].

At degree three the constraint $a_1 + 2a_2 + 6a_3 = 0$ with $|a| \le 3$ and odd coordinate sum forces $a_3 = 0$ (the value 6 is too large for $|a_1| + 2|a_2|$ at this degree) and $a_1 = -2a_2$; the solution is unique up to the sign change $a \mapsto -a$: $a = (2, -1, 0)$, realised by $z_1^2\bar{z}_2$, whence the complex conjugate $\bar{z}_1^2 z_2$ with $a = (-2, 1, 0)$. Thus the cubic kernel is one indecomposable coupling, the 1:2 coupling, with two real generators

$$\sigma_1 = \tfrac{1}{2}\operatorname{Re}(z_1^2\bar{z}_2), \qquad \sigma_2 = \tfrac{1}{2}\operatorname{Im}(z_1^2\bar{z}_2), \tag{7}$$

which are the real and imaginary parts of the same complex monomial, not two independent couplings. They are not algebraically free: since $\sigma_1 + i\sigma_2 = \frac{1}{2}z_1^2\bar{z}_2$,

$$\sigma_1^2 + \sigma_2^2 = 2\,\tau_1^2\tau_2, \tag{8}$$

the syzygy that will reappear as the equation of the reduced phase space. This is the local counterpart of the 1:2 resonance invariant computation of [5, §VII.7] and the integrity-basis method of [6, §2.1].

At degree four two new families enter the kernel: the six pure-action products $\tau_i\tau_j$ ($a = 0$), and the indecomposable coupling $z_2^3\bar{z}_3$ with $a = (0, 3, -1)$, resonant because $2 \cdot 3 - 6 = 0$ — the $\alpha_2{:}\alpha_3 = 2{:}6 = 1{:}3$ relation, and the first term ever to involve oscillator 3. At degree five the genuinely three-oscillator coupling $z_1^2 z_2^2 \bar{z}_3$ appears, $a = (2, 2, -1)$, since $1 \cdot 2 + 2 \cdot 2 - 6 = 0$; the list of indecomposable couplings continues beyond it with $z_1^4 z_2 \bar{z}_3$ ($a = (4, 1, -1)$, degree six) and $z_1^6\bar{z}_3$ ($a = (6, 0, -1)$, degree seven), both resonant and neither a product of lower generators. A real basis of each kernel is obtained by replacing each conjugate pair $\{M, \bar{M}\}$ by $\operatorname{Re} M, \operatorname{Im} M$; every term of the real normal form is therefore a polynomial in the actions τ_1, τ_2, τ_3 and the resonant generators $\sigma_1, \sigma_2, \operatorname{Re}/\operatorname{Im}(z_2^3\bar{z}_3), \ldots$, constrained by their syzygies [5, 6].

5 The cubic normal form: forced integrability, unavoidable degeneracy

The cubic normal form is $\overline{H} = H_0^0 + H_0^1$ with $H_0^1 \in \ker\mathcal{A} \cap G_3$, and a phase rotation of (z_1, z_2) reduces its cubic part to a single real multiple of σ_1,

$$\overline{H} = \tau_1 + 2\tau_2 + 6\tau_3 + c\,\sigma_1, \qquad c \ge 0. \tag{9}$$

Such $\overline{H}$ *is* integrable, and integrable not by genericity but by force. The single coupling imposes a single cancellation: from $\{\tau_1, z_1^2\bar{z}_2\} = 2i\, z_1^2\bar{z}_2$ and $\{\tau_2, z_1^2\bar{z}_2\} = -i\, z_1^2\bar{z}_2$ one reads that $N := \tau_1 + 2\tau_2$ commutes with the coupling ($2-2=0$), while τ_3 commutes with it trivially ($a_3 = 0$). Hence $\overline{H}$, N and τ_3 are three independent integrals in involution; oscillator 3 is a spectator and oscillators $1, 2$ form a 1:2 Fermi resonance. Because there is only one cubic coupling, there is simply not enough interaction to break integrability — in sharp contrast to the 1:2:4 chain, where $2\alpha_1 = \alpha_2$ and $2\alpha_2 = \alpha_3$ produce *two* indecomposable cubic couplings, two cancellation conditions, and generically only $\beta \propto \alpha$ surviving, so that the cubic normal form is already non-integrable [3].

It cannot, however, be non-degenerate. Non-degeneracy in Kolmogorov's sense is the invertibility of the action-Hessian, $\det D_\tau^2 \overline{H} \neq 0$ [2, 7], computed in action–angle variables of the integrable system at hand. The dynamics of (9) do carry their own action–angle variables: away from the singular fibres one may take, besides N and τ_3, the action I of the reduced periodic orbit of §6, and the corresponding frequency genuinely varies — it tends to 0 as the reduced orbit approaches the homoclinic loop through the apex, the equatorial slice of Figure 1. But the "fast" frequency conjugate to N and the spectator frequency $\alpha_3 = 6$ do not vary at all, so the frequency map has rank one and its Hessian is singular. Dynamically, each invariant $\mathbb{T}^3 = \overline{H}^{-1}(h) \cap N^{-1}(n) \cap \tau_3^{-1}(t_3)$ consists of invariant 2-tori — no orbit is dense in a 3-torus. The quadratic-in-action terms $\tau_i\tau_j$ that could bend the constant frequencies live in G_4, not G_3. The cubic truncation is thus, at best, a degenerate integrable system — a phenomenon common to these resonances at cubic order.

These statements may be verified on the cubic potential $H_1^0(x, y) = \frac{x_2}{2}\left(x_1^2 + x_2x_3\right)$ of [1]. Writing $x_j = \frac{1}{2}(z_j + \bar{z}_j)$ and projecting onto $\ker \mathcal{A}$, the term $x_2^2 x_3$ has no resonant content whatsoever — resonance would require $a_2 = \mp 3$, unreachable at degree three — and is removed entirely by the coordinate change, while the resonant content of $\frac{1}{2}x_1^2x_2$ is read off from

$$\tfrac{1}{2}\, x_1^2 x_2 = \tfrac{1}{16}(z_1 + \bar{z}_1)^2(z_2 + \bar{z}_2) = \tfrac{1}{16}\left(z_1^2\bar{z}_2 + \bar{z}_1^2 z_2\right) + \text{non-resonant} = \tfrac{1}{8}\,\mathrm{Re}(z_1^2\bar{z}_2) = \tfrac{1}{4}\sigma_1,$$

with the factors of 2 tracked explicitly at every step. The cubic normal form is therefore

$$\overline{H} = \tau_1 + 2\tau_2 + 6\tau_3 + \tfrac{1}{16}\left(z_1^2\bar{z}_2 + \bar{z}_1^2 z_2\right) = \tau_1 + 2\tau_2 + 6\tau_3 + \tfrac{1}{4}\sigma_1, \tag{10}$$

with the non-resonant remainder removed by the explicit generator $F_1 \in G_3$ solving $\{F_1, H_0^0\} = H_1^0 - H_0^1$, each non-resonant monomial $c_m M_m$ contributing $c_m M_m/(-i\langle \alpha, a\rangle)$. As anticipated, (10) carries the integrals H, $N = \tau_1 + 2\tau_2$, τ_3 but has a linear action part, hence is integrable yet degenerate. The generator F_1 is not merely bookkeeping: being nonlinear, it will generate the quartic terms of §7.

6 Reduction and the reduced dynamics: the teardrop

The two integrals reduce the system. Conserving τ_3 removes the spectator oscillator 3, which merely rotates at frequency 6; reducing by the S^1 generated by N removes a further degree of freedom. From six phase-space dimensions, fixing $N = n$ and τ_3 subtracts two and quotienting the two circles subtracts two more, leaving a two-dimensional reduced space — one degree of freedom. Since $H_0^0 = N + 6\tau_3$ is constant on each leaf, the reduced Hamiltonian is just the coupling, $h = \frac{1}{4}\sigma_1$.

The reduced Poisson structure closes on the 1:2 invariants $(\tau_1, \sigma_1, \sigma_2)$,

$$\{\tau_1, \sigma_1\} = -2\sigma_2, \qquad \{\tau_1, \sigma_2\} = 2\sigma_1, \qquad \{\sigma_1, \sigma_2\} = \tau_1(2n - 3\tau_1), \tag{11}$$

a Lie–Poisson structure whose Casimir, obtained by eliminating $\tau_2 = \frac{n-\tau_1}{2}$ from the syzygy (8), yields the surface of revolution

$$\sigma_1^2 + \sigma_2^2 = \tau_1^2\,(n - \tau_1), \qquad 0 \le \tau_1 \le n. \tag{12}$$

This “teardrop” is the standard reduced phase space of the 1:2 resonance: a topological sphere carrying one conical singularity and one smooth pole, exactly the surface and singular structure of the singular reduction in [5, §VII.7, Fig. 7.1] and [6, §2.1]. The conical apex at $\tau_1 = 0$ is the $\mathbb{Z}_2$-isotropy stratum of the reduced circle action — there $z_1 = 0$ and only the oscillator-2 plane rotates — and is the oscillator-2 normal mode; the smooth cap at $\tau_1 = n$ is the oscillator-1 normal mode (Figure 1).

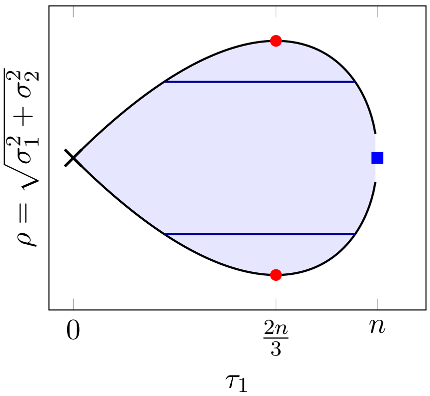

Figure 1: Meridian ($\sigma_2 = 0$) of the reduced phase space (12); revolve about the τ_1-axis for the teardrop. Conical apex $\tau_1 = 0$ ($\times$, oscillator-2 mode; unstable, but of index 0 rather than a saddle, see §6); two mixed-mode centres ($\bullet$) of the reduced Hamiltonian flow at $\tau_1 = \frac{2n}{3}$; smooth cap $\tau_1 = n$ ($\blacksquare$, oscillator-1 mode, not an equilibrium). Orbits are the $\sigma_1 = \text{const}$ slices (blue).

Under $h = \frac{1}{4}\sigma_1$ the reduced equations are

$$\dot{\tau}_1 = -\tfrac{1}{2}\sigma_2, \qquad \dot{\sigma}_1 = 0, \qquad \dot{\sigma}_2 = -\tfrac{1}{4}\,\tau_1(2n - 3\tau_1), \tag{13}$$

so σ_1 is conserved and the orbits are the closed curves $\{\sigma_1 = \text{const}\} \cap \{\text{teardrop}\}$. The relative equilibria satisfy $\sigma_2 = 0$ and $\tau_1(2n - 3\tau_1) = 0$: the apex $\tau_1 = 0$, an unstable equilibrium through which the homoclinic separatrix runs — though of Poincaré index 0 rather than the -1 of a saddle, which is possible only because the apex is a singular point of the reduced space — and the two mixed-mode centres at $\tau_1 = \frac{2n}{3}$, $\sigma_1 = \pm\frac{2n^{3/2}}{3\sqrt{3}}$. The cap $\tau_1 = n$ is conspicuously *not* an equilibrium: there $\dot{\sigma}_2 = \frac{1}{4}n^2 \neq 0$, equivalently $\dot{z}_2 = -\frac{i}{8}z_1^2 \neq 0$ when $z_2 = 0$. This asymmetry between the two normal modes — one a centre, the other not even stationary — is the dynamical fingerprint of the 1:2 resonance: the fundamental seeds its second harmonic, the classical second-harmonic energy exchange. Reconstruction lifts the reduced orbits to invariant 2-tori in the $(1, 2)$-subsystem and, with the spectator restored, to 3-tori [5].

7 The quartic normal form: non-degeneracy and conditional integrability

A quartic normal form adds $H_0^2 \in \ker \mathcal{A} \cap G_4$, so the general expression is

$$\overline{H} = \tau_1 + 2\tau_2 + 6\tau_3 + c\,\sigma_1 + \tfrac{1}{2}\sum_{i \le j} b_{ij}\,\tau_i\tau_j + d_1\,\mathrm{Re}(z_2^3\bar{z}_3) + d_2\,\mathrm{Im}(z_2^3\bar{z}_3), \tag{14}$$

with two genuinely new structures relative to the cubic order: the quadratic-in-action terms $\tau_i\tau_j$ — in Hanßmann's notation the Hessian coefficients α_{ij}, and what twist-map theory [4] would call the twist — and the 1:3 coupling $z_2^3\bar z_3$ that first reaches oscillator 3. Expression (14) is written at exact resonance; a full unfolding would in addition carry detuning terms $\delta_1\tau_1+\delta_2\tau_2+\delta_3\tau_3$ in the linear part, which we suppress by fixing $\alpha=(1,2,6)$ precisely.

Both verdicts of §5 now move. Non-degeneracy becomes available: with the $\tau_i\tau_j$ present, $D^2_\tau\overline{H}$ is a genuine quadratic form, and the Kolmogorov condition $\det D^2_\tau\overline{H}\neq 0$ holds off a hypersurface in coefficient space; where it fails one may fall back on iso-energetic non-degeneracy or Rüssmann's weaker curvature conditions [2]. Geometrically, the frequency map $\omega(\tau)=\partial_\tau\overline{H}$ — whose value at the origin is the linear $\alpha=(1,2,6)$, and whose dependence on τ is precisely $D^2_\tau\overline{H}$ — becomes a local diffeomorphism and sweeps a positive-measure set of frequencies; at cubic order it was the constant α, which is exactly the degeneracy. If $c=0=d_1=d_2$ the variables τ_1,τ_2,τ_3 are genuine actions of action–angle variables of the quartic normal form, and $\det D^2_\tau\overline{H}=\det(b_{ij})_{ij}$ is exactly the determinant entering the Kolmogorov condition. Dynamically, the foliation of the invariant $\mathbb{T}^3$ into invariant 2-tori that characterised the cubic order is destroyed: as soon as $b_{33}\neq 0$ the spectator frequency varies with the actions, and in general so does the fast frequency; and because the frequency map now sweeps a positive-measure set, many non-resonant $\mathbb{T}^3$ means mostly Diophantine $\mathbb{T}^3$.

Integrability, by contrast, is no longer automatic. A linear action combination $\sum_j\beta_j\tau_j$ is conserved by the σ_1 coupling iff $2\beta_1-\beta_2=0$ and by the $z_2^3\bar z_3$ coupling iff $3\beta_2-\beta_3=0$; with both couplings active the only solution is $\beta\propto(1,2,6)$, i.e. H_0^0 itself, yielding no new integral. The generic quartic normal form therefore retains only H and H_0^0 — one short of Liouville integrability in three degrees of freedom. No third integral survives among the natural candidates; genuine non-integrability is what one expects, though — as such statements go — it is not easy to prove. The integrable case is the codimension-two locus $d_1=d_2=0$, where N and τ_3 persist; combined with the now-available twist, this is a non-degenerate integrable system. Thus, unlike at cubic order, the quartic normal form *can* be non-degenerate and integrable, but only while oscillator 3 stays decoupled.

The worked potential realises exactly this fortunate case, and by a mechanism worth isolating. Its input has no quartic part, $H_2^0=0$; yet the nonlinear transformation ϕ_{F_1} removing the cubic remainder generates one — the one genuinely second-order step — through

$$\widetilde{H}_2^0=\tfrac12\{H_1^0+H_0^1,F_1\},\qquad H_0^2=\operatorname{Proj}_{\ker\mathcal{A}}\widetilde{H}_2^0,\tag{15}$$

the degree-three bracket landing in degree four. Carrying out (15) and projecting, the coefficient of $z_2^3\bar z_3$ vanishes — so this potential lands in the integrable branch — and the surviving term is purely quadratic in the actions,

$$H_0^2=-\tfrac{9}{128}\tau_1^2-\tfrac{1}{32}\tau_1\tau_2-\tfrac{19}{480}\tau_2^2+\tfrac{1}{20}\tau_2\tau_3,\qquad \det D^2_\tau H_0^2=\tfrac{9}{25600}\neq 0.\tag{16}$$

The non-vanishing determinant certifies Kolmogorov non-degeneracy, so the quartic normal form

$$\overline{H}=\tau_1+2\tau_2+6\tau_3+\tfrac14\sigma_1-\tfrac{9}{128}\tau_1^2-\tfrac{1}{32}\tau_1\tau_2-\tfrac{19}{480}\tau_2^2+\tfrac{1}{20}\tau_2\tau_3$$

is a non-degenerate integrable system with integrals H,N,τ_3. The numerical coefficients in (16) depend on the bookkeeping of the cubic term through a simple scaling law, which makes precise how much of the computation is convention-independent. Replacing the cubic part H_1^0 by λH_1^0 — the slip recorded in the footnote of §5, carried through consistently, amounts to $\lambda=\frac12$ — scales the generator to λF_1 by linearity of the homological equation, and hence the quartic normal form to $\lambda^2H_0^2$ by bilinearity of the bracket (15): every coefficient in (16) is multiplied by λ^2, the determinant by $\lambda^6\neq 0$, and the coefficient of $z_2^3\bar z_3$, a λ^2-multiple of zero, still vanishes. The transformation ϕ_{F_1}, and with it the action–angle coordinates in which the normal form is read, do change with λ; the conclusions — integrability and Kolmogorov non-degeneracy — do not.

On a leaf $\{N = n, \tau_3 = \zeta\}$ the reduction of §6 applies unchanged, save that the reduced Hamiltonian acquires the action-quadratic contribution restricted to the leaf,

$$h(\tau_1, \sigma_1) = \tfrac{1}{4}\sigma_1 + Q(\tau_1), \qquad Q(\tau_1) = -\tfrac{31}{480}\tau_1^2 + \tfrac{n-6\zeta}{240}\,\tau_1, \tag{17}$$

a downward parabola whose vertex — and hence the mixed-mode equilibria — can be controlled by the conserved values n and ζ. The flat slices of the cubic picture become Q-curved cuts of the teardrop, the relative equilibria move, and as roots collide one sees centre–saddle and pitchfork bifurcations of the normal and mixed modes [2, 6]. Because the system is integrable, reduced to one explicit degree of freedom, and non-degenerate, this bifurcation analysis is complete, and KAM theory applies to the full Hamiltonian: the invariant tori carrying frequencies that satisfy a Diophantine condition

$$|\langle k, \omega(\tau)\rangle| \geq \gamma\,|k|^{-\kappa} \quad \text{for all } k \in \mathbb{Z}^3 \setminus \{0\} \qquad (\gamma > 0,\ \kappa > 2) \tag{18}$$

persist and fill most of phase space [2, 7].

8 Higher-order normalisation and the limits of the picture

The decoupling of oscillator 3 in the worked example is an accident of low order and of this particular potential, not a feature of the resonance: the vanishing of the $z_2^3\bar{z}_3$ coefficient is forced by the specific F_1, not by any symmetry. Generically oscillator 3 enters, either through $z_2^3\bar{z}_3$ at degree four or, failing that, through the genuine three-oscillator term $z_1^2 z_2^2 \bar{z}_3$ at degree five. That this term is within reach of the normalisation is elementary: iterated brackets of cubic monomials already produce it, e.g.

$$\big\{\{z_1\bar{z}_1 z_2, z_1^2 z_3\}\,,\ z_2\bar{z}_3^2\big\} = 16\, z_1^2 z_2^2 \bar{z}_3$$

(in the convention $\{z_j, \bar{z}_j\} = -2i$ of §2; the coefficient is 4 in the convention $\{z_j, \bar{z}_j\} = -i$), two bracketing steps mirroring two normalisation steps applied to cubic data. The moment any such term is present, the cancellation argument of §7 eliminates the third independent integral — H itself and H_0^0 always survive — and the truncation becomes generically non-integrable: the normal form still has H_0^0 as second integral, so reduction of its circle action leaves a genuinely two-degree-of-freedom system, with separatrix splitting and thin chaotic layers near the homoclinic structures.

What survives is the robust core. The 1:2 energy-exchange skeleton and the quartic non-degeneracy already established imply, via KAM, that the invariant tori carrying Diophantine frequencies (18) persist under both the higher normal-form terms and the non-normalised remainder; the qualitative picture of §6–§7, including the teardrop's bifurcation diagram, endures. The essential caveat is that Birkhoff normalisation is asymptotic: the formal series is generically divergent, and the meaningful object is an optimal finite truncation whose remainder is exponentially small in the distance to the equilibrium, in the Nekhoroshev manner [2, App. C]. Persistence, moreover, is not confinement: the original $H = H_0^0 + H_1^0 + \cdots$ has no second integral, so the surviving 3-tori, three-dimensional in a five-dimensional energy shell, do not divide it, and motion can in principle get very far away (Arnol'd diffusion); since the remainder is exponentially small, such diffusion takes an exponentially long time. Further normalisation refines and confirms the dynamics already described; it does not overturn it.

The contrast with the 1:2:4 chain is instructive. There the accidental decoupling that we enjoy at one order is replaced by a genuine discrete symmetry: the chain of [3] is a periodic four-particle chain, and when two diametrically opposite particles carry equal masses the chain is invariant under a $\mathbb{Z}_2$ reflection; this symmetry makes the cubic terms vanish identically, keeps the analogue of τ_3 an integral and defers the breakdown of integrability to all orders, while the generic case is non-integrable already at cubic order (equal *adjacent* masses give no such symmetry). In 1:2:6 the same protective role is played, but only once and only by the arithmetic that pushes the 2:6 = 1:3 coupling out to degree four.

9 Conclusion

The 1:2:6 resonance is a tame second-order resonance whose local dynamics — for the worked potential $\frac{x_2}{2}(x_1^2 + x_2 x_3)$ of [1] — can be described essentially completely; the completeness is a property of that Hamiltonian, not of the resonance as such. The structural reason is arithmetic: among the low-degree resonant lattice points, the frequency 6 admits only the single cubic coupling $z_1^2 \bar{z}_2$, delaying any coupling to the third oscillator until degree four. This single fact propagates through the entire analysis. It forces the cubic normal form to be integrable — with H, $N = \tau_1 + 2\tau_2$ and τ_3 in involution and oscillator 3 a spectator — while the linearity of the action part makes it necessarily degenerate. It leaves room for the quartic normal form, once the quadratic-in-action terms appear, to be non-degenerate — whether this is realised depends on the Hamiltonian at hand — and to remain integrable precisely on the locus where oscillator 3 stays uncoupled; the worked potential lands on that locus and yields a bona fide non-degenerate integrable approximation, with $\det D_\tau^2 H_0^2 = \frac{9}{25600} \neq 0$. And it concedes, beyond quartic order, that oscillator 3 generically couples in, integrability is lost, and only the KAM skeleton — the 1:2 exchange and the persistent Diophantine tori — survives.

Geometrically the whole story is carried by one object, the reduced phase space of the 1:2 resonance: the singular teardrop (12), with its conical normal-mode apex and its smooth cap, on which the reduced Hamiltonian $\frac{1}{4}\sigma_1 + Q(\tau_1)$ picks out the two mixed-mode centres — equilibria of the Hamiltonian flow, not features of the surface — and plays out the second-harmonic exchange and its bifurcations [5, 6]. The same arithmetic that makes the picture clean also places 1:2:6 one block below 1:2:4 in Hanßmann's classification [2, Table 2.2], and the comparison with the particle chain [3] shows what a single coupling, an accidental zero, or a genuine discrete symmetry can each contribute to integrability. The limits of the description are exactly those of normal-form theory itself: the truncation is an asymptotic, not convergent, approximation, and the non-normalised tail — though exponentially small at optimal order — is never removed.

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